REACHABLE STATES FOR INFINITE-DIMENSIONAL LINEAR SYSTEMS: OLD AND NEW

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ABSTRACT

This work describes some recent results on the reachable spaces for infinite dimensional linear time invariant systems. The focus is on systems described by the constant coefficients heat equation, when the question is shown to be intimately connected to the theory of Hilbert spaces of analytic functions.

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1. INTRODUCTION

Determining the reachable states of a controlled dynamical system is a major question in control theory. The set formed by these states measures our capability of acting on a system and provides important information for safety verifications. This fundamental question is well understood for linear finite-dimensional systems but much less is known in an infinite-dimensional context (namely for systems governed by partial differential equations). Most of the known results concern the case when the system is exactly controllable, which means, as reminded below, that the reachable state coincides with the state space of the system. When the reachable space is a strict subspace of the state space, its description is generally far from being complete. Note that for infinite-dimensional systems, as recalled below, the reachable space also serves to define the main controllability types in a precise and condensed manner.

The present work aims at describing some of the major advances in this field, with focus on those involving interactions with complex and harmonic analysis techniques. With no claim of exhaustiveness, we first briefly discuss some of the interactions which are by now classical (such as those based on Ingham–Beurling-type theorems) and then we describe recent advances involving various complex analysis techniques, such as the theory of reproducing kernel Hilbert spaces (namely of Bergman type) or separation of singularities for spaces of holomorphic functions.

The study of the reachable space and of the controllability of finite-dimensional linear control systems have been set at the center of control theory by the works of R. Kalman in the 1960s (see, for instance, [20]). Controllability theory for infinite-dimensional linear control systems emerged soon after. Among the early contributors we mention D. L. Russell, H. Fattorini, T. Seidman, A. V. Balakrishnan, R. Triggiani, W. Littman, and J.-L. Lions. The latter gave the field an enormous impact with his book [26], which opened the way to fascinating interactions of controllability theory with various fields of analysis.

The related question of the study of the *reachable space of infinite-dimensional linear control systems*, namely those governed by partial differential equations, has been initiated, as far as we know, by the papers of Russell [31] and Fattorini and Russell [11]. In these famous papers the authors provide relevant information on the reachable space of systems described by hyperbolic and parabolic partial differential equations in one space dimension controlled from the boundary.

The techniques generally employed for one-dimensional wave or Euler–Bernoulli plate equations are quite close to those used for the corresponding controllability problems, in particular Ingham–Beurling-type theorems, and they often provide full characterizations of the reachable space. To give the reader a flavor of the techniques used for systems describing one-dimensional elastic structures, we give an abstract result in Section 3 and an illustrating example in Section 4. The situation is much more complicated for the wave equation in several space dimensions where (with the exception of the exactly controllable case) characterizing the reachable spaces is essentially an open question.

On the other hand, determining the reachable states for systems described by the heat equation with boundary control is an extremely challenging question, on which major advances have been obtained within the last years. Indeed, due to the smoothing effect of the heat kernel, the reachable states are expected to be very smooth functions. However, since the control functions are in general only in L^2, the characterization of the reachable space, even in apparently very simple situations, is a difficult question, solved only very recently. To be more precise, consider the system

$$\begin{cases} \dfrac{\partial \theta}{\partial t}(t,x) = \dfrac{\partial^2 \theta}{\partial x^2}(t,x), & t \geqslant 0,\ x \in (0,\pi), \\ \theta(t,0) = u_0(t), \quad \theta(t,\pi) = u_\pi(t), & t \in [0,\infty), \\ \theta(0,x) = 0, & x \in (0,\pi), \end{cases} \tag{1.1}$$

which models the heat propagation in a rod of length π, controlled by prescribing the temperature at both ends. It is well known that for every $u_0, u_\pi \in L^2[0,\infty)$, problem (1.1) admits a unique solution θ and that the restriction of this function to $(0,\infty) \times (0,\pi)$ is an analytic function. The *input-to-state maps* (briefly, input maps) $(\Phi_\tau^{\text{heat}})_{\tau \geqslant 0}$ are defined by

$$\Phi_\tau^{\text{heat}} \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix} = \theta(\tau,\cdot) \quad \big(\tau \geqslant 0,\ u_0, u_\pi \in L^2[0,\tau]\big). \tag{1.2}$$

Determining the reachable space at instant τ of the system determined by the 1D *heat equation with Dirichlet boundary control consists in determining* $\operatorname{Ran} \Phi_\tau^{\text{heat}}$.

The first result on this space goes back to [11], where it is shown that the functions which extend holomorphically to a horizontal strip containing $[0,\pi]$ and vanishing, together with all their derivatives of even order, at $x = 0$ and $x = \pi$, belong to $\operatorname{Ran} \Phi_\tau^{\text{heat}}$. The fact that some other types of functions (like polynomials), not necessarily vanishing at the extremities of the considered interval, are in the reachable space has been remarked in a series of papers published in the 1980s (see, for instance, Schmidt [36] and the references therein). A significant advance towards such a characterization was reported only in 2016, in the work by Martin, Rosier, and Rouchon [27], where it was shown that any function which can be extended to a holomorphic map in a disk centered in $\frac{\pi}{2}$ and of diameter $\pi \mathrm{e}^{(2\mathrm{e})^{-1}}$ lies in the reachable space. This result has been further improved in Dardé and Ervedoza [8], where it has been shown that any function which can be extended to a holomorphic one in a neighborhood of the square D defined by

$$D = \big\{ s = x + iy \in \mathbb{C} \mid |y| < x \text{ and } |y| < \pi - x \big\} \tag{1.3}$$

lies in the reachable space.

On the other hand, it is not difficult to check (see, for instance, [27, THEOREM 1]) that if $\psi \in \operatorname{Ran} \Phi_\tau^{\text{heat}}$ then ψ can be extended to a function holomorphic in D, so that the assertion in [8] suggests that the reachable space could in this case be connected to a classical space of holomorphic functions defined on D. This has been confirmed by a series of recent papers (see, Hartmann, Kellay, and Tucsnak [13], Normand, Kellay, and Tucsnak [21], Orsoni [29], and Hartmann and Orsoni [14]) which led to a full characterization of this space to be described in Section 6.

2. SOME BACKGROUND ON WELL-POSED LINEAR CONTROL SYSTEMS

The concept of a well-posed linear system, introduced in Salamon [35] and further developed in Weiss [44], plays an important role in control theory for infinite-dimensional systems. We briefly recall below some basic facts about these systems, including the definition of the reachable space and the three main controllability types.

Let U (the input space) and X (the state space) be Hilbert spaces (possibly infinite-dimensional). The spaces U and X will be constantly identified with their duals and, if there is no risk of confusion, the inner product and norm in these spaces will be simply denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively.

From a system-theoretic viewpoint, the simplest way to define a linear well-posed time-invariant system in a possibly infinite-dimensional setting is to introduce families of operators satisfying the properties in the definition below.

Definition 2.1. Let U and X be Hilbert spaces. A *well-posed linear control system* with input space U and state space X is a couple $\Sigma = (\mathbb{T}, \Phi)$ of families of operators such that

(1) $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ is an operator semigroup on X, i.e.,

- $\mathbb{T}_t \in \mathcal{L}(X)$ for every $t \geqslant 0$,
- $\mathbb{T}_0\psi = \psi$ for every $\psi \in X$,
- $\mathbb{T}_{t+\tau} = \mathbb{T}_t\mathbb{T}_\tau$ $(t, \tau \geqslant 0)$,
- $\lim_{t\to 0+} \mathbb{T}_t\psi = \psi$ $(\psi \in X)$;

(2) For every $t \geqslant 0$, we have $\Phi_t \in \mathcal{L}(L^2([0,\infty); U), X)$ and

$$\Phi_{\tau+t}(u \underset{\tau}{\diamondsuit} v) = \mathbb{T}_t\Phi_\tau u + \Phi_t v \quad (t, \tau \geqslant 0), \tag{2.1}$$

where the τ-*concatenation* of two signals u and v, denoted by $u \underset{\tau}{\diamondsuit} v$, is the function

$$u \underset{\tau}{\diamondsuit} v = \begin{cases} u(t) & \text{for } t \in [0,\tau), \\ v(t-\tau) & \text{for } t \geqslant \tau. \end{cases} \tag{2.2}$$

It can be shown that the above properties imply that the map

$$(t, u) \mapsto \Phi_t u,$$

is continuous from $[0,\infty) \times L^2([0,\infty); U)$ to X.

Let $A : \mathcal{D}(A) \to X$ be the generator of $\mathbb{T} = (\mathbb{T}_t)_{t\geqslant 0}$ on X. We denote by $\mathbb{T}^*$ the adjoint semigroup, which is generated by the adjoint of A^* of A. The operator domain $\mathcal{D}(A)$, when endowed with norm

$$\|\varphi\|_{X_1}^2 = \|\varphi\|^2 + \|A\varphi\|^2 \quad (\varphi \in X_1), \tag{2.3}$$

is a Hilbert space. This Hilbert space is denoted by X_1. Similarly, we denote by X_1^d the Hilbert space obtained by endowing $\mathcal{D}(A^*)$ with the norm

$$\|\varphi\|_{X_1^d}^2 = \|\varphi\|^2 + \left\|A^*\varphi\right\|^2 \quad \left(\varphi \in X_1^d\right). \tag{2.4}$$

Let X_{-1} be the dual of X_1^d with respect to the pivot space X, so that $X_1 \subset X \subset X_{-1}$ with continuous and dense embeddings. Note that, for each $k \in \{-1, 1\}$, the original semigroup $\mathbb{T}$ has a restriction (or an extension) to X_k that is the image of $\mathbb{T}$ through the unitary operator $(\beta I - A)^{-k} \in \mathcal{L}(X, X_k)$, where $\beta \in \rho(A)$ (the resolvent set of A). We refer to **[41, REMARK 2.10.5]** for a proof of the last statement. This restriction (extension) will be still denoted by $\mathbb{T}$.

An important consequence of Definition 2.1 is (see, for instance, **[44]**) that there exists a unique $B \in \mathcal{L}(U, X_{-1})$, called the *control operator* of Σ, such that

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-\sigma} B u(\sigma)\, \mathrm{d}\sigma \quad \big(\tau \geqslant 0,\ u \in L^2\big([0,\infty); U\big)\big). \tag{2.5}$$

Notice that in the above formula, $\mathbb{T}$ acts on X_{-1} and the integration is carried out in X_{-1}. The operator B can be found by

$$B\mathrm{v} = \lim_{\tau\to 0} \frac{1}{\tau} \Phi_\tau(\chi \cdot \mathrm{v}) \quad (\mathrm{v} \in U), \tag{2.6}$$

where χ denotes the characteristic function of the interval $[0, 1]$. We mention that it follows from the above definitions that if $(\mathbb{T}, \Phi)$ is a well-posed control system then for all $u \in L^2([0,\infty); U)$, $t \mapsto \Phi_t u$ is a continuous function from $[0, \infty)$ to X.

From the above facts, it follows that a well-posed control system can alternatively be described by a pair (A, B), where $A : \mathcal{D}(A) \to X$ generates a C^0-semigroup $\mathbb{T}$ on X and $B \in \mathcal{L}(U, X_{-1})$ is an *admissible control operator for* $\mathbb{T}$. This latter property means that for some $t > 0$, the operator Φ_t defined by (2.5) has its range contained in X. We refer to Tucsnak and Weiss **[41, SECTIONS 4 AND 5]** for more material on this concept.

We also recall (see, for instance, **[41, PROPOSITION 4.2.5]**) that the families $\mathbb{T}$ and Φ can also be seen as the solution operators for the initial value problem

$$\dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0, \tag{2.7}$$

in the following sense:

Proposition 2.1. *Let $\tau > 0$. Then for every $z_0 \in X$ and every $u \in L^2([0,\tau]; U)$, the initial value problem* (2.7) *has a unique solution*

$$z \in C\big([0,\tau]; X\big) \cap H^1\big((0,\tau); X_{-1}\big).$$

This solution is given by

$$z(t) = \mathbb{T}_t z_0 + \Phi_t u \quad \big(t \in [0,\tau]\big). \tag{2.8}$$

In most of the remaining part of this work, we describe a well-posed control system either by a couple $(\mathbb{T}, \Phi)$ as in Definition 2.1 or by a couple (A, B), where A is the generator of $\mathbb{T}$ and B is the unique operator in $\mathcal{L}(U, X_{-1})$ satisfying (2.5).

Given a well-posed control system $\Sigma = (\mathbb{T}, \Phi)$ and $\tau > 0$, *the reachable space in time τ of* Σ is defined as the range $\operatorname{Ran}\Phi_\tau$ of the operator Φ_τ. This space can be endowed with the norm induced from $L^2([0,\tau]; U)$, which is

$$\|\eta\|_{\operatorname{Ran}\Phi_\tau} = \inf_{\substack{u \in L^2([0,\tau];U)\\ \Phi_\tau u = \psi}} \|u\|_{L^2([0,\tau];U)} \quad (\eta \in \operatorname{Ran}\Phi_\tau). \tag{2.9}$$

Remark 2.1. From the above construction of the reachable space, it easily follows (see, for instance, Saitoh and Sawano [34, THEOREM 2.36]) that, when endowed with the norm (2.9), $\operatorname{Ran}\Phi_\tau$ becomes a Hilbert space, isomorphic to the orthogonal complement in $L^2([0,\tau];U)$ of $\operatorname{Ker}\Phi_\tau$.

Remark 2.2. We obviously have that Φ_τ is onto from $L^2([0,\tau];U)$ onto $\operatorname{Ran}\Phi_\tau$. Moreover, we have

$$\|\Phi_\tau\|_{\mathcal{L}(L^2([0,\tau];U),\operatorname{Ran}\Phi_\tau)} = 1. \tag{2.10}$$

Indeed, we clearly have that

$$\|\Phi_\tau\|_{\mathcal{L}(L^2([0,\tau];U),\operatorname{Ran}\Phi_\tau)} \leqslant 1.$$

Moreover, if $\eta \in \operatorname{Ran}\Phi_\tau \setminus \{0\}$ there exists a sequence $(u_n)_{n\geqslant 0}$ in $(L^2([0,\tau];U)\setminus\{0\})^{\mathbb{N}}$ such that $\Phi_\tau u_n = \eta$ for every $n \in \mathbb{N}$ and $\|u_n\|_{L^2([0,\tau];U)} \to \|\eta\|_{\operatorname{Ran}\Phi_\tau}$ as $n \to \infty$. We thus have that

$$\lim_{n\to\infty} \frac{\|\Phi_\tau u_n\|_{\operatorname{Ran}\Phi_\tau}}{\|u_n\|_{L^2([0,\tau];U)}} = 1,$$

and, consequently, we have (2.10).

If the spaces U and X are finite-dimensional then there exists $A \in \mathcal{L}(X)$ such that $\mathbb{T}_t = \exp(tA)$ for every $t \geqslant 0$ and $B \in \mathcal{L}(U,X)$. In this case the following result, known as the *Kalman rank condition* for controllability, holds:

Proposition 2.2. *If U and X are finite-dimensional then we have, for every $\tau > 0$,*

$$\operatorname{Ran}\Phi_\tau = \operatorname{Ran}\begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}. \tag{2.11}$$

Remark 2.3. From Proposition 2.2, it follows in particular that for finite-dimensional systems the reachable space does not depend on the time horizon $\tau > 0$. Moreover, it is not difficult to check (see, for instance, Normand, Kellay, and Tucsnak [21]) that Proposition 2.2 implies that $\operatorname{Ran}\Phi_\tau$ coincides with the range of the restriction of Φ_τ to signals which can be extended to entire functions from $\mathbb{C}$ to U.

Unlike the finite-dimensional case, for general well-posed linear control systems, there is no simple characterization of the reachable space in terms of the operators A and B. Moreover, this space depends in general on τ and, for most systems described by partial differential equations, we have only a small amount of information on the reachable space. Another difference with respect to the finite-dimensional case is that the range $\operatorname{Ran}\Phi_\tau^\infty$ of the restriction of Φ_τ to a smaller space (such as $L^\infty([0,\tau];U)$) is in general a strict subset of $\operatorname{Ran}\Phi_\tau$.

The concept of reachable space appears, in particular, in the definition of the main three controllability concepts used in the infinite-dimensional system theory.

Definition 2.2. Let $\tau > 0$ and let the pair $(\mathbb{T},\Phi)$ define a well-posed control LTI system.

- The pair $(\mathbb{T},\Phi)$ is *exactly controllable in time* τ if $\operatorname{Ran}\Phi_\tau = X$.

- $(\mathbb{T}, \Phi)$ is *approximately controllable in time* τ if $\operatorname{Ran} \Phi_\tau$ is dense in X.
- The pair $(\mathbb{T}, \Phi)$ is *null-controllable in time* τ if $\operatorname{Ran} \Phi_\tau \supset \operatorname{Ran} \mathbb{T}_\tau$.

From the above definition, we see that for systems which are approximately controllable in some time $\tau > 0$ we can define the dual $(\operatorname{Ran} \Phi_\tau)'$ of $\operatorname{Ran} \Phi_\tau$ with respect to the pivot space X (we refer to Tucsnak and Weiss [41, SECTION 2.9] for the general definition of this concept). More precisely:

Definition 2.3. Let $\Sigma = (\mathbb{T}, \Phi)$ be approximately controllable in time τ and let $(\operatorname{Ran} \Phi_\tau)'$ be the dual of $\operatorname{Ran} \Phi_\tau$ with respect to the pivot space X, so that we have

$$\operatorname{Ran} \Phi_\tau \subset X \subset (\operatorname{Ran} \Phi_\tau)',$$

with continuous and dense inclusions.

The dual $\Phi_\tau' \in \mathcal{L}((\operatorname{Ran} \Phi_\tau)', L^2([0, \tau]; U))$ of the operator Φ_τ introduced in (2.5) is defined by

$$\langle \Phi_\tau u, \eta \rangle_{\operatorname{Ran} \Phi_\tau, (\operatorname{Ran} \Phi_\tau)'} = \left\langle u, (\Phi_\tau)' \eta \right\rangle_{L^2([0,\tau];U)},$$

for every $u \in L^2([0, \tau]; U)$ and $\eta \in (\operatorname{Ran} \Phi_\tau)'$.

It can be easily checked that the norm in the space $\operatorname{Ran} \Phi_\tau$ can be characterized as follows:

Proposition 2.3. *Assume that (A, B) is approximately controllable in some time $\tau > 0$. Then*

$$\|\eta\|_{(\operatorname{Ran} \Phi_\tau)'} = \|\Phi_\tau^* \eta\|_{L^2([0,\tau];U)} \quad (\eta \in X), \tag{2.12}$$

where $\Phi_\tau^* \in \mathcal{L}(X, L^2([0, \tau]; U))$ is the adjoint of Φ_τ defined by

$$\langle \Phi_\tau u, \eta \rangle_X = \left\langle u, \Phi_\tau^* \eta \right\rangle_{L^2([0,\tau];U)} \quad \left(u \in L^2\left([0, \tau]; U\right), \ \eta \in X\right).$$

Note that the fact that the right-hand side of (2.12) defines a norm follows from the fact that $\operatorname{Ran} \Phi_\tau$ is dense in X.

A direct consequence of Proposition 2.3 is the following characterization of $(\operatorname{Ran} \Phi_\tau)'$:

Proposition 2.4. *If (A, B) is approximately controllable in time $\tau > 0$ then $(\operatorname{Ran} \Phi_\tau)'$ coincides with the completion of X with respect to the norm $\eta \mapsto \|\Phi_\tau^* \eta\|_{L^2([0,\tau];U)}$.*

By combining the above result with a classical duality argument (see, for, instance, [41, PROPOSITION 4.4.1]), we obtain:

Corollary 2.1. *If (A, B) is approximately controllable in time $\tau > 0$ then $(\operatorname{Ran} \Phi_\tau)'$ coincides with the completion of $\mathcal{D}(A^*)$ with respect to the norm $\eta \mapsto (\int_0^\tau \|B^* \mathbb{T}_t^* \eta\|^2 \, \mathrm{d}t)^{\frac{1}{2}}$.*

As already mentioned, in the infinite-dimensional case the reachable space generally depends on the time horizon τ. However, as precisely stated below, there exists an important class of infinite-dimensional systems for which the reachable space is independent of the time horizon.

Proposition 2.5. *If the well-posed linear control system* $(\mathbb{T}, \Phi)$ *is null-controllable in any positive time then* $\operatorname{Ran} \Phi_\tau$ *does not depend on* $\tau > 0$.

Following the ideas of [37], a very short proof of the above result is provided in [21].

3. SINGLE INPUT SYSTEMS WITH SKEW-ADJOINT GENERATOR

In this section we consider, for the sake of simplicity, a class of systems which can be seen as a "toy model" for many linear control problems involving the dynamics of flexible structures. In fact, our abstract result in Theorem 3.1 below can be directly applied only to problems in one space dimension. Nevertheless, estimates similar to the inequality in Theorem 3.2 below can be used when tackling some problems in several space dimensions, at least in particular geometric configurations (see, for instance, Allibert [2], Jaffard [17], Jaffard and Micu [18], or Komornik and Loreti [22]). The situation is much more complicated, requiring different techniques, in several space dimensions and with arbitrary shapes of the domain filled by the elastic structure, see, for instance, Avdonin, Belishev, and Ivanov [3].

Let $A : \mathcal{D}(A) \to X$ be a skew-adjoint operator, with nonempty resolvent set $\rho(A)$ and with compact resolvents. We denote by $(\phi_k)_{k\in\mathbb{Z}^*}$ an orthonormal basis of X consisting of eigenvectors of A. For every $k \in \mathbb{Z}^*$, we denote by $i\lambda_k$ the eigenvalue associated to the eigenvector ϕ_k, so that λ_k is real for all $k \in \mathbb{Z}^*$. Without loss of generality, we can assume that $\lambda_1 \geqslant \lambda_{-1}$ and

$$\lambda_{n+1} - \lambda_n \geqslant 0 \quad \big(n \in \mathbb{Z}^* \setminus \{-1\}\big). \tag{3.1}$$

According to Stone's theorem, the operator A generates a strongly continuous group of unitary operators on X. This group, denoted by $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$, is described by the formula

$$\mathbb{T}_t \psi = \sum_{k\in\mathbb{Z}^*} \langle \psi, \phi_k \rangle \exp(i\lambda_k t)\phi_k \quad (t \in \mathbb{R},\ \psi \in X). \tag{3.2}$$

Assume that the control space U is one-dimensional (i.e., that $U = \mathbb{C}$) and that the control operator $B \in \mathcal{L}(U; X_{-1})$ is given by

$$B\mathrm{u} = \mathrm{u}b \quad (\mathrm{u} \in U), \tag{3.3}$$

with b a fixed element of X_{-1}, where, as mentioned in Section 2, X_{-1} is the dual of $\mathcal{D}(A^*)$ with respect to the pivot space X. For b as above and $\psi \in \mathcal{D}(A)$, the notation $\langle b, \psi \rangle$ stands for the duality product of b and ψ. For every $k \in \mathbb{N}$, we set

$$b_k := \langle b, \phi_k \rangle. \tag{3.4}$$

The main result in this section is:

Theorem 3.1. *Let A be a skew-adjoint operator with compact resolvents on X with spectrum $\sigma(A) = i\Lambda$, where $\Lambda = (\lambda_k)_{k\in\mathbb{Z}^*}$ is a* regular *sequence of real numbers, i.e., with*

$$\gamma_1 := \inf_{\substack{n\in\mathbb{Z}^* \\ n\neq -1}} |\lambda_{n+1} - \lambda_n| > 0. \tag{3.5}$$

Moreover, assume that there exist $p \in \mathbb{N}$ and $\gamma_p > 0$ such that

$$\gamma_p := \inf_{\substack{n\in\mathbb{Z}^* \\ n\neq -p}} \left(\frac{\lambda_{n+p}-\lambda_n}{p}\right) > 0. \tag{3.6}$$

Finally, suppose that the sequence (b_k) defined in (3.4) is bounded and that $b_k \neq 0$ for every $k \in \mathbb{Z}^*$. Then for every $\tau > \frac{2\pi}{\gamma_p}$, the input map Φ_τ of the system (A, B) (with B defined in (3.3)) satisfies

$$\operatorname{Ran}\Phi_\tau = \left\{\eta \in X \;\middle|\; \sum_{k\in\mathbb{Z}^*} |b_k|^{-2}\big|\langle \eta, \phi_k\rangle\big|^2 < \infty\right\}. \tag{3.7}$$

Remark 3.1. The assumption that $b_k \neq 0$ for every $k \in \mathbb{Z}^*$ is not essential. Indeed, it is not difficult to check that for every $b \neq 0$ we have that $\operatorname{Ran}\Phi_\tau$ is contained in the closed span $\tilde{X}$ of the set $\{\varphi_k \mid b_k \neq 0\}$. Consequently, we can apply Theorem 3.6 to the restriction of our original system to $\tilde{X}$ and obtain that

$$\operatorname{Ran}\Phi_\tau = \left\{\eta \in \tilde{X} \;\middle|\; \sum_{\substack{k\in\mathbb{Z}^* \\ b_k\neq 0}} |b_k|^{-2}\big|\langle \eta, \phi_k\rangle\big|^2 < \infty\right\}.$$

The proof of Theorem 3.1 is based on a class of results playing, more generally, an important role in the study of reachability questions for the 1D elastic structures. More precisely, we refer here to several inequalities coming from the theory of nonharmonic Fourier series, introduced in Ingham [16]. In particular, we use below the following generalization of Parseval's inequality:

Proposition 3.1 (Ingham, 1936). *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}^*}$ be a real sequence satisfying (3.5). Then for any interval I with length $|I|$ there exists a constant c, depending on $|I|$ and γ_1, such that*

$$\int_I \left|\sum_{n\in\mathbb{Z}^*} a_n \exp(i\lambda_n t)\right|^2 \mathrm{d}t \leqslant c \sum_{n\in\mathbb{Z}^*} |a_n|^2,$$

for any sequence $(a_n) \in \ell^2(\mathbb{Z}^*, \mathbb{C})$.

It is not difficult to check that the proposition above implies the following admissibility result for (3.3) (note that the result below can also be seen as a particular case of the admissibility conditions given in Ho and Russell [15] and Weiss [43]).

Proposition 3.2. *Let A be a skew-adjoint operator with compact resolvents on X with spectrum $\sigma(A) = i\Lambda$, where $\Lambda = (\lambda_k)_{k\in\mathbb{Z}^*}$ satisfies (3.5). Assume that $b \in X_{-1}$ is such that for every $k \in \mathbb{Z}^*$, the number b_k defined in (3.4) is nonzero. Moreover, suppose that $\sup_{k\in\mathbb{N}} |b_k| < \infty$ (recall that the sequence (b_k) has been defined in (3.4)). Then B defined by (3.3) is an admissible control operator for $\mathbb{T}$.*

The main analytical tool in the proof of Theorem 3.1 is a lower bound for exponential sums, in the spirit of classical inequalities of Ingham [16], Beurling [5], and Kahane [19]. We give below the quantitative version proved in Tenenbaum and Tucsnak [39], making the dependency of the involved constants explicit in terms of various parameters.

Theorem 3.2. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}^*}$ be a real sequence satisfying (3.5) and (3.6). Then, for any $\gamma \in (0, \gamma_p)$ and interval I with length $|I| \geqslant \frac{2\pi}{\gamma}$, there exists a constant $\kappa = \kappa(\gamma_1) > 0$ such that, writing $\varepsilon := \frac{1}{2}\{1/\gamma - 1/\gamma_p\}$, we have*

$$\int_I \left| \sum_{n\in\mathbb{Z}^*} a_n \exp(i\lambda_n t) \right|^2 \mathrm{d}t \geqslant \frac{\kappa \varepsilon^{5p+2}}{p^{12p}} \sum_{n\in\mathbb{Z}^*} |a_n|^2$$

for any sequence $(a_n) \in \ell^2(\mathbb{Z}^*, \mathbb{C})$.

We are now in a position to prove Theorem 3.1.

Proof of Theorem 3.1. It is not difficult to check that our standing assumptions imply that the system (A, B) is approximately controllable in time τ. Thus, according to Corollary 2.1, it suffices to identify the completion of $\mathcal{D}(A^*) = \mathcal{D}(A)$ with respect to the norm

$$\eta \mapsto \left(\int_0^\tau \left\| B^* \mathbb{T}_t^* \eta \right\|^2 \mathrm{d}t \right)^{\frac{1}{2}}. \tag{3.8}$$

After some simple calculations, we obtain that

$$B^* \mathbb{T}_t^* \eta = \sum_{k\in\mathbb{Z}^*} b_k \langle \eta, \phi_k \rangle \exp(-i\lambda_k t) \quad \big(t \geqslant 0,\ \eta \in \mathcal{D}(A^*)\big).$$

By combining Proposition 3.1 and Theorem 3.2, it follows that the norm defined in (3.8) is equivalent to the norm

$$\eta \mapsto \left(\sum_{k\in\mathbb{Z}^*} |b_k|^2 \left| \langle \eta, \phi_k \rangle \right|^2 \right)^{\frac{1}{2}}. \tag{3.9}$$

We can thus use Corollary 2.1 to conclude that the dual $(\mathrm{Ran}\, \Phi_\tau)'$ of $\mathrm{Ran}\, \Phi_\tau$ with respect to the pivot space X is the completion of $\mathcal{D}(A)$ with respect to the norm defined in (3.9).

On the other hand, the completion of $\mathcal{D}(A)$ with respect to the norm defined in (3.9) clearly coincides with the dual with respect to the pivot space X of the space

$$\left\{ \eta \in X \;\middle|\; \sum_{k\in\mathbb{Z}^*} |b_k|^{-2} \left| \langle \eta, \phi_k \rangle \right|^2 < \infty \right\},$$

so that we obtain the conclusion (3.7). ■

4. AN EXAMPLE COMING FROM ELASTICITY

In this section we show how the abstract result in Theorem 3.1 can be applied to determine the reachable space of a system describing the vibrations of an Euler–Bernoulli beam with piezoelectric actuator. More precisely, we consider the initial and boundary value problem modeling the vibrations of an Euler–Bernoulli beam which is subject to the action of a piezoelectric actuator. Most of the results in this section appear, using a different terminology, in Tucsnak [40].

If we suppose that the beam is hinged at both ends and that the actuator is excited in a manner so as to produce pure bending moments, the model for the controlled beam can be written as (see, for instance, Crawley [7] or Destuynder et al. [10]):

$$\ddot{w}(t,x) + \frac{\partial^4 w}{\partial x^4}(t,x) = u(t)\frac{\mathrm{d}}{\mathrm{d}x}\left[\delta_b(x) - \delta_a(x)\right] \quad (0 < x < \pi,\ t > 0), \tag{4.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad \frac{\partial^2 w}{\partial x^2}(t,0) = \frac{\partial^2 w}{\partial x^2}(t,\pi) = 0 \quad (t \geqslant 0), \tag{4.2}$$

$$w(0,x) = 0, \quad \dot{w}(0,x) = 0 \quad (0 < x < \pi). \tag{4.3}$$

In the equations above, w represents the transverse deflection of the beam, $a, b \in (0,\pi)$ stand for the ends of the actuator, and δ_y is the Dirac mass at the point y. Moreover, $\dot{w}$, $\ddot{w}$ denote the partial derivatives of w with respect to time. The control is the function u representing the time variation of the voltage applied to the actuator.

It is easily seen that equations (4.1)–(4.3) can be written, using the standard notation for Sobolev spaces, using a second-order abstract form in the space $H = H^{-1}(0,\pi)$. More precisely, the system (4.1)–(4.3) can be rephrased as

$$\ddot{w}(t) + A_0^2 w(t) = B_0 u(t) \quad (t > 0), \tag{4.4}$$

$$w(0) = 0, \quad \dot{w}(0) = 0, \tag{4.5}$$

where A_0 is the Dirichlet Laplacian on $(0,\pi)$ defined by

$$\mathcal{D}(A_0) = H_0^1(0,\pi), \tag{4.6}$$

$$A_0\varphi = -\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} \quad \big(\varphi \in \mathcal{D}(A_0)\big), \tag{4.7}$$

and the operator B_0 is defined by

$$B_0 \mathrm{u} = \mathrm{u}\frac{\mathrm{d}}{\mathrm{d}x}(\delta_b - \delta_a) \quad (u \in \mathbb{C}). \tag{4.8}$$

We first recall the following well-posedness result from [40]:

Proposition 4.1. *Equations* (4.1)–(4.3) *determine a well-posed control system with state space $X = \mathcal{D}(A_0) \times H$ and control space $U = \mathbb{C}$. The corresponding semigroup generator and control operator are defined by*

$$\mathcal{D}(A) = \mathcal{D}\big(A_0^2\big) \times \mathcal{D}(A_0), \quad A = \begin{bmatrix} 0 & \mathbb{I} \\ -A_0^2 & 0 \end{bmatrix}, \tag{4.9}$$

respectively

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \tag{4.10}$$

where the operators A_0 and B_0 have been defined in (4.6)–(4.8).

Let Φ^{beam} be the input maps associated to the well-posed system from Proposition 4.1, defined by

$$\Phi_\tau^{\mathrm{beam}} u = \begin{bmatrix} w(\tau,\cdot) \\ \dot{w}(\tau,\cdot) \end{bmatrix} \quad \big(u \in L^2([0,\tau];U)\big).$$

The main result in this section is

Proposition 4.2. *For* $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$*, we define the Fourier coefficients* (a_n) *and* (b_n) *by*

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx), \quad g(x) = \sum_{n=1}^{\infty} n^2 d_n \sin(nx), \tag{4.11}$$

with (nc_n) *and* (nd_n) *in* $\ell^2(\mathbb{N}, \mathbb{C})$*. Moreover, assume that*

$$\frac{a+b}{\pi}, \ \frac{a-b}{\pi} \in \mathbb{R} - \mathbb{Q}. \tag{4.12}$$

Then for every $\tau > 0$*, we have that* $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$ *lies in* $\operatorname{Ran} \Phi_\tau^{\text{beam}}$ *if and only if*

$$\sum_{n \in \mathbb{N}} n^2 \sin^{-2}\left[\frac{n(a+b)}{2}\right] \sin^{-2}\left[\frac{n(a-b)}{2}\right] \left(|c_n|^2 + |d_n|^2\right) < \infty. \tag{4.13}$$

Proof. It is known that the operators A_0 and A_0^2, where A_0 has been defined in (4.6) and (4.7), are self-adjoint and positive on H (see, for instance, [41, SECTIONS 3.3 AND 3.4]). From this it follows that the operator A defined in (4.9) is skew-adjoint on $X = \mathcal{D}(A_0) \times H$, see [41, SECTION 3.7]. Moreover, we know from Proposition 4.1 that B is an admissible control operator for the unitary group $\mathbb{T}$ generated by A, so that the system (A, B) is eligible for the application of Theorem 3.1. To check that all the assumptions in this theorem are satisfied, let

$$\varphi_k(x) = k\sqrt{\frac{2}{\pi}} \sin(kx) \quad \big(k \in \mathbb{N}, \ x \in (0, \pi)\big).$$

It is easily seen that $(\varphi_k)_{k \in \mathbb{N}}$ is an orthonormal basis of H comprising eigenvectors of A_0 with corresponding eigenvalues $(\lambda_k^2)_{k \in \mathbb{N}}$, where $\lambda_k = k^2$ for every $k \in \mathbb{N}$. This enables us, according to [41, PROPOSITION 3.7.7], to construct an orthonormal basis in X consisting of eigenvectors of A. More precisely, for every $k \in \mathbb{N}$, we set $\varphi_{-k} = -\varphi_k$ and $\lambda_{-k} = -\lambda_k$, and

$$\phi_k = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{i\lambda_k} \varphi_k \\ \varphi_k \end{bmatrix}. \tag{4.14}$$

Then for every $k \in \mathbb{Z}^*$, we have that ϕ_k is an eigenvector of A corresponding to the eigenvalue $i\lambda_k$ and $(\phi_k)_{k \in \mathbb{Z}^*}$ is an orthonormal basis in X.

Let us note at this stage that the sequence $(\lambda_k)_{k \in \mathbb{Z}^*}$ obviously satisfies assumption (3.5) from Theorem 3.1 and that for every $\gamma > 0$ there exists $p \in \mathbb{N}$ such that $(\lambda_k)_{k \in \mathbb{Z}^*}$ satisfies assumption (3.6) with $\gamma_p > \gamma$.

To compute the coefficients (b_k) defined in (3.4), we note that from (4.10) and (4.14) it follows that

$$\langle B\mathrm{u}, \phi_k \rangle_{X_{-1}, X_1} = \frac{1}{\sqrt{2}} \langle B_0 u, \varphi_k \rangle_{[\mathcal{D}(A_0)]', \mathcal{D}(A_0)},$$

where $[\mathcal{D}(A_0)]'$ is the dual of $\mathcal{D}(A_0)$ with respect to the pivot space H. Recalling that $H = H^{-1}(0, \pi)$ and using (4.8), it follows that, for every $u \in \mathbb{C}$ and $k \in \mathbb{Z}^*$, we have

$$\langle B\mathrm{u}, \phi_k \rangle_{X_{-1}, X_1} = \frac{1}{\sqrt{2}} \mathrm{u} \left(\left[\frac{\mathrm{d}}{\mathrm{d}x} \left(A_0^{-1} \varphi_k \right) \right]_{x=a} - \left[\frac{\mathrm{d}}{\mathrm{d}x} \left(A_0^{-1} \varphi_k \right) \right]_{x=b} \right),$$

so that

$$b_k = \frac{1}{\sqrt{2}}\left(\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(A_0^{-1}\varphi_k\right)\right]_{x=a} - \left[\frac{\mathrm{d}}{\mathrm{d}x}\left(A_0^{-1}\varphi_k\right)\right]_{x=b}\right) \quad \left(k \in \mathbb{Z}^*\right).$$

After some simple calculations, we obtain that

$$b_k = \frac{1}{\sqrt{\pi}}\left(\cos(ka) - \cos(kb)\right) = \frac{2}{\sqrt{\pi}} \sin\left[\frac{k(b+a)}{2}\right] \sin\left[\frac{k(b-a)}{2}\right] \quad \left(k \in \mathbb{Z}^*\right).$$

From the above formula, it follows that the sequence (b_k) is bounded and, recalling (4.12), that $b_k \neq 0$ for every $k \in \mathbb{Z}^*$.

We have thus checked all the assumptions of Theorem 3.1. Applying this theorem to the system described by the operators of A and B defined in this section, it this follows that $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in X$ indeed belongs to the reachable space of the considered system iff (4.13) holds. ■

Remark 4.1. The result in Proposition 4.2 can be combined with some simple diophantine approximation results to obtain more explicit information on $\operatorname{Ran} \Phi_\tau^{\mathrm{beam}}$. Some of these properties are:

- There exist no locations a and b for which the system is exactly controllable. Indeed, from (4.13) it follows that the system (A, B) is exactly controllable iff the sequences $(|\sin[\frac{n(a\pm b)}{2}]|)_{n\in\mathbb{N}}$ are bounded away from zero. Or, using the continuous fraction approximation of real numbers, it is easy to check (see **[40]**) that there are no real numbers a and b with this property.

- The largest reachable spaces are obtained when $\frac{a\pm b}{\pi}$ can be "badly" approximated by rational numbers. In particular, if $\frac{a\pm b}{\pi}$ are quadratic irrationals (i.e., solutions of a second-order equation with integer coefficients), then

$$\operatorname{Ran} \Phi_\tau^{\mathrm{beam}} \supset \mathcal{D}(A).$$

- On the other hand, choosing a and b such that $\frac{a\pm b}{\pi}$ can be well approximated by rational numbers, the reachable space diminishes. We think, in particular, of Liouville numbers (see Valiron **[42]**). More precisely, for every $m \in \mathbb{N}$, there exist $a, b \in (0, \pi)$ such that $\frac{a\pm b}{\pi} \notin \mathbb{Q}$ and $\mathcal{D}(A^m)$ contains states which are not reachable.

5. THE HEAT EQUATION ON A HALF-LINE

The properties of the system we consider in this section strongly contrast those encountered in the finite-dimensional context. We just mention here that its reachable space depends on time and that the system is approximately controllable but not null-controllable. The results presented in this section are not new, but we chose to describe them in detail for two reasons. Firstly, the study of the reachable space of this system brought in new techniques in control theory for infinite-dimensional systems, essentially coming from the theory of reproducing kernel Hilbert spaces. Secondly, as it has been very recently discovered, these results have an important role in characterizing the reachable space for the controlled heat equation on a bounded interval, as it will be shown in Section 6.

Consider the initial and boundary value problem

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2 v}{\partial x^2}(t,x) & (t \geqslant 0,\ x \in (0,\infty)), \\ v(t,0) = u_0(t), & (t \in [0,\infty)), \\ v(0,x) = 0 & (x \in (0,\infty)), \end{cases} \tag{5.1}$$

and the associated input maps $(\Phi_\tau^{\text{left}})_{\tau>0}$ defined by

$$\Phi_\tau^{\text{left}} u = v(\tau,\cdot) \quad \big(u \in L^2[0,\infty), \tau > 0\big). \tag{5.2}$$

As far as we know, the first paper with explicit control-theoretic purposes tackling the system described by the first two equations in (5.1) is Micu and Zuazua [28]. The main results in [28] assert that the first two equations in (5.1) determine a well-posed control system in appropriate spaces and that this system is not null-controllable in any time $\tau > 0$ (concerning this last assertion, we also refer to Dardé and Ervedoza [9] for an elegant proof and extensions to related PDE systems). Combining the above mentioned lack of controllability property with Proposition 2.5 suggests that $\operatorname{Ran} \Phi_\tau^{\text{left}}$ depends on the time τ. This dependence was, in fact, already made explicit in a series of papers driven by complex analysis motivations, see Aikawa et al. [1] and Saitoh [32,33]. These results came to the attention of the control-theoretic community only very recently, when they became an important ingredient in proving the main results in [13].

Before stating some of the main results from [1] and [33], we first recall some definitions concerning Bergman spaces. More precisely, for $\Omega \subset \mathbb{C}$ an open set and $\omega \in C(\Omega)$, with $|\omega(x)| > 0$ for every $x \in \Omega$, the *Bergman space on* Ω *with weight* ω, denoted $A^2(\Omega,\omega)$ is formed by all the functions f holomorphic on Ω such that $f\sqrt{|\omega|}$ is in $L^2(\Omega)$. For $\omega = 1$, this space is simply denoted by $A^2(\Omega)$. Note that $A^2(\Omega,\omega)$ becomes a Hilbert space when endowed with the norm

$$\|\psi\|^2_{A^2(\Omega,\omega)} = \int_\Omega \big|\psi(x+iy)\big|^2 \big|\omega(x+iy)\big| \,\mathrm{d}x\,\mathrm{d}y.$$

We also recall (see, for instance, [6, SECTION 4.1]) that the input maps defined in (5.2) can be alternatively described by the integral formula

$$\big(\Phi_\tau^{\text{left}} u\big)(x) = -\int_0^\tau \frac{\partial \kappa}{\partial x}(\tau-\sigma,x)u(\sigma)\,\mathrm{d}\sigma \quad \big(u \in L^2[0,\infty), \tau > 0, x \in (0,\pi)\big), \tag{5.3}$$

where

$$\kappa(t,x) = \sqrt{\frac{1}{\pi t}}\exp\left(-\frac{x^2}{4t}\right) \quad (t > 0,\ x \in \mathbb{R}) \tag{5.4}$$

is the heat kernel on $\mathbb{R}$.

For each $\tau > 0$, the range of the input map Φ_τ^{left} defined in (5.2) has been completely described in [32] as an appropriate subspace of the space of functions continuous on $(0,\pi)$ and which can be extended to a function which is holomorphic on the set Δ defined by

$$\Delta = \left\{ s \in \mathbb{C} \,\middle|\, -\frac{\pi}{4} < \arg s < \frac{\pi}{4} \right\}. \tag{5.5}$$

The precise description given in [32] of this space involves the sum of a two Hilbert spaces of holomorphic functions defined on Δ, one of them being of Bergman type. To avoid extra notational complexity, we choose to omit the precise statement of this result and to focus on the characterization of the range of the restriction of Φ_τ^{left} to the space of inputs $u(t) = \sqrt{t}\, f(t)$ with $f \in L^2[0,\tau]$. Recalling (5.3) and (5.4), this means that for every $\tau > 0$ we focus on the range of the operator defined by

$$(P_\tau f)(x) = \int_0^\tau \frac{x \exp(-\frac{x^2}{4(\tau-\sigma)})}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} f(\sigma)\sqrt{\sigma}\,\mathrm{d}\sigma \quad \big(f \in L^2[0,\tau],\ x \in (0,\pi)\big). \tag{5.6}$$

We are now in a position to state the main result in [1].

Theorem 5.1. *For every $\tau > 0$, the operator P_τ defined in (5.6) is an isometry from $L^2[0,\tau]$ onto $A^2(\Delta, \omega_{0,\tau})$, where Δ has been defined in (5.5) and*

$$\omega_{0,\delta}(s) = \frac{\exp(\frac{\operatorname{Re}(s^2)}{2\delta})}{\delta} \quad (\delta > 0,\ s \in \Delta). \tag{5.7}$$

The proof of Theorem 5.1 is a very nice application of the theory of linear operators in reproducing kernel spaces, as described, for instance, in [34]. More precisely, the main steps of the proof from [1] are:

- remarking that, by elementary calculus, if in the definition (5.3) of Φ_τ^{left} we replace $x \in (0,\pi)$ by $s \in \Delta$ then the right-hand side of (5.3) defines a function which is holomorphic on Δ;
- using general results on the range of integral operators on RKHS and appropriate integrations, deduce that Ran P_τ is an RKHS of holomorphic functions on Δ whose kernel is

$$K_\tau(s, \overline{w}) = \exp\left(-\frac{s^2 + \overline{w}^2}{4\tau}\right) \frac{4s\overline{w}}{\pi(s^2 + \overline{w}^2)^2}; \tag{5.8}$$

- finally, remarking that the kernel K_τ in (5.8) coincides with the reproducing kernel of $A^2(\Delta, \omega_{0,\tau})$.

6. THE HEAT EQUATION ON AN INTERVAL

In this section we come back to equations in (1.1), already briefly discussed in Section 1. More precisely, we describe below very recent advances which have lead to several equivalent characterizations of the system described by the first two equations in (1.1). In other words, our aim is to characterize the range of the operator Φ_τ^{heat} introduced in (1.2). We continue using the notation introduced in Section 5, namely for Bergman spaces (possibly weighted).

We first state the following remarkably simple characterization, proved in [14] and confirming a conjecture formulated in [13]:

Theorem 6.1. *Let $\tau > 0$ and let Φ_τ^{heat} be the input map introduced in* (1.2)*. Then*

$$\operatorname{Ran} \Phi_\tau^{\text{heat}} = A^2(D), \tag{6.1}$$

where D is the square introduced in (1.3)*.*

It is quite natural to postpone the discussion of the main steps of the proof of Theorem 6.1 to the end of this section. Indeed, this proof is based, in particular, on another characterization of $\operatorname{Ran} \Phi_\tau^{\text{heat}}$, as a sum of two Bergman spaces on two symmetric infinite sectors. Besides being used in the proof of Theorem 6.1, this type of characterization is of independent interest.

To state these results, we need some notation. We first introduce the set

$$\tilde{\Delta} = \pi - \Delta, \tag{6.2}$$

where Δ has been defined in (5.5), and the weight function

$$\omega_{\pi,\delta}(\tilde{s}) = \frac{\exp(\frac{\operatorname{Re}[(\pi-\tilde{s})^2]}{2\delta})}{\delta} \quad (\delta > 0,\ \tilde{s} \in \tilde{\Delta}). \tag{6.3}$$

Note that

$$\omega_{0,\delta}(s) = \omega_{\pi,\delta}(\pi - s) \quad (s \in \Delta), \tag{6.4}$$

where $\omega_{0,\delta}$ is the weight introduced in (5.7).

We also introduce the space X_δ defined for every $\delta > 0$ by

$$X_\delta = \left\{ \psi \in C(0,\pi) \,\middle|\, \begin{matrix} \exists \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \exists \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{matrix},\ \psi = \varphi_0 + \varphi_\pi \text{ on } (0,\pi) \right\}, \tag{6.5}$$

which is endowed with the norm

$$\|\varphi\|_\delta = \inf \left\{ \|\varphi_0\|_{A^2(\Delta,\omega_{0,\delta})} + \|\varphi_\pi\|_{A^2(\tilde{\Delta},\omega_{\pi,\delta})} \,\middle|\, \begin{matrix} \varphi_0 + \varphi_\pi = \varphi \\ \varphi_0 \in A^2(\Delta, \omega_{0,\delta}) \\ \varphi_\pi \in A^2(\tilde{\Delta}, \omega_{\pi,\delta}) \end{matrix} \right\}. \tag{6.6}$$

We are now in a position to formulate the main result in [21].

Theorem 6.2. *With the above notation, for every $\tau, \delta > 0$, we have*

$$\operatorname{Ran} \Phi_\tau^{\text{heat}} = X_\delta = A^2(\Delta) + A^2(\tilde{\Delta}). \tag{6.7}$$

Let us mention that the equality $\operatorname{Ran} \Phi_\tau^{\text{heat}} = A^2(\Delta) + A^2(\tilde{\Delta})$ has been obtained independently in [29].

We briefly describe below the main steps of the proof of Theorem 6.2.

- We first remark that Φ_τ^{heat} is the sum of a series of integral operators involving the heat kernel. More precisely, we have

$$\left(\Phi_\tau \begin{bmatrix} u_0 \\ u_\pi \end{bmatrix}\right)(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau - \sigma, x) u_0(\sigma)\, \mathrm{d}\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma)\, \mathrm{d}\sigma$$
$$\big(\tau > 0,\ u_0, u_\pi \in L^2[0,\tau],\ x \in (0,\pi)\big), \tag{6.8}$$

where

$$K_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m\in\mathbb{Z}} \exp\left(-\frac{(x+2m\pi)^2}{4\sigma}\right) \quad \big(\sigma > 0,\ x \in [0,\pi]\big), \quad (6.9)$$

$$K_\pi(\sigma, x) = K_\pi(\sigma, \pi - x) \quad \big(\sigma > 0, x \in [0,\pi]\big). \quad (6.10)$$

Formula (6.8) can be derived, using symmetry considerations, from (5.3). An alternative proof is proposed in **[13]** by combining the Fourier series expression of the solution of (1.1) and the Poisson summation formula.

- The second step consists in remarking that Ran Φ_τ^{heat} coincides with the range of the map (still defined on $(L^2[0,\tau])^2$)

$$\begin{bmatrix} f \\ g \end{bmatrix} \mapsto \Phi_\tau^{\text{heat}} \begin{bmatrix} \sqrt{t} f \\ \sqrt{t} g \end{bmatrix} \quad \big(f, g \in L^2[0,\tau]\big).$$

  This can be easily proved using the fact that the considered system is null-controllable in any positive time, see **[21, PROPOSITION 3.2]**.

- For the third step, we first prove that from (6.8) it follows that for every $f, g \in L^2[0,\tau]$, we have

$$\Phi_\tau^{\text{heat}} \begin{bmatrix} \sqrt{t} f \\ \sqrt{t} g \end{bmatrix} = P_\tau f + Q_\tau g + R_\tau \begin{bmatrix} f \\ g \end{bmatrix},$$

  where P_τ has been defined in (5.6),

$$(Q_\tau g)(x) = (P_\tau g)(\pi - x) \quad \big(x \in (0,\pi)\big),$$

  and R_τ is an operator whose norm tends to zero when $\tau \to 0+$. In other words, Φ_τ^{heat} decomposes into the sum of the input maps of the system describing the boundary controlled heat equation on $[0,\infty)$ and $(-\infty,\pi]$, respectively (for which the ranges are known from the previous section) and a remainder term R_τ which becomes "negligible" for small τ.
  Combined with Theorem 5.1, this fact implies, recalling (6.5), that

$$\operatorname{Ran} \Phi_\tau^{\text{heat}} = X_\tau \quad (\tau > 0).$$

- The last step of the proof consists in showing that

$$X_\tau = A^2(\Delta) + A^2(\tilde{\Delta}) \quad (\tau > 0).$$

  This can be accomplished by combining Proposition 2.5 and the construction of appropriate multipliers (see **[21]** for details).

We end this section by coming back to the proof of Theorem 6.1. In view of Theorem 6.2, the conclusion of Theorem 6.1 is equivalent to the equality

$$A^2(D) = A^2(\Delta) + A^2(\tilde{\Delta}). \quad (6.11)$$

This a question which is part of a class of problems with a quite long history in complex analysis: the *separation of singularities* for holomorphic functions. A general formulation of this type of problems is: denoting by $\mathrm{Hol}(\mathcal{O})$ the space of holomorphic functions on an open set $\mathcal{O} \subset \mathbb{C}$ (in particular, the Banach space of analytic functions) and given two open sets $\Omega_1, \Omega_2 \subset \mathbb{C}$ with $\Omega_1 \cap \Omega_2 \neq \emptyset$, is it true that $\mathrm{Hol}(\Omega_1 \cap \Omega_2) = \mathrm{Hol}(\Omega_1) + \mathrm{Hol}(\Omega_2)$? We refer to [14] for detailed historical information on this issue, mentioning here just that [14] is the first work considering this question in a Bergman space context. Moreover, using a methodology involving sophisticated analytical techniques, like Hörmander-type L^p-estimates for the solution of the $\bar{\partial}$ equation, the main results in [14] assert that the separation of singularities for Bergman spaces holds in a geometrical context more general than that in (6.11).

7. CONCLUSIONS, REMARKS, AND OPEN QUESTIONS

This work gives an overview, far from being exhaustive, of the applications of complex and harmonic analysis methods in the study of the reachable space of infinite-dimensional systems. In most of the presented results, the analytical tools appearing in the previous sections have been developed for purposes having a priori nothing to do with the infinite-dimensional system theory. This is the case, for instance, for the Ingham–Beurling–Kahane-type inequalities appearing in Section 3, which began to be applied in controllability and reachability questions only several decades after their publication. The situation is similar for the methods coming from the theories of RKHS and spaces of analytic functions, namely those described in Sections 5 and 6: their penetration in the control-theoretic community took place 20 years after their first publication. An important fact is that these interactions raised new problems and allowed significant progress in the concerned fields of analysis. The separation of singularities for Bergman spaces, briefly discussed in Section 6, is a remarkable example illustrating these mutual interactions.

We conclude this work by briefly describing some open questions which are, at least in the author's opinion, of major interest in the infinite-dimensional system theory.

7.1. Time reversible systems

We think here of linear control systems described by the wave, Schrödinger, or Euler–Bernoulli equations. As already mentioned, the characterization of the reachable space of these systems is quite well understood in the case of one space dimension, but essentially open in several space dimensions. Taking the example of a system described by the wave equation in a bounded domain in $\mathbb{R}^n$ ($n \geqslant 2$), we should mention the famous paper by Bardos, Lebeau, and Rauch [4], where it is proved that the exact controllability (in sufficiently large time) holds iff the control support satisfies the so-called *geometric optics* condition. On the other hand, using a duality argument and Holmgren's uniqueness theorem, it is not difficult to see that if the control support is an arbitrary open subset of the boundary, then the system is approximately controllable, again in sufficiently large time. As far as we know, the question of characterizing the reachable space when the control support does not satisfy the geometric optics condition is essentially open and it seems an extremely challenging one. Some information on these spaces can be found in [18], where the wave equation holds in a rectangular domain, or in [3]. Possible tools for tackling a more general geometry can be found in Lebeau [25], Robbiano [30], or Laurent and Léautaud [23].

7.2. Systems described by parabolic equations

As described in Section 6, the reachable space for systems described by the constant coefficient heat equation on a bounded interval has been recently completely characterized in terms of Hilbert spaces of analytic functions. However, for systems described by variable coefficient parabolic equations, even in one space dimension, many natural questions are still open. We think, in particular, of the sharp identification of the domain of analyticity of the reachable space when all the coefficients of the parabolic equation are entire functions of the space variable, see Laurent and Rosier [24] for several remarkable results in this direction.

Coming back to the system described by the one-dimensional constant coefficient heat equations, it would be important to understand the action of the heat semigroup on the reachable space. In particular, is the semigroup obtained by restricting the heat semigroup to the reachable space strongly continuous on $\operatorname{Ran} \Phi_\tau$ (when endowed with the norm defined in (2.9))? A positive answer to this question would be a good departure point in studying the robustness of the reachable space with respect to various perturbations (linear or nonlinear), in the vein of the corresponding theory for exactly controllable systems.

Finally, let us briefly discuss the state-of-the-art for systems described by the constant coefficient heat equation in several space dimensions. An early result in this direction has been provided in Fernández-Cara and Zuazua [12], where it is shown that a class of functions which are holomorphic in an appropriate infinite strip are in the reachable space. A very recent and important contribution to this question has been recently brought in a work by Strohmaier and Waters [38]. In this work, assuming that the spatial domain is a ball and that the control acts on the whole boundary, the authors provide detailed information on the reachable space, similar to that obtained in [8] for systems described by the one-dimensional heat equation. As far as we know, with the exception of the above-mentioned situation, the study of the reachable space described by boundary-controlled parabolic equation in $\mathbb{R}^n$, with $n \geqslant 2$, is a widely open question.

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