QUANTITATIVE ANALYSIS OF FIELD CONCENTRATION IN PRESENCE OF CLOSELY LOCATED INCLUSIONS OF HIGH CONTRAST

HYEONBAE KANG

ABSTRACT

In composites consisting of inclusions and a matrix of different materials, some inclusions are located closely to each other. If the material properties of inclusions are of high contrast with that of the matrix, field concentration occurs in the narrow region between closely located inclusions. Understanding the field concentration quantitatively is important in the theory of composites and imaging since it represents stress or field enhancement. The last 30 years or so have witnessed significant progress in analyzing this phenomena of field concentration: optimal estimates and asymptotic characterization capturing the field concentration have been derived in the contexts of the conductivity equation (or antiplane elasticity), the Lamé system of linear elasticity, and the Stokes system. The purpose of this paper is to review some of them in a coherent manner.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 35J15; Secondary 35B65

KEYWORDS

Transmission problem, conductivity equation, Lamé system, Stokes system, high contrast, stress, field concentration, optimal estimates, spectrum, Neumann–Poincaré operator

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 7, pp. 5680–5699
DOI 10.4171/ICM2022/83

Published by EMS Press

1. INTRODUCTION

Typical composites consist of inclusions imbedded in the matrix (the background medium), where the inclusions have material properties different from that of the matrix. In some composites, two inclusions are located closely to each other, and if their material properties are in high contrast with that of the matrix, then a strong concentration of the field or stress may occur in the narrow region between two inclusions. It is important to quantitatively analyze the field concentration or the stress since it may cause material failure (see, for example, [5]).

Composites may have multiple inclusions. But, since the region of interest is the local narrow area in-between two inclusions which are closely located, all other inclusions except for the considered two are ignored and the mathematical problem is formulated with just two inclusions, that is, the problem is formulated in terms of disjoint bounded domains D_1 and D_2 in $\mathbb{R}^d$ $(d = 2, 3)$ representing the two inclusions. They are assumed to have Lipschitz continuous boundaries, and the interface conditions along ∂D_j $(j = 1, 2)$ are given by the perfectly bonding conditions, namely, continuity of the flux and the potential (see (2.5) and below). With these interface conditions, we consider the homogeneous and inhomogeneous transmission problems of various equations such as the equation of conductivity or antiplane elasticity, the Lamé system for linear elasticity, and the Stokes system for fluid flow. The inclusions represent conductors or insulators for conductivity equations, elastic inclusions for antiplane elasticity equations or Lamé systems, and suspensions for Stokes systems.

Throughout this paper, ε denotes the distance between two inclusions, namely,

$$\varepsilon := \operatorname{dist}(D_1, D_2). \tag{1.1}$$

The characteristic feature of the configuration for the problem is that ε is arbitrarily small. The mathematical problem here is to capture in a quantitative way the behavior of the field (the gradient of the solution) and its derivatives in the narrow region between D_1 and D_2 in terms of ε and, if possible, the contrast of material parameters. As mentioned before, this problem arises from the stress analysis in composites. It also arises from the effective medium theory [12,25] (see also [19]): in order to compute the effective properties of composites with the periodic array of densely packed inclusions, it is necessary to capture the asymptotic behavior of the field in-between inclusions. Sometimes the two inclusions are designed to create the field concentration to achieve a desired enhancement of the field.

During the last three decades or so, significant development on the problem has been made: optimal estimates for the gradient and its derivatives have been obtained and asymptotic characterizations of the field concentration have been derived. The purpose of this paper is to review them. Despite all this progress, some outstanding and challenging problems remain unsolved. We discuss them as well.

The rest of this paper consists of three sections, reviewing the conductivity equation, the Lamé system, and the Stokes systems in turn. A short discussion is added at the end of the paper.

2. THE CONDUCTIVITY EQUATION

Let D_1 and D_2 be disjoint bounded domains in $\mathbb{R}^d$ $(d = 2, 3)$ whose boundaries are assumed to be Lipschitz continuous. Let k_j be the conductivity of D_j for $j = 1, 2$, while that of $\mathbb{R}^d \setminus (D_1 \cup D_2)$ is assumed to be 1. So the conductivity distribution is given by

$$\sigma = \chi_{\mathbb{R}^d \setminus (D_1 \cup D_2)} + k_1 \chi_{D_1} + k_2 \chi_{D_2}, \tag{2.1}$$

where χ denotes the characteristic function on the respective set. We assume that $0 < k_j \neq 1 < \infty$ $(j = 1, 2)$.

We consider the inhomogeneous transmission problem: for a given function f,

$$\begin{cases} \nabla \cdot \sigma \nabla u = f & \text{in } \mathbb{R}^d, \\ u(x) = c \ln |x| + O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{2.2}$$

for some constant c. The constant c can be nonzero if $d = 2$, and it is zero if $d = 3$. We also consider the homogeneous transmission problem

$$\begin{cases} \nabla \cdot \sigma \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(x) - H(x) = O(|x|^{-d+1}) & \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

where H is a given function harmonic in $\mathbb{R}^d$. Instead of the free-space problems (2.2) and (2.3), one may consider the corresponding boundary value problems, which are equivalent to the above problems. However, the free-space problems seem more natural since the problems arise from the composite theory, and all but the two closely located inclusions are ignored.

When the conductivities k_1 and k_2 simultaneously tend ∞ or 0, it is expected for the ∇u of the solution u to become arbitrarily large as the distance ε between the two inclusions tends to 0. The problem is to derive estimates for ∇u in terms of ε (and k_1, k_2, if possible) as ε tends to 0. The conductivity being ∞ means that the inclusion is perfectly conducting, while 0 means insulating. The two-dimensional equation may represent the antiplane elasticity, and in such a case it means that the inclusion is either stiff or void. When $k_1 = 0$ and $k_2 = \infty$, or the other way around, a quite different singular behavior (blow-up) occurs as ε tends to 0 as we will see later.

When the distance ε tends to 0, the numerical computation of u becomes quite difficult since the blow-up of ∇u forces us to use a refined mesh. In this respect, an asymptotic characterization of the singularity of ∇u has an important role. By an asymptotic characterization, as ε tends to 0, we mean a decomposition of the form

$$u = s + r, \tag{2.4}$$

where s is the singular part, namely, ∇s carries the full information about the singularity of ∇u, while r is the regular part, namely, ∇r is bounded. To be used effectively for numerical computations, the singular part s needs to be the solution of the conductivity equation, and explicit.

The problem (2.2) can be expressed as

$$\begin{cases} \Delta u = f & \text{in } \mathbb{R}^d \setminus \overline{D}, \\ \Delta u = k_j^{-1} f & \text{in } D_j,\ j = 1,2, \\ u|_+ - u|_- = 0 & \text{on } \partial D_j,\ j = 1,2, \\ \partial_\nu u|_+ - k_j \partial_\nu u|_- = 0 & \text{on } \partial D_j,\ j = 1,2, \\ u(x) = c \ln|x| + O\big(|x|^{-1}\big) & \text{as } |x| \to \infty. \end{cases} \tag{2.5}$$

Here and throughout this paper, ∂_ν denotes the outward normal derivative on ∂D_j and the subscripts $\pm$ denote the limits from outside and inside of D_j, respectively. The third and fourth lines in (2.5) represent the perfect-bonding conditions along ∂D, namely the continuity of the potential and flux, respectively.

Let F be the (weighted) Newtonian potential of f, namely,

$$F(x) = \int_{\mathbb{R}^d \setminus D} \Gamma(x-y) f(y)\, dy + \sum_{j=1}^{2} \frac{1}{k_j} \int_{D_j} \Gamma(x-y) f(y)\, dy, \quad x \in \mathbb{R}^d, \tag{2.6}$$

where $\Gamma(x)$ is the fundamental solution to the Laplacian, i.e.,

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln|x|, & d = 2, \\ -\dfrac{1}{4\pi} |x|^{-1}, & d = 3. \end{cases} \tag{2.7}$$

Since $\Delta F = f$ in $\mathbb{R}^d \setminus \overline{D}$ and $\Delta F = k_j^{-1} f$ in D_j, $v := u - F$ (u is the solution to (2.2)) is the solution to

$$\begin{cases} \Delta v = 0 & \text{in } D \cup (\mathbb{R}^d \setminus \overline{D}), \\ v|_+ - v|_- = 0 & \text{on } \partial D_j,\ j = 1,2, \\ \partial_\nu v|_+ - k_j \partial_\nu v|_- = (k_j - 1)\eta_j & \text{on } \partial D_j,\ j = 1,2, \\ v(x) = c \ln|x| + O\big(|x|^{-1}\big) & \text{as } |x| \to \infty, \end{cases} \tag{2.8}$$

with $\eta_j = \partial_\nu F|_{\partial D_j}$ ($j = 1, 2$). That is, the inhomogeneous problem (2.2) is reduced to (2.8). By putting $v := u - H$, we see that the homogeneous (2.3) is reduced to (2.8) with $\eta_j = \partial_\nu H|_{\partial D_j}$.

The solution to (2.8) can be represented in terms of the single-layer potentials, and if it is done so, the problem is reduced to a system integral equations for the Neumann–Poincaré operator on $\partial D_1 \times \partial D_2$. In a recent paper [14], explicit solutions to (2.2) and (2.3) have been constructed when inclusions are circular using the complete knowledge of the spectrum for the Neumann–Poincaré operator on two circles. In Section 2.1, we review them and optimal estimates of the derivatives of the solution as consequences. We then review in Section 2.2 important generalizations to inclusions, of more general shape in two and three dimensions, of results for circular inclusions. These are actually results established earlier than the circular case of [14]; the review of this section is in reverse historical order. The merit in doing so is that the fine results for the case of circular inclusions may serve as milestones of which problems have been solved and which still need to be solved.

In the last subsection, we review the results on the asymptotic characterizations of the singular behavior of the gradient of the solution.

2.1. Estimates for circular inclusions

2.1.1. Explicit representation of the solution

Suppose that D_1 and D_2 are disks of radii r_1 and r_2, respectively. Explicit solutions are constructed in [14] by transforming circles ∂D_1 and ∂D_2 to two concentric circles. In order for the transformation to take a simple form, we make some necessary translations and rotations so that after them centers of D_1 and D_2 are located at $(c_1, 0)$ and $(c_2, 0)$, where

$$c_1 = \frac{r_2^2 - r_1^2 - (r_1 + r_2 + \varepsilon)^2}{2(r_1 + r_2 + \varepsilon)} - \frac{\beta}{2}, \quad c_2 = c_1 + r_1 + r_2 + \varepsilon, \tag{2.9}$$

with

$$\beta = \frac{\sqrt{\varepsilon}\sqrt{(2r_1 + \varepsilon)(2r_2 + \varepsilon)(2r_1 + 2r_2 + \varepsilon)}}{r_1 + r_2 + \varepsilon}. \tag{2.10}$$

Then, ∂D_1 and ∂D_2 are mapped onto two concentric circles by the transformation

$$z^* = Tz := \frac{\beta}{z} + 1, \tag{2.11}$$

namely, $T(\partial D_j)$ $(j = 1, 2)$ is the circle of the radius R_j centered at 0, where R_j is given by

$$R_1^2 = 1 + \frac{\beta}{c_1}, \quad R_2^2 = 1 + \frac{\beta}{c_2}. \tag{2.12}$$

Let

$$D_1^* := T(D_1) = \{|\zeta| < R_1\}, \quad D_2^* := T(D_2) = \{|\zeta| > R_2\}. \tag{2.13}$$

Let $H^{-1/2}(\partial D_j)$ denote the Sobolev space of order $-1/2$ on ∂D_j and $H_0^{-1/2}(\partial D_j)$ be the subspace of $H^{-1/2}(\partial D_j)$ whose element f satisfies $\int_{\partial D_j} f = 0$. Suppose that the function η_j appearing in (2.8) belongs to $H_0^{-1/2}(\partial D_j)$ and let H_j be the unique solution to the following Neumann boundary value problem:

$$\begin{cases} \Delta H_j = 0 & \text{in } D_j, \\ \partial_\nu H_j = \eta_j & \text{on } \partial D_j. \end{cases} \tag{2.14}$$

Let h_j be the analytic function in D_j^* such that $h_1(0) = 0$, $\lim_{|\zeta|\to\infty} h_2(\zeta) = 0$, and

$$H_j(z) = \Re(h_j \circ T)(z) + C_j, \quad z \in D_j, \tag{2.15}$$

for some constant C_j. Here and afterwards, $\Re$ indicates the real part. Let

$$\rho := \frac{R_1}{R_2} \quad \text{and} \quad \lambda_j := \frac{k_j + 1}{2(k_j - 1)}, \quad j = 1, 2, \tag{2.16}$$

and define functions w_j by

$$w_1(\zeta) = \sum_{l=0}^{\infty} \frac{h_1(\rho^{2l}\zeta)}{(4\lambda_1\lambda_2)^{l+1}}, \quad |\zeta| < R_1, \tag{2.17}$$

and

$$w_2(\zeta) = \sum_{l=0}^{\infty} \frac{h_2(\rho^{-2l}\zeta)}{(4\lambda_1\lambda_2)^{l+1}}, \quad |\zeta| > R_2. \tag{2.18}$$

Using functions w_1 and w_2, we define

$$A_1(\zeta) := \begin{cases} (\lambda_1+\lambda_2)w_1(\zeta) \\ \quad + (\lambda_1-\lambda_2)w_1(\rho\zeta) - w_1(\rho^2\zeta), & |\zeta| \le R_1, \\ (\lambda_1+\lambda_2)w_1(R_1^2\bar{\zeta}^{-1}) \\ \quad + (\lambda_1-\lambda_2)w_1(\rho\zeta) - w_1(\rho^2\zeta), & R_1 < |\zeta| \le R_2, \\ (\lambda_1+\lambda_2)w_1(R_1^2\bar{\zeta}^{-1}) \\ \quad + (\lambda_1-\lambda_2)w_1(R_1R_2\bar{\zeta}^{-1}) - w_1(R_1^2\bar{\zeta}^{-1}), & R_2 < |\zeta|, \end{cases} \tag{2.19}$$

and

$$A_2(\zeta) := \begin{cases} (\lambda_1+\lambda_2)w_2(R_2^2\zeta^{-1}) \\ \quad - (\lambda_1-\lambda_2)w_2(R_1R_2\zeta^{-1}) - w_2(R_2^2\zeta^{-1}), & |\zeta| \le R_1, \\ (\lambda_1+\lambda_2)w_2(R_2^2\zeta^{-1}) \\ \quad - (\lambda_1-\lambda_2)w_2(\rho^{-1}\bar{\zeta}) - w_2(\rho^{-2}\bar{\zeta}), & R_1 < |\zeta| \le R_2, \\ (\lambda_1+\lambda_2)w_2(\bar{\zeta}) \\ \quad - (\lambda_1-\lambda_2)w_2(\rho^{-1}\bar{\zeta}) - w_2(\rho^{-2}\bar{\zeta}), & R_2 < |\zeta|. \end{cases} \tag{2.20}$$

We have the following representation formula for the solution to (2.8).

Proposition 2.1. *Suppose $\eta_j \in H_0^{-1/2}(\partial D_j)$ ($j = 1, 2$). The solution v to (2.8) is given by*

$$v(z) = \Re\big(A_1(T(z)) + A_2(T(z))\big), \quad z \in \mathbb{R}^2. \tag{2.21}$$

For the inhomogeneous problem (2.2), $\eta_j = \partial_\nu F|_{\partial D_j}$, and hence the condition that η_j belongs to $H_0^{-1/2}(\partial D_j)$ ($j = 1, 2$) amounts to

$$\int_{D_1} f = \int_{D_2} f = 0. \tag{2.22}$$

Thus we have the following corollary for (2.2).

Corollary 2.2. *Suppose that f satisfies (2.22). The solution u to (2.2) is represented as*

$$u(z) = F(z) + \Re\big(A_1(T(z)) + A_2(T(z))\big) + const. \tag{2.23}$$

For the general case when f does not necessarily satisfy condition (2.22), we can (explicitly) construct functions V_1 and V_2 such that the function f_0, defined by

$$f_0 = f - \left(\int_{D_1} f\right)\nabla\cdot\sigma\nabla V_1 - \left(\int_{D_2} f\right)\nabla\cdot\sigma\nabla V_2,$$

satisfies (2.22), and hence the solution u to (2.2) takes the form

$$u = \left(\int_{D_1} f\right)V_1 + \left(\int_{D_2} f\right)V_2 + u_0, \tag{2.24}$$

where u_0 is the solution to (2.2) of the form (2.21). The construction of functions V_1 and V_2 in [14] heavily uses the fact that D_1 and D_2 are disks.

For the homogeneous problem (2.3), $\eta_j = \partial_\nu H|_{\partial D_j}$ and hence $H_j = H$. Thus, we have the following corollary:

Corollary 2.3. *The solution u to (2.2) is represented as*

$$u(z) = H(z) + \Re\big(A_1\big(T(z)\big) + A_2\big(T(z)\big)\big). \tag{2.25}$$

2.1.2. Optimal estimates for the solution

We now present estimates for the solutions and their derivatives. These estimates are optimal and derived from the explicit representations of the solution presented in the previous subsection. The derivation is far from trivial.

We first introduce some norms for regularity of functions. A function g defined on $\mathbb{R}^2$ (with inclusions D_1 and D_2) is said to be piecewise $C^{n,\alpha}$ for some nonnegative integer n and $0 < \alpha < 1$ if g is $C^{n,\alpha}$ on $\overline{D_1}$, $\overline{D_2}$ and $\mathbb{R}^2 \setminus D$ ($D = D_1 \cup D_2$) separately. For piecewise $C^{n,\alpha}$ functions g, the norm is defined by

$$\|g\|_{n,\alpha} := \|g\|_{C^{n,\alpha}(\overline{D_1})} + \|g\|_{C^{n,\alpha}(\overline{D_2})} + \|g\|_{C^{n,\alpha}(\mathbb{R}^2\setminus D)}. \tag{2.26}$$

When $\alpha = 0$, we denote it by $\|g\|_{n,0}$. We also use the following norm:

$$\|g\|^*_{n,\alpha} := \frac{1}{k_1}\|g\|_{C^{n,\alpha}(\overline{D_1})} + \frac{1}{k_2}\|g\|_{C^{n,\alpha}(\overline{D_2})} + \|g\|_{C^{n,\alpha}(\mathbb{R}^2\setminus D)}. \tag{2.27}$$

When $(k_1 - 1)(k_2 - 1) > 0$ which includes the case when $k_1 = k_2 = \infty$ or $k_1 = k_2 = 0$ in limits, we obtain the following theorems for the inhomogeneous and homogeneous transmission problems. Here and throughout this paper, we put

$$r_* := \sqrt{\frac{2(r_1 + r_2)}{r_1 r_2}}. \tag{2.28}$$

We assume that the inhomogeneity f is given by $f = \nabla \cdot g$ for some g. It is assumed that g is compactly supported in $\mathbb{R}^2$ for the sake of simplicity.

Theorem 2.4. *Suppose $(k_1 - 1)(k_2 - 1) > 0$ and $f = \nabla \cdot g$ for some piecewise $C^{n-1,\alpha}$ function g with the compact support (n is a positive integer and $0 < \alpha < 1$). There is a constant $C > 0$ independent of k_1, k_2, ε, and g such that the solution u to (2.2) satisfies*

$$\|u\|_{n,0} \le C\|g\|^*_{n-1,\alpha}(4\lambda_1\lambda_2 - 1 + r_*\sqrt{\varepsilon})^{-n}. \tag{2.29}$$

This estimate is optimal in the sense that there is g such that the reverse inequality (with a different constant C) holds when $n = 1$.

Theorem 2.5. *Let Ω be a bounded set containing $\overline{D_1 \cup D_2}$. Let u be the solution to (2.3). If $(k_1 - 1)(k_2 - 1) > 0$, then there is a constant $C > 0$ independent of k_1, k_2, ε, and the function H such that*

$$\|u\|_{n,\Omega} \le C\|H\|_{C^n(\Omega)}(4\lambda_1\lambda_2 - 1 + r_*\sqrt{\varepsilon})^{-n}. \tag{2.30}$$

This estimate is optimal in the sense that there is a harmonic function H such that the reverse inequality (with a different constant C) holds for the case $n = 1$. Here, $\|u\|_{n,\Omega}$ denotes the piecewise C^n norm on Ω, namely,

$$\|u\|_{n,\Omega} := \|u\|_{C^n(\overline{D_1})} + \|u\|_{C^n(\overline{D_2})} + \|u\|_{C^n(\Omega\setminus D)}. \tag{2.31}$$

The estimates (2.29) and (2.30) are not new. The estimate (2.29) (for the inhomogeneous problem with circular inclusions) was obtained in **[11]**. The estimate (2.30) for the gradient for the homogeneous problem (with circular inclusions), namely, for $n = 1$, is obtained in **[3, 4]**, while that for higher n in **[11]**.

Since

$$4\lambda_1\lambda_2 - 1 = \frac{2(k_1 + k_2)}{(k_1 - 1)(k_2 - 1)},$$

the estimate (2.29) shows that if either k_1 or k_2 is finite (away from 0 and ∞), then $\|u\|_{n,0}$ is bounded regardless of the distance ε, while if both k_1 and k_2 tend to ∞, then the right-hand side of (2.29) is of order $\varepsilon^{-n/2}$. As explained at the end of this subsection, ∇u may actually blow up at the order of $\varepsilon^{-1/2}$. If k_1 and k_2 tend to 0, then the right-hand side of (2.29) is also of order $\varepsilon^{-n/2}$ provided that $\|g\|^*_{n-1,\alpha}$ is bounded, in particular, if there is no source in $D_1 \cup D_2$, namely, $g = 0$ in $D_1 \cup D_2$. The estimate (2.30) yields the same findings.

If $(k_1 - 1)(k_2 - 1) < 0$ which includes the case when $k_1 = 0$ and $k_2 = \infty$ (or the other way around) in limits, then $4\lambda_1\lambda_2 < 0$. Thus the right-hand sides of (2.29) and (2.30) are bounded and cannot be the right estimates for this case. Instead, we obtain the following theorems.

Theorem 2.6. *Suppose $(k_1 - 1)(k_2 - 1) < 0$ and $f = \nabla \cdot g$ for some piecewise $C^{n,\alpha}$ function g with compact support (n is a positive integer and $0 < \alpha < 1$). There is a constant $C > 0$ independent of k_1, k_2, ε, and g such that the solution u to (2.2) satisfies*

$$\|u\|_{n,0} \le C\|g\|^*_{n,\alpha}\big(4|\lambda_1\lambda_2| - 1 + r_*\sqrt{\varepsilon}\big)^{-n+1}. \tag{2.32}$$

This estimate is optimal in the sense that there is f such that the reverse inequality (with a different constant C) holds for $n = 2$.

Theorem 2.7. *Let Ω be a bounded set containing $\overline{D_1 \cup D_2}$. Let u be the solution to (2.3). If $(k_1 - 1)(k_2 - 1) < 0$, then there is a constant $C > 0$ independent of k_1, k_2, ε, and the function H such that*

$$\|u\|_{n,\Omega} \le C\|H\|_{C^{n+1}(\Omega)}\big(4|\lambda_1\lambda_2| - 1 + r_*\sqrt{\varepsilon}\big)^{-n+1}. \tag{2.33}$$

This estimate is optimal in the sense that there is a harmonic function H such that the reverse inequality (with a different constant C) holds for $n = 2$.

Estimates (2.32) and (2.33) show that if $(k_1 - 1)(k_2 - 1) < 0$, then ∇u is bounded regardless of the k_1, k_2, and ε. But, the nth ($n \ge 2$) order derivative may blow up at the rate of $\varepsilon^{-(n-1)/2}$ if, for example, $k_1 = 0$ and $k_2 = \infty$. The second derivative of u actually blows up at the rate of $\varepsilon^{-1/2}$ in some cases as explained in the next subsection. These results are new and waiting to be generalized to inclusions of general shape and to higher dimensions.

2.1.3. Optimality of the estimates

Let F be a smooth function in $\mathbb{R}^2$ with a compact support such that $F(z) = x_1$ in a neighborhood of $\overline{D_1 \cup D_2}$. Let $f := \Delta F$. Then the following hold [14]:

(i) Let $k_1 = k_2 = \infty$. The solution u to (2.2) satisfies

$$\left|\nabla u(z)\right| \gtrsim \varepsilon^{-1/2} \tag{2.34}$$

for some $z \in \mathbb{R}^2 \setminus \overline{D}$.

(ii) For the case when $(k_1 - 1)(k_2 - 1) < 0$, we take either $k_1 = 0$, $k_2 = \infty$ or $k_1 = \infty$, $k_2 = 0$. The solution u to (2.2) satisfies

$$\left|\nabla^2 u(z)\right| \gtrsim \varepsilon^{-1/2} \tag{2.35}$$

for some $z \in \mathbb{R}^2 \setminus \overline{D}$, while ∇u is bounded.

Similar estimates hold for the solution to the homogeneous problem (2.3) with $H(x) = x_1$ (the optimality of the gradient estimate is also shown [3]).

2.2. Estimates for inclusions of general shape

The estimate (2.30) shows that if k_1, k_2 are finite, namely, $0 < C_1 \le k_1$, $k_2 \le C_2 < \infty$ for some constants C_1, C_2, then ∇u is bounded regardless of ε. This fact is known to be true in a more general setting where there are several inclusions of arbitrary shape [29] (see [10] for the case of circular inclusions).

If $k_1 = k_2 = \infty$ (the perfectly conducting case), then we see from (2.30) that

$$\left|\nabla u(z)\right| \lesssim \varepsilon^{-1/2}. \tag{2.36}$$

This estimate and its optimality for the case of strictly convex inclusions (more generally, if they are strictly convex near the points of the shortest distance) in two dimensions has been proved in [34]. In three dimensions, the optimal estimate for ∇u has been obtained in [6] as

$$\left|\nabla u(z)\right| \lesssim \frac{1}{\varepsilon|\ln \varepsilon|}. \tag{2.37}$$

(See [26, 31] for the case of spherical inclusions.) In [21], a bow-tie structure, where two vertices are points of the shortest distance, is considered. It is proved that two kinds of singularities appear, one due to the corners and the other due to the interaction between the two inclusions.

If $k_1 = k_2 = 0$ (the insulating case), the same estimate for $|\nabla u|$ as the perfectly conducting case holds in two dimensions. This is due to the existence of harmonic conjugates and does not extend to three dimensions. In fact, the three-dimensional case is completely different. It is proved in [7] that if $k_1 = k_2 = 0$, the estimate

$$\left|\nabla u(z)\right| \lesssim \varepsilon^{-s} \tag{2.38}$$

holds with $s = 1/2$ when inclusions are strictly convex inclusions in three dimensions. It is then proved in [35] that the surprising estimate with $s = \frac{2-\sqrt{2}}{2}$ holds on the shortest line segment between two spherical inclusions of the same radii. Recently in [30] the estimate with $s = 1/2 - \gamma$ for some $\gamma > 0$ was derived on strictly convex inclusions and for dimensions $d \geq 3$. An upper bound of γ for $d \geq 4$ has been derived in [33].

It is likely that in the three-dimensional insulating case the behavior of the gradient depends heavily on the geometry of inclusions, and it is not clear at all what the best possible s is in (2.38). It is not even clear if such a number exists; it may depend on the position x of the estimate. Clarifying this is now an outstanding open problem to be solved.

For the inhomogeneous problem, estimates on conducting inclusions of circular and bow-tie shapes in two dimensions and of spherical shape in three dimensions when the source function is an emitter, namely, $f = a \cdot \delta_z$ for some z outside inclusions, have been obtained [22–24]. Here, δ_z denotes the Dirac-delta function. Such a problem is considered in relation to the patched antenna where the field excited by an emitter of the dipole-type is enhanced by closely located antenna (see, for example, [32]).

Theorems 2.6 and 2.7 for the case $(k_1 - 1)(k_2 - 1) < 0$ are new and unexpected, and their extension to inclusions of general shape and to higher dimensions is wide open. Particular interest lies in the high contrast case, namely, $k_1 = 0$ and $k_2 = \infty$; we do not know whether the gradient is bounded and the higher order derivatives blow up, if so at what rate. The case of spherical inclusions seems already quite challenging.

2.3. Asymptotic characterizations of the gradient blow-up

The problem (2.3) in the limit $k_1 \to \infty$ and $k_2 \to \infty$ can be rewritten as

$$\begin{cases} \Delta u = 0 & \text{in } D^e, \\ u = \lambda_j \text{ (constant)} & \text{on } \partial D_j,\ j = 1, 2, \\ u(x) - H(x) = O\big(|x|^{1-d}\big) & \text{as } |x| \to \infty, \end{cases} \tag{2.39}$$

where $D^e := \mathbb{R}^d \setminus \overline{(D_1 \cup D_2)}$. The problem (2.39) is not an exterior Dirichlet problem since the constants λ_j are not prescribed. Rather, they are determined by the conditions

$$\int_{\partial D_j} \partial u|_+ \, dS = 0, \quad j = 1, 2. \tag{2.40}$$

The constants λ_1 and λ_2 may or may not be the same depending on the given H (and the configuration of inclusions). When they are different, a sharp gradient occurs if the distance between D_1 and D_2 is short.

The singular behavior of ∇u where u is the solution to (2.39) can be characterized by the singular function $q = q_D$ which is the solution to

$$\begin{cases} \Delta q = 0 & \text{in } D^e, \\ q = \text{constant} & \text{on } \partial D_j,\ j = 1, 2, \\ \displaystyle\int_{\partial D_j} \partial q|_+ dS = -(-1)^j, & j = 1, 2, \\ q(x) = O\big(|x|^{1-d}\big) & \text{as } |x| \to \infty. \end{cases} \tag{2.41}$$

For general inclusions D_1 and D_2, there is a unique solution to (2.41) (see [1]).

Using the singular function q_D, the solution u to (2.39) can be decomposed as

$$u = \alpha q_D + r, \tag{2.42}$$

where

$$\alpha = \frac{u|_{\partial D_2} - u|_{\partial D_1}}{q_D|_{\partial D_2} - q_D|_{\partial D_1}}. \tag{2.43}$$

Here the constant α and functions q_D, r depend on ε. Observe that r attains constant values on ∂D_1 and ∂D_2, and $r|_{\partial D_1} = r|_{\partial D_2}$, so that ∇r is bounded on D^e (see [16]). Thus the term $\alpha \nabla q_D$ characterizes the blow-up of ∇u as $\varepsilon \to 0$. In particular, since ∇q_D is of order $\varepsilon^{-1/2}$, α represents the magnitude of the blow-up, and hence is called the stress concentration factor.

If $D_1 = B_1$ and $D_2 = B_2$ are two disjoint disks, the solution q (we denote it by q_B in this case) can be found explicitly. Let R_j be the inversion with respect to ∂B_j ($j = 1, 2$), and let δ_1 and δ_2 be the unique fixed points of the combined inversions $R_1 \circ R_2$ and $R_2 \circ R_1$, respectively. Let

$$q_B(x) = \frac{1}{2\pi}\big(\ln|x - \delta_1| - \ln|x - \delta_2|\big). \tag{2.44}$$

The function q_B is the solution to (2.41). In particular, q_B is constant on ∂B_j because ∂B_1 and ∂B_2 are circles of Apollonius of points δ_1 and δ_2. The function q_B appears in the bipolar coordinate system for ∂B_1 and ∂B_2 and was used for analysis of the field concentration for the first time in [34]. Using the explicit form of the function q_B, it is proved that

$$\|\nabla q_B\|_{L^\infty(\mathbb{R}^2 \setminus (B_1 \cup B_2))} \sim \varepsilon^{-1/2}. \tag{2.45}$$

Results on asymptotic characterizations of the gradient blow-up in two dimensions may be summarized as follows:

(i) If $D_1 = B_1$ and $D_2 = B_2$ are disks, then

$$\alpha = \frac{4\pi r_1 r_2}{r_1 + r_2} \frac{(z_2 - z_1) \cdot \nabla H(\frac{z_1 + z_2}{2})}{|z_2 - z_1|} + O(\sqrt{\varepsilon}) \quad \text{as } \varepsilon \to 0, \tag{2.46}$$

where r_j is the radius of D_j, $j = 1, 2$ [16].

(ii) Suppose that ∂D_j is $\mathcal{C}^{2,\gamma}$ for some $\gamma \in (0, 1)$. We further suppose that there are unique points $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$ such that $|z_1 - z_2| = \text{dist}(D_1, D_2)$ and there is a common neighborhood U of z_1 and z_2 such that $D_j \cap U$ is strictly convex for $j = 1, 2$. Let B_j be the disk osculating to D_j at z_j ($j = 1, 2$). Then,

$$\nabla q_D = \nabla q_B\big(1 + O(\varepsilon^{\gamma/2})\big) + O(1), \tag{2.47}$$

and

$$\alpha = \frac{\sqrt{2}\pi}{\sqrt{\kappa_1 + \kappa_2}} \frac{1}{\sqrt{\varepsilon}} \int_{\partial D_1 \cup \partial D_2} H \partial_\nu q_D \, d\sigma \big(1 + O(\varepsilon^{\gamma/2})\big). \tag{2.48}$$

In particular, α is bounded regardless of ε [1].

(iii) Let D_1^0 and D_2^0 be the touching inclusions obtained as the limit of D_1 and D_2 as $\varepsilon \to 0$ (D_1 and D_2 are still assumed to satisfy assumptions of (ii)), and let u_0 be the solution for the touching case, namely,

$$\begin{cases} \Delta u_0 = 0 & \text{in } D_0^e, \\ u_0 = \lambda_0 & \text{on } \partial D_0^e, \\ u_0(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{2.49}$$

where $D_0^e := \mathbb{R}^2 \setminus \overline{(D_1^0 \cup D_2^0)}$ and λ_0 is a constant determined by the additional condition

$$\int_\Omega |\nabla(u_0 - H)|^2 dA < \infty. \tag{2.50}$$

Then,

$$\alpha = \int_{\partial D_1^0} \partial_\nu u_0 + O\big(\varepsilon |\log \varepsilon|\big) \tag{2.51}$$

as $\varepsilon \to 0$ [15].

The decomposition formula (2.42) (together with (2.47) and (2.51)) has some important consequences. Since ∇q_D is bounded from below and above by $\varepsilon^{-1/2}$ (up to constant multiples), the blow-up estimates for ∇u can be obtained from the formula. It can be used to compute u numerically. Since the formula extracts the leading singular term in an explicit way, it suffices to compute the residual term b for which only regular meshes are required. This idea appeared and was exploited in [16] in the special case when D_j are disks.

The formula (2.42) has another very interesting implication. The quantity $\nabla u \cdot n$ represents the charge density on $\partial D_1 \cup \partial D_2$ induced by the field $-\nabla H$, and $\nabla u_0 \cdot n$ does that on $\partial D_1^0 \cup \partial D_2^0$. Note that the charge densities on the separated inclusions have a singular part $\alpha \nabla q_D \cdot n$ and a regular part $\nabla r \cdot n$. It is proved in [15] that $\nabla r \cdot n$ converges to $\nabla u_0 \cdot n$ as $\varepsilon \to 0$, that is, as the separated inclusions approach the touching ones. So the singular part suddenly disappears when the two inclusions become touching. It is reminiscent of the electrical spark occurring between two separated conductors which suddenly disappears when the conductors are touching.

The decomposition formula of the kind (2.42) when D_1 and D_2 are three-dimensional balls of the same radii has been derived in [17] (see [27] for the case of different radii). In this case the singular function is given as an infinite superposition of point charges.

3. LAMÉ SYSTEM

In this section we review results on the field concentration for the Lamé system of linear elasticity. If Lamé parameters are finite so that inclusions are of low contrast with the matrix, then the gradient of the solution is bounded regardless of the distance between inclusions. This is the well-known result of Li–Nirenberg [28]. The only known results for the high contrast case are when inclusions are hard and strictly convex. We review them here. Hard inclusions for the elasticity correspond to the perfect conductors for the electricity and are characterized by the boundary condition as explained blow.

As before, let D_1 and D_2 be bounded domains in $\mathbb{R}^2$. Let (λ, μ) be the pair of Lamé constants of $D^e = \mathbb{R}^2 \setminus \overline{(D_1 \cup D_2)}$ which satisfies the strong ellipticity conditions, $\mu > 0$ and $\lambda + \mu > 0$ (we only consider the two-dimensional case). The Lamé operator is given by

$$\mathcal{L}_{\lambda,\mu} u := \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u, \tag{3.1}$$

where $u = (u_1, u_2)^T$ (T for transpose) is a vector-valued function. Let

$$\Psi_1(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Psi_2(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \Psi_3(x) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}, \tag{3.2}$$

which are the displacement fields of the rigid motions.

The problem for the Lamé system is given as follows:

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u = 0 & \text{in } D^e, \\ u = \displaystyle\sum_{j=1}^{3} c_{ij} \Psi_j & \text{on } \partial D_i, i = 1, 2, \\ u(x) - H(x) = O(|x|^{-1}) & \text{as } |x| \to \infty, \end{cases} \tag{3.3}$$

where $H = (h_1, h_2)^T$ is a given function satisfying $\mathcal{L}_{\lambda,\mu} H = 0$ in $\mathbb{R}^2$. The boundary conditions to be satisfied by the displacement u on ∂D_j (the second line in (3.3)) indicate that D_1 and D_2 are hard inclusions. The constants c_{ij} there are not given but determined by the condition similar to (2.40), that is,

$$\int_{\partial D_i} \Psi_j \cdot \sigma[u] n \, ds = 0, \quad i = 1, 2, \ j = 1, 2, 3. \tag{3.4}$$

Here, $\sigma[u]$ denotes the stress tensor corresponding to the displacement vector u, defined by

$$\sigma[u] := \lambda (\nabla \cdot u) + 2\mu (\widehat{\nabla} u),$$

where $\widehat{\nabla} u = \frac{1}{2}(\nabla u + \nabla u^T)$.

An asymptotic characterization of the solution u to (3.3), which captures the singular behavior of ∇u, is obtained in [18]. It is given in terms of singular functions which are constructed by the singular function q_B for the conductivity problem given in (2.44). To describe them, let z_1, z_2, B_1, B_2 be as before (right before (2.47)), namely $z_1 \in \partial D_1$ and $z_2 \in \partial D_2$ are unique points such that $|z_1 - z_2| = \mathrm{dist}(D_1, D_2)$, there is a common neighborhood U of z_1 and z_2 such that $D_i \cap U$ is strictly convex for $i = 1, 2$, and B_i is the disk osculating to D_i at z_i ($i = 1, 2$). Let δ_1 and δ_2 be the points appearing in the definition (2.44) of q_B, namely the fixed points of the combined inversions. After a translation and a rotation if necessary, we may assume that $\delta_1 = (-a, 0)$ and $\delta_2 = (a, 0)$. This number a is actually satisfies $a = 2\beta$, where β is given in (2.10). If we denote the centers of B_i by $(c_i, 0)$ ($i = 1, 2$), then c_i satisfies the relation

$$c_i = (-1)^i \sqrt{r_i^2 + a^2}, \quad i = 1, 2. \tag{3.5}$$

Let $q = q_B$ and let

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{\lambda + 2\mu}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\lambda + 2\mu}\right). \tag{3.6}$$

Singular functions Q_1 and Q_2 for the elasticity problem of this section are defined by

$$Q_1 = \alpha_1 \begin{bmatrix} q \\ 0 \end{bmatrix} - \alpha_2 x_1 \nabla q \tag{3.7}$$

and

$$Q_2 = \alpha_1 \begin{bmatrix} 0 \\ q \end{bmatrix} + \alpha_2 x_1 (\nabla q)^{\perp}, \tag{3.8}$$

where $(a,b)^{\perp} = (-b,a)$. Actually, these functions were found in [18] as linear combinations of point-source functions in linear elasticity called nuclei of strain. It turns out that they can be expressed in simple forms using the function q (see also [20]).

One can easily see that Q_j are solutions to the Lamé system, namely

$$\mathcal{L}_{\lambda,\mu} Q_j = 0 \quad \text{in } \mathbb{R}^2 \setminus \{\delta_1, \delta_2\}. \tag{3.9}$$

It is shown in [18] that Q_j takes "almost" constant values Ψ_j on the osculating circles ∂B_i ($i = 1,2$). In fact, there are constants k_{ji} and l_{ji} such that for $i = 1,2$,

$$Q_1(x) = k_{1i} \Psi_1(x) + l_{1i} x, \quad x \in \partial B_i, \tag{3.10}$$

and

$$Q_2(x) = k_{2i} \Psi_2(x) + l_{2i} x^{\perp}, \quad x \in \partial B_i. \tag{3.11}$$

Actually, the constants k_{ji} and l_{ji} can be easily derived using the simple forms Q_j. Using the fact that q is constant on ∂B_i, one can show that

$$\nabla q(x) = -\frac{a}{2\pi r_i} \frac{1}{x_1} (x_1 - c_i, x_2), \quad x \in \partial B_i, \ i = 1,2.$$

It thus follows that for $i = 1,2$,

$$k_{1i} = \alpha_1 q|_{\partial B_i} - \frac{\alpha_2 a c_i}{2\pi r_i}, \quad l_{1i} = \frac{\alpha_2 a}{2\pi r_i}, \tag{3.12}$$

and

$$k_{2i} = \alpha_1 q|_{\partial B_i} + \frac{\alpha_2 a c_i}{2\pi r_i}, \quad l_{2i} = -\frac{\alpha_2 a}{2\pi r_i} \tag{3.13}$$

Another function related with the boundary value Ψ_3 on ∂B_1 and ∂B_2 is constructed in the same paper. But this function has nothing to do with the singular behavior of the field, so we omit it here. It is worth mentioning that the singular functions Q_1 and Q_2 are effectively utilized to prove the Flaherty–Keller formula [12] describing the effective property of densely packed elastic composites [19].

Using the singular functions Q_1 and Q_2, it is proved that the solution u to (3.3) admits the following decomposition:

$$u = C_1 Q_1 + C_2 Q_2 + b, \tag{3.14}$$

where C_1 and C_2 are constants depending on ε, but bounded independently of ε, and b is a function whose gradient is bounded on any bounded subset of D^e. The following estimate is obtained as an immediate consequence of the decomposition formula:

$$\|\nabla u\|_{L^\infty(D^e)} \lesssim \varepsilon^{-1/2}. \tag{3.15}$$

This estimate is also proved in [8]. This estimate is optimal in the sense that the reverse inequality holds in some cases. An extension to three dimensions has been achieved in [9].

We emphasize that the constants C_1 and C_2 appearing in formula (3.14) are not explicit. Thus further investigation on how to determine them (or compute them numerically) is desired.

4. STOKES SYSTEM

In this section we review the result in [2], that is, an asymptotic characterization of the stress concentration for the Stokes flow modeled by $\mu\Delta u = \nabla p$ and $\nabla \cdot u = 0$. Here, μ represents the constant viscosity of the fluid. Even if the result is only for the two-dimensional inclusions of circular shape, the result may serve as a milestone for further development.

Let D_1 and D_2 be disks and let $D^e = \mathbb{R}^2 \setminus \overline{D_1 \cup D_2}$ as before. Let (U, P) is a given background solution to the homogeneous Stokes system in $\mathbb{R}^2$, namely, $\mu\Delta U = \nabla P$ and $\nabla \cdot U = 0$ in $\mathbb{R}^2$. We consider the following problem of the Stokes system:

$$\begin{cases} \mu\Delta u = \nabla p & \text{in } D^e, \\ \nabla \cdot u = 0 & \text{in } D^e, \\ u = \displaystyle\sum_{j=1}^{3} d_{ij}\Psi_j & \text{on } \partial D_i,\ i = 1, 2, \end{cases} \tag{4.1}$$

with the conditions

$$(u - U)(x) = O\big(|x|^{-1}\big), \quad \nabla(u - U)(x) = O\big(|x|^{-2}\big), \quad (p - P)(x) = O\big(|x|^{-2}\big)$$

as $|x| \to \infty$. Here, Ψ_j are the functions given in (3.2), and d_{ij} are constants to be determined from the equilibrium conditions

$$\int_{\partial D_i} \Psi_j \cdot \sigma[u, p]n \, d\sigma = 0, \quad i = 1, 2,\ j = 1, 2, 3. \tag{4.2}$$

Here, $\sigma[u, p]$ is the stress field induced by the velocity-pressure pair (u, p), namely

$$\sigma[u, p] = -pI + 2\mu\widehat{\nabla} u, \tag{4.3}$$

where I is the identity matrix.

As the distance between D_1 and D_2 tends to 0, the solution to (4.1) exhibits singular behavior in its gradient which can be captured in terms of singular functions. The singular functions for (4.1) form the solution (V_j, p_j) $(j = 1, 2)$ to the following problem:

$$\begin{cases} \mu\Delta V_j = \nabla p_j & \text{in } \mathbb{R}^2 \setminus \{\delta_1, \delta_2\}, \\ \nabla \cdot V_j = 0 & \text{in } \mathbb{R}^2 \setminus \{\delta_1, \delta_2\}, \\ V_j = \dfrac{(-1)^i}{2}\Psi_j & \partial B_i,\ i = 1, 2, \end{cases} \tag{4.4}$$

with the conditions

$$V_j(x) = C_j + O\big(|x|^{-1}\big), \quad \nabla V_j(x) = O\big(|x|^{-2}\big), \quad p_j(x) = O\big(|x|^{-2}\big)$$

for some constant C_j as $|x| \to \infty$. Here δ_j is the point appearing in (2.44).

In [2], singular functions (V_j, p_j) are constructed using the stream function formulation for which the bipolar coordinate system is used. We assume $\delta_1 = (-a, 0)$ and $\delta_2 = (a, 0)$ as before. Then, the bipolar coordinates (ζ, θ) are defined by

$$\zeta = 2\pi q_D, \quad \theta = \arg(x-a, y) - \arg(x+a, y). \tag{4.5}$$

Let

$$e_\zeta = \frac{\nabla\zeta}{|\nabla\zeta|}, \quad e_\theta = \frac{\nabla\theta}{|\nabla\theta|}.$$

Suppose that D_1 and D_2 have the same radius, say R, and let

$$s = \sinh^{-1}(a/R).$$

Define two constants A_1 and B_1 by

$$A_1 := \frac{1}{2s - \tanh 2s}, \quad B_1 := -\frac{1}{2\cosh 2s} A_1. \tag{4.6}$$

Then, the velocity V_1 is given by $V_1 = v_{1\zeta} e_\zeta + v_{1\theta} e_\theta$ where

$$v_{1\zeta} = (A_1\zeta + B_1 \sinh 2\zeta) \frac{1 - \cosh\zeta\cos\theta}{\cosh\zeta - \cos\theta}, \tag{4.7}$$

$$v_{1\theta} = \sin\theta\left(A_1 + 2B_1\cosh 2\zeta - \frac{\sinh\zeta(A_1\zeta + B_1\sinh 2\zeta)}{\cosh\zeta - \cos\theta}\right), \tag{4.8}$$

and the pressure p_1 is given by

$$p_1 = \frac{2\mu}{a}\big((A_1 - 2B_1)\cosh\zeta\cos\theta + B_1\cosh 2\zeta\cos 2\theta\big) - \frac{2\mu}{a}(A_1 - B_1). \tag{4.9}$$

The formulas for (V_2, p_2) are quite involved. But it is proved in [2] that

$$V_2 = -A_2\begin{bmatrix} 0 \\ \zeta \end{bmatrix} + A_2 x(\nabla\zeta)^\perp + V_{2o} \tag{4.10}$$

and

$$p_2 = -\frac{2\mu}{a} A_2 \sinh\zeta\sin\theta + p_{2o}, \tag{4.11}$$

where (V_{2o}, p_{2o}) is a solution to the Stokes system whose gradient is bounded regardless of ε, and A_2 is the constant defined by

$$A_2 = -\frac{1}{2s + \sinh 2s}. \tag{4.12}$$

The function V_2 is similar to the function Q_2 for the Lamé system given in (3.8).

It is proved in the same paper that if the background velocity field U is given by

$$U(x_1, x_2) = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \big(\alpha^2 + (\beta+\gamma)^2 \neq 0\big) \tag{4.13}$$

for some constants α, β, and γ, the background pressure $P = 0$, and if D_1 and D_2 are disks of the same radius R, then the solution (u, p) admits a decomposition of the following form:

$$(u, p) = \alpha\frac{2}{\sqrt{R}}\varepsilon^{3/2}(V_1, p_1) + \frac{\beta+\gamma}{2}\sqrt{R\varepsilon}(V_2, p_2) + (u_0, p_0), \tag{4.14}$$

where (u_0, p_0) is a solution to the Stokes problem whose stress tensor is bounded. Thus we have

$$\sigma[u,p] = \alpha \frac{2}{\sqrt{R}} \varepsilon^{3/2} \sigma[V_1, p_1] + \frac{\beta + \gamma}{2} \sqrt{R\varepsilon} \sigma[V_2, p_2] + \sigma[u_0, p_0]. \tag{4.15}$$

Since $\|\sigma[V_1, p_1]\|_{L^\infty(D^e)} \approx \varepsilon^{-2}$ and $\|\sigma[V_2, p_2]\|_{L^\infty(D^e)} \approx \varepsilon^{-1}$ as proved in [2], we have

$$\|\sigma[u,p]\|_{L^\infty(D^e)} \approx \varepsilon^{-1/2}, \tag{4.16}$$

which says that the stress always blows up at the rate of $\varepsilon^{-1/2}$ provided that U is linear as given in (4.13) and inclusions are circular. It is quite interesting and challenging to extend this result to the noncircular case.

5. CONCLUSIONS

In this paper we review significant results on optimal estimates of the derivatives and asymptotic characterizations of the solution in the presence of two inclusions when the distance between them tends to zero. A special emphasis is laid on the case of high contrast. We review results on the conductivity equation, the Lamé system, and the Stokes system. Apart from these equations, the stress concentration factor for the p-Laplacian has been derived in [13].

As mentioned in the text, many challenging problems remain unsolved. Among them, the problem for the three-dimensional insulating case is outstanding. The case when the conductivities k_1 and k_2 satisfy the condition $(k_1 - 1)(k_2 - 1) < 0$ is also quite interesting. It goes without saying that the studies of problems for the Lamé and Stokes systems are in their early stage. Extensions to general shape and higher dimensions are quite challenging.

FUNDING

This work was partially supported by NRF (of S. Korea) grant No. 2019R1A2B5B0106-9967.

REFERENCES

[1] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and K. Yun, Spectral analysis of the Neumann–Poincaré operator and characterization of the stress concentration in anti-plane elasticity. *Arch. Ration. Mech. Anal.* **208** (2013), 275–304.

[2] H. Ammari, H. Kang, D. W. Kim, and S. Yu, Quantitative estimates for stress concentration of the Stokes flow between adjacent circular cylinders. 2020, arXiv:2003.06578.

[3] H. Ammari, H. Kang, H. Lee, J. Lee, and M. Lim, Optimal Estimates for the Electrical Field in Two Dimensions. *J. Math. Pures Appl.* **88** (2007), 307–324.

[4] H. Ammari, H. Kang, and M. Lim, Gradient estimates for solutions to the conductivity problem. *Math. Ann.* **332** (2005), 277–286.

[5] I. Babuška, B. Andersson, P. Smith, and K. Levin, Damage analysis of fiber composites. I. Statistical analysis on fiber scale. *Comput. Methods Appl. Mech. Engrg.* **172** (1999), 27–77.

[6] E. Bao, Y. Y. Li, and B. Yin, Gradient estimates for the perfect conductivity problem. *Arch. Ration. Mech. Anal.* **193** (2009), 195–226.

[7] E. Bao, Y. Y. Li, and B. Yin, Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions. *Comm. Partial Differential Equations* **35** (2010), 1982–2006.

[8] J. Bao, H. Li, and Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients. *Arch. Ration. Mech. Anal.* **215** (2015), 307–351.

[9] J. Bao, H. Li, and Y. Li, Gradient estimates for solutions of the Lamé system with partially infinite coefficients in dimensions greater than two. *Adv. Math.* **305** (2017), 298–338.

[10] E. Bonnetier and M. Vogelius, An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section. *SIAM J. Math. Anal.* **31** (2000), 651–677.

[11] H. Dong and H. Li, Optimal estimates for the conductivity problem by Green's function method. *Arch. Ration. Mech. Anal.* **231** (2019), 1427–1453.

[12] J. E. Flaherty and J. B. Keller, Elastic behavior of composite media. *Comm. Pure Appl. Math.* **26** (1973), 565–580.

[13] Y. Gorb and A. Novikov, Blow-up of solutions to a p-Laplace equation. *Multiscale Model. Simul.* **10** (2012), 727–743.

[14] Y. Ji and H. Kang, Spectrum of the Neumann–Poincaré operator and optimal estimates for transmission problems in presence of two circular inclusions. 2021, arXiv:2105.06093.

[15] H. Kang, H. Lee, and K. Yun, Optimal estimates and asymptotics for the stress concentration between closely located stiff inclusions. *Math. Ann.* **363** (2015), 1281–1306.

[16] H. Kang, M. Lim, and K. Yun, Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities. *J. Math. Pures Appl.* **99** (2013), 234–249.

[17] H. Kang, M. Lim, and K. Yun, Characterization of the electric field concentration between two adjacent spherical perfect conductors. *SIAM J. Appl. Math.* **74** (2014), 125–146.

[18] H. Kang and S. Yu, Quantitative characterization of stress concentration in the presence of closely spaced hard inclusions in two-dimensional linear elasticity. *Arch. Ration. Mech. Anal.* **232** (2019), 121–196.

[19] H. Kang and S. Yu, A proof of the Flaherty–Keller formula on the effective property of densely packed elastic composites. *Calc. Var. Partial Differential Equations* **59** (2020), 22.

[20] H. Kang and S. Yu, Singular functions and characterizations of field concentrations: a survey. *Anal. Theory Appl.* **37** (2021), 102–113.

[21] H. Kang and K. Yun, Optimal estimates of the field enhancement in presence of a bow-tie structure of perfectly conducting inclusions in two dimensions. *J. Differential Equations* **266** (2019), 5064–5094.

[22] H. Kang and K. Yun, Precise estimates of the field excited by an emitter in presence of closely located inclusions of a bow-tie shape. *J. Math. Anal. Appl.* **479** (2019), no. 2, 1670–1707.

[23] H. Kang and K. Yun, Quantitative estimates of the field excited by an emitter in a narrow region between two circular inclusions. *Quart. Appl. Math.* **LXXVII** (2019), no. 4, 861–873.

[24] H. Kang and K. Yun, Quantitative estimates for enhancement of the field excited by an emitter due to presence of two closely located spherical inclusions. *J. Differential Equations* **269** (2020), 2977–3002.

[25] J. B. Keller, Conductivity of a medium containing a dense array of perfectly conducting spheres or cylinders or nonconducting cylinders. *J. Appl. Phys.* **34** (1963), 991–993.

[26] J. Lekner, Electrostatics of two charged conducting spheres. *Proc. R. Soc. A* **468** (2012), 2829–2848.

[27] H. Li, F. Wang, and L. Xu, Characterization of electric fields between two spherical perfect conductors with general radii in 3D. *J. Differential Equations* **267** (2019), 6644–6690.

[28] Y. Y. Li and L. Nirenberg, Estimates for elliptic system from composite material. *Comm. Pure Appl. Math.* **LVI** (2003), 892–925.

[29] Y. Y. Li and M. Vogelius, Gradient estimates for solution to divergence form elliptic equation with discontinuous coefficients. *Arch. Ration. Mech. Anal.* **153** (2000), 91–151.

[30] Y. Y. Li and Z. Yang, Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two. 2020, arXiv:2012.14056.

[31] M. Lim and K. Yun, Blow-up of electric fields between closely spaced spherical perfect conductors. *Comm. Partial Differential Equations* **34** (2009), 1287–1315.

[32] V. Pacheco-Penã, M. Beruete, A. I. Fernández-Domínquez, Y. Luo, and M. Navarro-Cía, Description of bow-tie nanoantennas excited by localized emitters using conformal transformation. *ACS Photonics* **3** (2016), 1223–1232.

[33] B. Weinkove, The insulated conductivity problem, effective gradient estimates and the maximum principle. 2021, arXiv:2103.14143.

[34] K. Yun, Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape. *SIAM J. Appl. Math.* **67** (2007), 714–730.

[35] K. Yun, An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions. *J. Differential Equations* **261** (2016), 148–188.

HYEONBAE KANG
Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon 22212, S. Korea, hbkang@inha.ac.kr