THE LONG WAY FROM MATHEMATICS TO MATHEMATICS EDUCATION: HOW EDUCATIONAL RESEARCH MAY CHANGE ONE'S VISION OF MATHEMATICS AND OF ITS LEARNING AND TEACHING

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ABSTRACT

Mathematicians and mathematics educators are united by their deep care for mathematics. This said, they are sometimes like parents who have differing ideas about what is good for the child. To improve communication between these two communities, I am telling the story of my own transformation from mathematics to mathematics education. In this account, I explain why I was compelled to revise my vision of mathematics and how I eventually arrived at the "commognitive" conceptualization, according to which mathematics is an activity of telling stories that produce their own objects. This change of vision brought many insights about learning mathematics and about factors that may slow students' progress. I illustrate some of the gains that come with commognitive conceptualization by showing how this approach allowed my colleagues and me to come to grips with some learning-related phenomena that have long been puzzling mathematicians and educators.

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Let me begin with introducing myself: I am a mathematics education researcher, particularly interested in how people learn mathematics. If I am here, at the convention of mathematicians, it is because our two communities, that of mathematicians and that of mathematics educators, have something centrally important in common: for all of us, 'mathematics' is the keyword around which our professional activities evolve. True, mathematicians spend their time *within* mathematical universe investigating its objects, whereas my colleagues and I sit in school and university classrooms observing those who try to enter that universe. Yet, understanding mathematics is the basic requirement for both of us. Our two communities are also united in their deep care for mathematics. This said, mathematicians and mathematics educators are sometimes like parents who have differing ideas about what is good for the child. The main sources of our occasional disagreements, it seems, are our dissimilar perspectives. Mathematicians never take their eyes off mathematical objects, whereas educational researchers constantly vacillate between this abstract universe and the outside world, populated by human beings. When invited to this conference, I felt this may be a good occasion to take a closer look at similarities and differences of these two outlooks. Getting acquainted with your interlocutor's thinking, even if it is unlikely to turn into your own, is the necessary first step in bridging potential communicational gaps. The best way to do this, I thought, would be by reflecting on what changed in my and my colleagues' journey from mathematics to mathematics education.

For me, this trip began years ago. As I traveled, the view before my eyes evolved all the time, changing from time to time almost beyond recognition. Today, I consider myself as a mathematical insider-turned-outsider, or a participant-turned-observer. I believe that my first-hand experiences as a member of both research communities makes me well equipped for the job of explaining and justifying my current perspective. Retracing the events that transformed me and my colleagues from people-who-think-like-mathematicians into those-who-think-like-educators may do the job best. This is, indeed, what I intend to do in this paper. Mine will be a story of an evolving vision of mathematics and of the deepening understanding of how children and young people turn into mathematical thinkers – or fail to do so. As my narrative unfolds, please keep in mind that if I occasionally speak in the first person singular, it is not because I consider my own history as in any way special or unique. On the contrary, it is because of its being rather common that I find it worth telling. My perspective may not be the only one with some traction within the community of mathematics education, but it can be considered as generic, in that it reflects concerns and sensitivities common to most of those who teach mathematics. The resulting story, therefore, which is not unlike those many of my colleagues could tell, should not be read as an autobiographical exercise but rather as a general reflection on how answers to the questions of what mathematics is and how people learn may change with the change of the storyteller's perspective. As you go through the following pages, please remember that whatever I found in this journey was generated in a collective effort of numerous people.[1]

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1 I cannot list here the countless encounters with colleagues and texts that contributed to the ideas to be presented in this paper, but I wish to mention the *Haifa Discourse Group*, whose role in this project has been central.

1. CONUNDRUMS THAT TRIGGERED THE TRANSFORMATION

The proper way to begin this "travelogue" is to mention those special events that got me started on the journey and then kept me going. I will present here just three out of the many formative occurrences that raised questions and made me think.

1.1. Why doesn't logic suffice to understand mathematics?

I was still in the middle school when, as I was reading Henri Poincaré's[2] seminal book *Science and method*, I came across a paragraph that gave me pause:

> *One ... fact must astonish us, or rather would astonish us if we were not too much accustomed to it: How does it happen that there are people who do not understand mathematics? If the science invokes only the rules of logic, those accepted by all well-formed minds, how does it happen that there are so many people who are entirely impervious to it?* [14, P. 47]

Poincare's words resonated with what I had been wondering about myself. My classmates seemed split into two groups: some students could clearly grasp mathematical ideas in no time, at a glance; others complained incessantly about their inability to make sense of what was going on in the classroom. The higher the grade, the sharper the split appeared. Those from the first camp, the fluent speakers of the language of mathematical symbols, wondered with Poincaré about the other students' imperviousness to the logic of this language; those from the group of nonspeakers could not understand how this language could ever be mastered.

I agreed with Poincaré about the puzzling nature of this difference and, like him, wondered how this split could be explained. Saying that mathematics, unlike other school subjects, is uniformly abstract did not satisfy me as an explanation. The word "abstract" has been offered as if it was clear what it meant, but was it? For most people, the term signals the intangibility of the mathematical universe, its being inaccessible to senses. But saying what abstract thing *is not* hardly solves Poincare's puzzle. Indeed, the question remains of why and how some people manage to get into this abstract universe despite of its intangibility; and what it is that keeps the rest of humanity behind its closed doors.

1.2. What is so complex about complex numbers?

The formative event to be presented now sharpened this latter question. It took place when I was already a graduate student in mathematics and served as a teaching assistant to a well-known mathematician specializing in mathematical logic. One day, I was briefing the professor about my recent classroom experiences: "The students could recite the definition of complex numbers, but they constantly complained about 'not understanding anything'

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2 The French thinker Henri Poincaré is known mainly as a mathematician, but he was a polymath who made important contributions also to theoretical physics, engineering, and philosophy of science.

and not being able to cope with the tasks I gave them". And, indeed, these students' minds seemed to be going blank even in the face of problems that would have yielded to just properly applied definition. The professor seemed puzzled. And then, suddenly, he said: "Well, this may be merely a matter of the teaching method. If I was their tutor, I would just discuss the definition and show that it is free from contradiction and consistent with the axioms of a number field. This, I am sure, would have opened their eyes."

I knew intuitively that this simple solution had little chance to work. Just as verbal instructions for juggling would not suffice to make a person able to juggle balls, clubs, and rings, repeating the definition of operations on complex numbers would also be insufficient to make the learners able to juggle a complex number. One mathematician whom I interviewed years ago told me that he could act with only those mathematical objects that appeared to him as having a clear "physiognomy" [17]. This metaphor brought back the issue of abstraction, but this time, it made me zero-in on the idea of a *mathematical object*: whereas it was clear how one develops an image of a person, how does one accord a distinct physiognomy to a new mathematical object, such as a complex number?

All this seemed to constitute at least a partial response to Poincare's question: Only those seem to be doing well in mathematics who have their ways to work out for themselves a good sense of mathematical objects. It is the ability to "see" these objects as they are being juggled by the teacher that allows one to make sense of the teacher's movements; and this is the inability to imagine them that turns these movements into incomprehensible. This was an important insight, and yet, it left me with new questions. Above all, I was now wondering about what mathematical object is, where it comes from, and how it can be turned into "one's own."

1.3. Why cannot children see as the same what grownups cannot see as different?

I was already a beginning researcher in mathematics education when an encounter with two four-year old girls put me and my colleagues on the path toward an all-new vision of mathematical universe. The search began when one of my Masters' student got interested in young children's numerical thinking, which she decided to investigate by watching her four-year old daughter Roni and Roni's 7-month older friend, Einat, performing some numerical comparisons. The girls were presented with pairs of boxes with marbles and then asked "In which box are there more marbles?" It soon became clear that the children could count properly. With a little prodding, they also managed, in most cases, to produce proper answers. And yet, even their successful solutions were accompanied by actions and utterances that we found strange and difficult to account for [21]. The greatest surprise came when the girls faced the pair of boxes with two marbles each. Upon seeing the two pairs of little balls, Roni smiled and said: "In none." Visibly pleased with the girl's answer, the interviewer closed the conversation: "There are more marbles in none of the boxes? Right." And yet, Roni's father, who watched the scene from behind the camera, was not yet fully satisfied. He asked for explanation, and the following conversation between him and his daughter took place:

1.	Father:	Why? Why do you say this?
2.	Roni:	Because there is [are] 2 in one, and in [this] one there is [are] another 2.
3.	Father:	So, this is why there is more in none of them? So, in both of them there is... what?
4.	Roni:	Two.
5.	Father:	And this is... more or less?
6.	Roni:	Less
7.	Father:	Less than what?
8.	Roni:	Than... than... than big numbers.
9.	Father:	Than big numbers? That means... If there is [are] 2 in one box and 2 also in the other, then what is there in the two boxes?
10.	Roni:	4.
11.	Father:	Aha. Together, there is [are] 4?
12.	Roni:	Yes.
13.	Father:	And in each box there is the sa...
14.	Roni:	Because it is between...
15.	Father:	I see. And there is the same [thing] in each box?
16.	Roni:	.....
17.	Father:	How many in each box?
18.	Roni:	2.
19.	Father:	Oh well...

At the first sight, what happened here, while quite amusing, could have been dismissed as too commonplace to merit a serious investigation: The little girl was unable to guess her father's intentions and did her best to satisfy his expectations by offering any guess she could muster. Anybody who has ever taught mathematics seems familiar with situations such as this. Yet, we were wondering about the futility of the father's multiple attempts to make his daughter use the expression "the same" (as, for instance, in "There is the same number of marbles in these two boxes"[3]). Why were they ineffective, in spite of their versatility? Why did even his "there is the sa..." (see turn 13 in the transcript), which left only one syllable to Roni's discretion,[4] fail to do the trick? And finally, why did his explicit formulation of the desired

3 The conversation was in Hebrew, where "the same" translates into an idiomatic expression "oto davar," verbally equivalent to "the same thing" ("the same" cannot be stated without being followed by "thing" or any other noun, such as "number"). Note, therefore, that to fulfill the father's expectation, Roni could use the generic "the same thing" rather than the more specific "the same number."

4 Father said "oto da... ", which had to be completed to "oto davar." This single syllable would have also completed Roni's answer because it would have produced a more or less full sentence.

answer (15) leave the girl visibly bewildered (16)? Our own bafflement was not any lesser: Why was this simple expression inaccessible to this obviously intelligent girl in this task, even though, as had already been repeatedly demonstrated, she was perfectly able to use it in other contexts?

After much deliberation, we concluded that our 4-year old participants could not think about any two objects to which the words "the same" could be applied. Evidently, Roni's father wanted this expression to be referred to *numbers*, or *amounts* of marbles in the two boxes. But these two italicized nouns, both of them used by the adult as signifiers of mathematical objects, were nothing of the kind in the eyes of the children. This event sharpened our interest in the nature and origins of mathematical objects. Whereas the previous story was about learners who have not yet developed a sense of a new mathematical object, this one was about students who did not even suspect the existence of such an object. The question now was how to bring this object to their awareness. If the query regarded concrete material objects, the response would have been clear. Objects such as those investigated in physics, biology, or astronomy are pretty straightforward and can be experienced by a person through his/her senses, either directly or indirectly, even before her being able to say anything about them. But the case of mathematical objects is quite different. Numbers, functions, and derivatives, unlike stones, stars, and living creatures, do not wait for the learner out there to be first detected, and investigated only later. So, how to even start talking about such an object?

Let me summarize. This last event, as well as the previous two, although brief and seemingly unremarkable, can be called formative: all three of them made us realize that to teach mathematics we can no longer ignore the question of the nature and origins of mathematical objects. We now needed to confront foundational queries head-on. After the iterative process of proposing tentative answers, which we would then critically examine, put to empirical tests and reject or modify, a far-reaching change in our vision of things eventually occurred. In the rest of this paper, I tell the story of transformations that led us to our current conceptualization. For reasons to be explained latter, we call this framework *discursive* or *commognitive*. The commognitive way of thinking has been working well for us for some time now. It made us able to formulate an answer to Poincare's query, to explain what the learners needed in order to reconcile themselves with complex numbers, and to account for the fact that four-year old children do not consider the expression "the same" as applicable within the context of numerical comparisons. These answers, while probably not the only possible, helped us make sense of what we saw and gave rise to pedagogical decisions that subsequently proved themselves in practice. We thus hold to the commognitive vision, at least for now, fully aware that it may be replaced one day with another, potentially more powerful way of thinking about mathematics, its objects, and its learning.

2. WHAT CHANGED ON THE WAY FROM MATHEMATICS TO MATHEMATICS EDUCATION

In this part, I explain what commognition is, while also telling the story of how this framework came into being. In the beginning, our thinking about mathematics was shaped exclusively by our own first-hand mathematical experience. It then evolved in a series of decisive steps, the first of which was the recognition of the very need to engage with the onto-epistemology of mathematics. Next came a series of small conceptual earthquakes, some of which have been presented above. One after another, these events effectively shook and transformed our foundational approach. I will now present each of these transformations in some detail.

2.1. Recognition of the need to elucidate onto-epistemological foundations

Although deliberations on the ontology and epistemology of mathematics have a long history, meta-mathematical questions usually fail to attract those who actually investigate numbers, functions, and abstract algebraic or geometric constructs. Preoccupied with the study of mathematical universe, they have little patience for conundrums labeled as "philosophical." This unwillingness to engage with foundational issues may be accounted for in a couple of ways.

In some cases, the lack of openness toward a serious conversation on foundational issues comes in a form of a quiet certainty about the mind-independent nature of mathematics. According to thinkers known as Platonists, mathematical objects, although inaccessible to our senses, are as much a part of the mind-independent reality as are stars, trees, and computers. Questioning the origins of mathematical universe would thus be an idle game. Since the times of the eponymous Plato, this view has been voiced over and over again, and most recently was reiterated by some of the most distinguished mathematicians of our times. Thus, for instance, the logician Kurt Gödel stated that "Mathematics describes a non-sensual reality, which exists independently both of the acts and [of] the dispositions of the human mind" [7, P. 311]. René Thom, the founder of catastrophe theory, sounded even more categorical when he stated that "mathematicians should have courage of their most profound convictions and thus affirm that mathematical forms indeed have an existence that is independent of the mind considering them" [22, P. 695].

Another reason that has been keeping mathematicians from engaging in serious foundational debates has been the view, shared by many, that onto-epistemological questions are irrevocably ill-defined and thus cannot lead to verifiable, useful answers. To save yourself embarrassment, it is better to remain silent on these issues, and thus agnosticism may be the safest option. This, indeed, is the spirit of Bertrand Russell's famous description of mathematics "as a subject in which we never know what we are talking about, nor whether what we are saying is true" [16, P. 84].

But this widespread disdain for foundational issues may also be explained in another way. If mathematicians may allow themselves the luxury of ignoring onto-epistemological infrastructure of their research, it is because no foundational resolutions seem necessary to investigate mathematical reality. Reuben Hersh and Philip Davis, two mathematicians turned philosophers of mathematics, speak explicitly about mathematicians' unwillingness to make a serious ontological commitment while stating, tongue in cheek, that "the typical working mathematician is a Platonist on weekdays and a formalist[5] on Sundays" [2, P. 321]. In short, theories on the nature and origins of mathematical universe seem as irrelevant to those who juggle mathematical objects as the theory of big-bang is to those who juggle balls, rings and clubs.

Well, some may doubt if it is really so. After all, the disbelief with which new mathematical objects have usually been greeted throughout history could usually be traced to uncertainties about the ontological status, and thus legitimacy, of these entities. On the face of it, this kind of problem should have prodded foundational reflections. Historical facts, however, undermine this claim. As explained by the British logician and historian of mathematics, Philip Jourdain, whenever "logically-minded men" objected to such "absurd" notions as a negative number and imaginary numbers, the struggle for the recognition was eventually settled not by rational argument but simply by mathematicians' stubborn application of the problematic entity and their eventual "getting used" to its presence. To put it in Jourdain's own words, "mathematicians simply ignored [the objectors] and said 'Go on; faith will come to you' ... So [the new objects] were used with faith that ... was justified much later" [10, PP. 29–30].

These days, the mathematicians' indifference toward the question of the origins and nature of mathematical objects spreads to education, and the foundational issues remain an elephant also in mathematics classroom. As long as I was involved in mathematical research myself, I was accepting this situation uncritically. My position changed, however, when I started introducing others to the world of mathematics. As explained above, I soon realized that without coming to grips with the sticky foundational questions I would not be able to address properly any of the conundrums I encountered while teaching. Taking exception with the agnostic attitude was the necessary first step on my way toward the kind of understandings that are indispensable for well-reasoned pedagogical decisions. Upon this realization, my colleagues and I began talking about things that, so far, went without saying. In the rest of this section, I present the insights gradually gained on these occasions, especially those of them that withstood empirical tests and have been deemed helpful enough to be retained as a part of our theory of learning mathematics.

2.2. Mathematical object as a *mode de parler* rather than a part of mind-independent reality

The story of our journey toward the commognitive conceptualization of mathematics will now be told as a series of three transformations that resulted from our foundational

5 Formalism, yet another school in the philosophy of mathematics, has been embraced, among others, by Gottlob Frege and David Hilbert. According to formalists, mathematics is, basically, a symbolic game – the art of manipulating "empty" symbols according to well-defined rules.

deliberations. The first of these changes was due to the doubt about the signifier–signified dichotomy. We decided that rather than treating mathematical objects as self-sustained entities, ontologically different from the discursive constructs used to "describe" them, it might be more useful to see them as mere fictitious interpretation of certain communicational forms.

What are mathematical objects? The Platonic stance implies that mathematical objects are entities in their own right, not to be confused with mathematical words, symbols, diagrams, and graphs, all of which play an only the auxiliary role of these objects' "representations" – the mere communicational means. Or, as stated by the French mathematician Alain Connes, "Conceptual tools [signs, representations] aren't to be confused with the mathematical reality itself" [1, P. 182].

You do not need to be the declared Platonist, however, to live in the world of this signifier–signified dichotomy. The idea that words and symbols are mere avatars of the "real things" is entrenched in the way we speak. For instance, we make statements such as

The symbols 13*, XIII, and* $5 + 6$ *represent the same number*.

The expression x^2 *and the basic parabola represent the same function*.

The word *represent* appearing in both these sentences implies that there are two categories of things, one of which comprises the entities that constitute the proper object of mathematical conversation (in this case, these are the number called "thirteen" and the function called "quadratic", respectively), and the other one composed of signifiers – the communicational counterparts of the former (in this case, these are the symbols 13, XIII, $5 + 6$ and x^2 and the words "number" or "function"[6]). The message about the independent existence of numbers or functions is implied by the fact that, as indicated by these last two utterances, a single mathematical object can have many sharply differing representations.

Being inscribed in the expressions we use, and thus in the ways we think, the signifier–signified dualism is difficult to argue with. It is unlikely to become an explicit topic of conversation in the first place. If the issue ever caught my attention, it was because of questions I began asking myself when, as a novice teacher, I was charged with the task of ushering other people to the world of mathematics. Before I could start introducing my students to the concept of negative number, for instance, I had to resolve the problem: How to talk with the class about entities that cannot be shown, while also claiming that these entities constitute products of operations that the young learners considered so far as "impossible"? The textbooks I was using suggested extending the number line to the left of the zero with the help of a symmetric half-line, whose integer points would now be given the names $-1, -2, -3, \ldots$ I was skeptical. Will the students believe me when I try to convince them that calling a point on a line with a new name suffices to conjure an all-new mathematical object? Will I be persuasive while claiming that by this simple act of baptism I had brought into being something

6 Here, I the quotation marks in the expressions "number" and "function" signal that I am speaking about *the words*, not about what is signified by these words.

that these young people had always considered as nonexistent and even "not allowed"? And if I put the new symbol -3 to the right of the equality sign in the expression "$5 - 8 = \dots$," saying "Now the operation $5 - 8$ can be performed and it gives a result," wouldn't they protest, asking what had been added in this act of arbitrary signification? While wondering about what is the point of all this, they will surely question our human power to conjure something out of nothing. Years later, when I got acquainted with a bunch of classroom studies on children learning about negative numbers, and especially when I also co-conducted one such study myself [18], I found out that all these fears were definitely justified.

The decisive turnaround in my implicitly Platonist vision of mathematics began taking place as a result of my encounter with the 4-year old Roni and Eynat. As I was deliberating on the children's inability to use the expression "the same" in the context of boxes with marbles, I realized: when I introduce a new mathematical object, such as number, to the conversation about what I see – in this case about boxes with marbles, – I add nothing. Rather, I am just changing the way I speak. Of course, there are reasons for this shift in my discourse, and in the longer run, this transformation is going to prove itself very useful. But the change in the way I talk is all that "introduction of a new object" may mean at this point to the uninitiated – to those who, like Roni and Eynat, are hearing the term "number" within this context for the first time. This change in the form of speech is bound to confuse a young person who cannot yet appreciate the prospective benefits of this move. This bafflement will be experienced not just during the introduction of negative numbers, but also when other types of numbers – fractional, irrational, "imaginary" or even the most basic one, the natural – enter the scene for the first time.

To explain, let me engage you in a thought exercise. Please, take a look at the four pictures in Figure 1. Although these images are very different from one another, we may still claim that they present the same person. What is it that justifies this last statement? The answer seems simple: the claim is true because a single person, Sigmunt Freud, served as the model for all of four them. If the four pictures seem dissimilar, this is because they were drawn at different times in his life.

FIGURE 1
What makes us say that these four pictures "present the same person"?.

FIGURE 2
What is it that is "the same" in these six pictures?.

Now, consider the six pictures in Figure 2. Here, too, we may speak in terms of a single thing represented in different ways: all these pictures represent the *number five*. But where is this common element, number five? The truth is that the only feature shared by the six figures is that whenever we count their elements, we end with the word "five." Thus, what makes these figures into "the same" is the five-word long *process of counting*, and not any common *object*, as was the case with Freud's pictures. Yes, only a shared procedure may become the basis for claiming a "sameness" of dissimilar figures. If this fact escapes our attention, it is because also in the case such as that in Figure 2, we use the form of speech that was applied, so far, only for stating the presence of a common object. Saming through common procedure rather than a common object brings results because of which the French mathematician Henri Poincaré defined mathematics as "the art of giving the same name to different things" (quoted in [24, P. 154]).

All this makes us aware of the fact that we are using number-word only *as if* they were names of some independently existing objects, in a metaphorical way. But metaphors have their entailments, and in this case, one of the metaphorical entailments is that such object as "number five" is "represented" in all these very different images here the way Freud was represented in the four photos. Now it became obvious why young children must be able to count long before they can speak about numbers as anything else than the sounds used in counting. Indeed, counting is probably where the very idea of the abstract object called "number" has its roots. Following this insight, we decided to investigate the processes of objectifying the operation of counting, with the term *objectification* to be understood as *a discursive transformation that makes us use mathematical words and symbols as if they signified discourse-independent objects*.

We soon realized that the change in the way of talking called objectification is a combination of two lexico-grammatical transformations. First, there is *nominalization* – the act of replacing lengthy portions of text with a single noun. This is what you do, for instance, when you replace the talk about counting with number-words used as nouns. This is also what happens when you transit from the proposition

(A) *If I extract a square root from x and raise the result to the third power, I get the same result as when I raise x to the 3rd power and extract square root from it.*

to the equivalent objectified sentence

(B) *The third power of square root equals square root of the third power.*

(Note that both propositions can be expressed symbolically as $\sqrt{x}^3 = \sqrt{x^3}$). The verb clauses from (A), "I extract square root from . . . " and "I raise ... to the third power" have been replaced in (B) by the noun phrases "square root of . . . " and "third power of . . . ," respectively.

The second component of objectification is *alienation*, that is, the removal of the human subject. Thus, in the example just given, the grammatical subject of (A) is "I," which implies that it is a human being who performs the operation given by the subsequent verb phrase "extract square root." In (B), it is the noun phrase "The third power of square root" that plays the role of grammatical subject. In result, (B) sounds as if it was speaking about a self-sustained entity that does its own thing, without an involvement of any human agent. Only when we adopt this impersonal form of speech, we also begin saying that the nouns or symbols "represent" the object.

It soon became clear to us that objectification is a common phenomenon, to be found almost everywhere, not just in mathematics. You build on the metaphor of object also when you use words such as "velocity," "energy," "identity," "class," "justice" or human "ego." And while the subsequent research taught us that the transition to this objectified form of talk is never straightforward or easy, it also made us aware of the reasons why so many people, in so many domains, are prepared to invest the necessary effort.

Why do we need MOs? So, why do we objectify, in the first place? What do we gain when making transition from talking about actions and operations to talking about objects? The theoretical and empirical scrutiny of what happens in this transition brought to our attention two beneficial consequences of objectification: first, it improves the effectiveness of communication by allowing us to say more with less; second, it widens the range of things we can do, and in particular, of practical tasks we can perform.

To make my first point, let me, once again, compare propositions (A) and (B), the first of them expressed as a story of a series of actions (extracting square root and raising a number to the third power), and the other as a description of properties of mathematical objects (of the square root, of the third power). One difference between the two is readily visible: the objectified statement (B) is much shorted, more concise, than its unobjectified equivalent, (A). Thus, this example clearly corroborates my first claim: objectification allowed us to express ourselves more briefly, whatever it was we wished to say.

To illustrate the compressing power of objectification in an even more dramatic way, I will engage you in the following *thought exercise*:

> *Suppose you cannot use number words "one," "two," "three,"... except in counting. How would you then present in words the general truths expressed in this equality:* $3 + 4 = 7$*?*

Let me explain: in your response, you are allowed to use the number words, but only as "empty" signifiers, that is, as just strings of letters or of phonemes. Thus, you can say: "I counted the marbles in this box and got 'five' as the last number word", but you cannot say "There are five marbles in this box." I suggest that you give some thought to possible answers before you read my own response below.

And here is my answer. Not allowed to say things like "There are four marbles in the box" or "4 plus 3 equals 7," I would translate the symbolic equality $3 + 4 = 7$ into the following statement:

> *If I have a set so that whenever I count its elements I stop at the word "three,"*
> *and I have yet another set such that whenever I count its elements I stop at the word "four"*
> *and if I put these two sets together,*
> *then,*
> *if I count the elements of the new set, I will always stop at the word "seven."*

This is a very long sentence. Without condensing it and similar ones into objectified expressions such as "$3 + 4 = 7$," or even just, in words, *three plus four equals seven*, how would we be able to develop mathematics at large, and its numerical algorithms in particular? This example shows with particular force how the discursive device called objectification impacts the efficiency of mathematical communication by compressing lengthy expressions into very short ones.

And now, let me substantiate the second claim, according to which objectification extends the range of things we can do. I will help myself with an example that may appear so familiar, commonplace, and simple that you may wonder why I even chose to deal with it. But this is exactly the point. The analysis of this seemingly trivial event will let you see things, of the existence of which you might have been always aware, but which you never scrutinized to see how and why they work. What we notice here can be extrapolated to even most complex cases.

The example is taken from one of our empirical studies, in which we observed young people performing tasks related to numbers. Consider the following conversation between the interviewer and the 18-year-old girl by the name Mira, who was asked to pay for an imaginary purchase with real coins[7] that have been given to her beforehand:

1.	Interviewer:	You bought 3 cookies from me; each one costs 75 agoras. Now you have to pay me.
2.	Mira:	Three times 75 . . .
3.		150 plus 3 times 25. . . 75. . .
4.		150 plus 75. . . 225.
5.		Here you are: 2 shekels and 25 agoras [*while saying this, Mira passes to the interviewer two coins of 1 shekel, two of 10 agoras, and one of 5 agoras*].

Let us take a close look at what Mira did. While saying "Three times 75" (utterance 2), she translated the required operation on coins into the numerical operation, multiplication of number 75 by 3. She did it by mapping the concrete objects (specific coins) onto mathematical objects (corresponding numbers) and by matching physical operations on the former with arithmetical operations on the latter. Then, in steps (3), (4), and (5), Mira implemented the operations on the mathematical objects, obtaining the number 225.[8] It is only then that she returned to the coins and composed the actual payment. Thus, the conversation that began as one about concrete objects (cookies and coins) has become one about mathematical objects (numbers), and then went back to concrete objects (coins). To sum up, the monetary transaction was a brief drama in three acts, with the middle one, the act of *planning* the action of paying, resulting in the *mediating story* about numbers, "three times 75 equals 225".

It is noteworthy that in a simple case such as that presented above, the task could have been performed also in an unmediated way. Such unmediated action is exemplified in another episode from our study:

1.	Interviewer:	Now you have to pay me. You bought 3 cookies from me; each one costs 75 agoras. Please, pay me.
2.	Talli:	Each one is 75 agoras. . . [*while saying this, hands a coin of 50 agoras (1/2 shekel), two of 10 agoras, and one of 5 agoras to the interviewer*].
3.	Interviewer:	What did you give me?
4.	Talli:	75.
5.	Interviewer:	Yes, you mean half and?
6.	Talli:	20 agoras and 5. Ok. And a shekel [*passes a coin of 1 shekel*]. One shekel and 75. Inside the shekel there is a 75, so there is 25 more. So, here is half a shekel more [*passes the coin of 50 agoras*]. And that's it.

7 The coins are in *shekels* and *agoras*, Israeli monetary units corresponding to dollars and cents, or pounds and pennies. Note that in the last sentence of the conversation (see line 5), the number names 2 and 25 are but labels for coins: the coin of one shekel and the set of coins including 2 coins of 10 agoras and one of five, respectively.

8 She took 50 out of the three 75s and added them together (3), and then, in (4) she first multiplied by 3 the remaining 25s and then added the products, 75, to the 150 obtained in (3).

Here, the required payment was performed directly on coins: Talli simply passed three sets of 75 agoras one by one. No mediating story has been told here and the payer ended up without necessarily knowing the total price of the purchase.

Considering this last example, the question may be asked why we should ever bother about mediating actions involving mathematical objects. Well, whereas this kind of action may appear just optional in simple tasks with which one is closely familiar, other tasks may be unfeasible without it. When the payment is made in the direct, immediate way, one relies on her memory of specific sets of coins that compose different basic values, such as that of 75 agoras. Sometimes, one's repertoire of memorized sums may not suffice to compose the required payment. Even more importantly, unmediated way of acting is applicable only in familiar situations, in which the performer can be guided by her previous experience. In contrast, mediating story used skillfully in one situation, may be appropriate also for a less familiar situation, involving concrete objects of a different kind. Thanks to their universality, therefore, mathematical objects make a person able to act in situations that are new to them, that is, involve objects – concrete or abstract – upon which she has never operated before. Indeed, mediating mechanisms of the kind of those exemplified here are at work even when you perform most complex and sophisticated practical tasks, such as building bridges or computers, flying to the moon, or designing vaccine for corona. One story about a single mathematical object allows us to deal with multiple situation that, so far, have not been considered as having anything in common. To sum up, mathematical objects are powerful tools, which not only make communication effective, but also allow us to deal with ever-new situations and to engaging in ever more complex forms of activity.

How are MOs discursively constructed? The interesting feature of these tools, and more specifically of mathematical objects, is that rather than being applied readymade, they are being constructed as we go. To put it differently, we conjure mathematical object by talking about them. This may sound as paradoxical as saying that a hammer is being put together during, and thanks to, the process of hammering. Yet, this is how it is. I will now take a closer look at the way in which the on-the-run object constructions take place.

After defining objectification as a discursive transformation that makes us use mathematical words and symbols as if they signified discourse-independent objects, I pointed out to two discursive operations that produce the objectifying effect: nominalization and alienation. Alienation has been briefly explained above, and I will now focus on nominalization, the process of replacing portions of text with a noun. Let us take a look at the different ways in which nominalization can be attained.

One of these ways has already been exemplified: I have shown how processes of counting turn into mathematical objects called numbers. Brief utterances with words such as *two* or *five* used as nouns may now replace long statements about human actions, such as "when I count the sides of pentagon, I arrive at the word 'five'." This move of replacing stories of processes with stories of objects is called *reification*. Reifying is also what I do when instead of speaking about my own action of multiplying, as in the narrative "When I multiply odd number by itself, I get odd number," I tell a story of an object: "The square of odd number is odd." And it is what I did above in transition from the proposition (A) to (B), when I disposed of *verb* phrases "extract square root" and "raise to the third power" appearing in proposition (A) and replaced them in (B) with *noun* phrases, "square root" and "third power." It should be stressed that reification is not restricted to mathematics. We apply it everywhere, even in everyday talk. I reify, for instance, when I replace the story employing the verb "move," as in "The antelope moves fast" with the one that uses the noun "movement," as in "The antelope's movement is fast."

Another nominalizing operation that may lead to the emergence of a new object takes place when we endow several different objects with the same name. As such, it may be called *saming*. Saming is what we do when we refer to things as different as, say, dog and cat with the same word, "domestic animal." We are saming in mathematics when we refer to both the expression x^2 and the curve known as parabola with the same name, "the basic quadratic function."

Finally, there is the operation of *encapsulating*, of replacing the plural form with the singular. This is what we do when instead of saying "The post-office workers *are* efficient," we declare "The post-office staff *is* efficient." Here, the word "staff," in singular, encapsulates "the workers," in plural. And in mathematics, we are encapsulating when, for instance, we replace the claim "The cubes of numbers *are* increasing" with "The function x^3 *is* increasing."

The following example, featuring the object called "the basic quadratic function," shows how these three operations, saming, reifying, and encapsulating, can be iteratively combined in the process of constructing a mathematical object. It is reasonable to conjecture that the idea of the quadratic function emerged when people realized that some stories about x^2 may be translated into narratives about the curve called "parabola" and also into those about a certain table – the one displaying a set of ordered number pairs, in each of which the second element is the square of the first. For instance, the claim that zero is the smallest possible value of x^2 can be translated into the story of the smallest second element of the pair and into one on the lowest point of the parabola.

The benefit of replacing all three signifiers, the algebraic expression, the parabola, and the table with the single term "basic quadratic function" is immediately obvious: this replacement allows us to make all these statements simultaneously, in the single sentence: "Zero is the smallest value of the basic quadratic function." Here, we used the new noun "function" to perform saming of the three original signifiers. Clearly, such saming makes our propositions incomparably more general, and thus more powerful, and it adds to the thriftiness of mathematical communication.

What we call "basic quadratic function" became a combination of three signifiers, which from now will be called *realizations* of the signifier "basic quadratic function." But the process of realizing signifiers with the help of other signifiers is recursive, and the three realizations of the basic quadratic function may themselves be realized by other signifiers. Thus, x^2 may be realized as a square of any specific number. It is obtained from these specific squares by saming. These square numbers, in turn, are reifications of the operation of multiplying numbers by themselves. Similarly, both the table and the parabola can be realized as

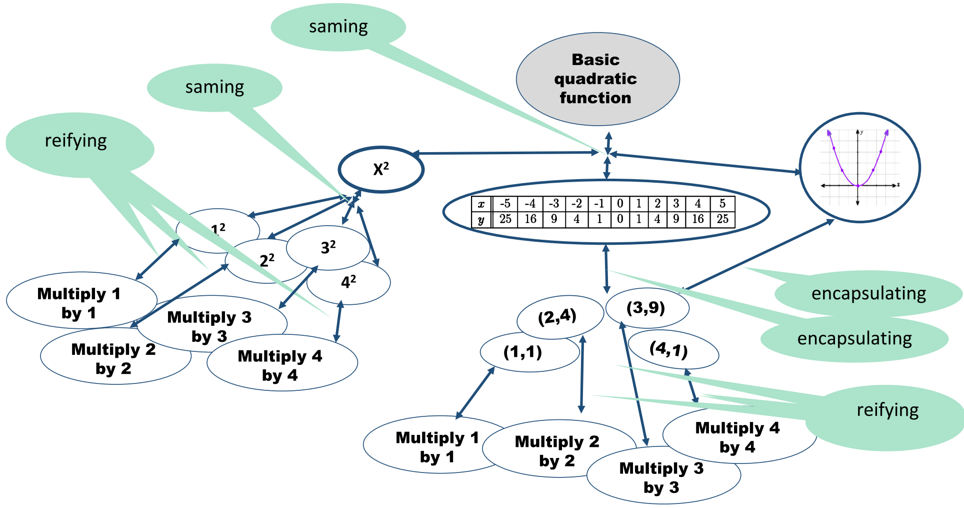

FIGURE 3

The realization tree of the signifier "Basic quadratic function" (adapted from [20]).

the set of ordered pairs of numbers and their squares, with encapsulation as the corresponding transformation. As before, these squares are reifications of the operation of squaring. The resulting diagram in Figure 3 presents the object called "basic quadratic function," composed of its name (the signifier) and its realization network (the signified).

Let me now summarize some of the insights, so far, about mathematical objects. First, the last statement of the previous paragraph can be generalized, and the *mathematical object* can now be defined as a *signifier*, such as word, written symbol or icon, together with its *realization network*. Here, the phrase "R is a *realization* of signifier S" is to be understood as saying that for a set of true proposition about S, there is an isomorphic set of true proposition about R (to avoid getting into technicalities, I will skip the definition of the relation of *isomorphism between sets of propositions*, but I hope the term is self-explanatory). Thus, the expression "x^2" is a realization of the signifier "basic quadratic function" and 3^2 is a realization of x^2.

Second, the set of all objects can be split into a pair of categories, and this can be done in two different ways. First, there is the distinction between *primary* and *discursive* objects – between those that exist in the world, independently from the human mind, and those that exist also, or only, in discourse. Second, there is the *concrete–abstract* dichotomy. These distinctions may be explained with the help of Figure 4, which shows in a schematic way how generations of new objects are being built, one after another, from those that precede them. The chain begins with concrete material objects, that is, those material things whose existence does not depend on whether somebody thinks or talks about them. These are the objects that are called *primary*. When a primary object is given a name or denoted with a symbol, we can start communicating about it. In this way, an atomic or elementary

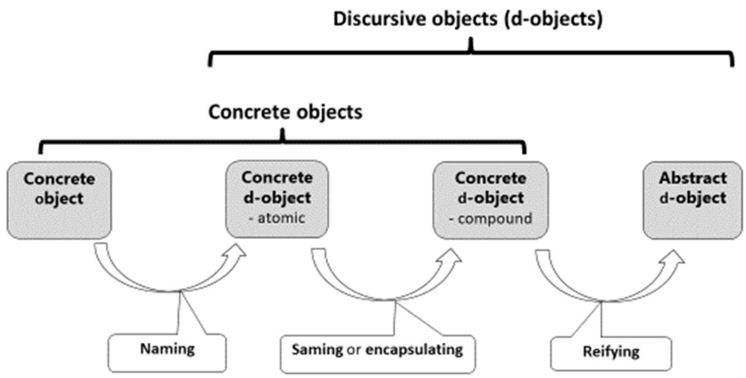

FIGURE 4
Types of objects (adapted from [20]).

discursive object, or *atomic d-object* for short, is created. In the third stage, some atomic d-objects are combined by saming or encapsulating into *compound discursive objects*. Finally, or perhaps in parallel to the stages of saming and encapsulating, additional compound d-objects are obtained by reifying processes that involve previously constructed objects. It is this last category that consists of objects called *abstract*. The other three, reification-free categories, contain objects considered as *concrete*.

Several conclusions follow from what was said so far. First, according to the above definitions, all *mathematical objects are abstract* because their construction involves reification, the operation that appears to be the act of adding a whole new entity but, in fact, introduces just a new figure of speech. Second, there is no ontological distinction anymore between signifier and signified. Any of the material means used for communicating – written or spoken words, visual devises, or touchable things – may serve as signifiers, and these are also the materials of which the signified, this dynamically expanding, never complete network of realizations is made. This means that all objects, whether primary or discursive, whether concrete or abstract, are basically material and accessible to senses, and the only difference between concrete and abstract entities is that in this latter case, the signified may be unbounded: it is always ready to accommodate new elements and is never perceivable in its entirety. Finally, discursive objects are personal constructs that develop gradually as a person learns mathematics. In this process, the realization network of signifiers such as "the basic quadratic function" or "rational number" is constantly expanding, sometimes deviating from the canonic version, accepted by the community of mathematicians.

The main idea to be taken from all that has been said so far is that mathematics is an *autopoietic* communicational system that creates its own objects while telling stories about them. Recognizing this inherently paradoxical nature of object construction is critical to our understanding of how mathematics is learned. Before I turn to new insights about learning that came to us with this recognition, let me add a few words about how this new vision of mathematical objects revolutionized our ideas about mathematics as an activity.

2.3. Mathematics: the activity of telling useful stories about reality rather than a search for the universal truth about it

It seems that the discursive nature of at least some mathematical objects has been already intimated by one of the greatest mathematicians of all times, Johann Carl Friedrich Gauss, who famously stated that "Infinity is merely a *façon de parler"* ([6, P. 216], quoted in [12, P. 337]). From referring to a particular mathematical object, this claim has now been extended to all of the abstract entities. An inescapable conclusion of this nondualist vision is that *mathematics is a form of communication, or discourse, that we adopt when constructing mathematical objects and telling potentially useful stories about them*. I will now unpack this assertion explaining what is meant here by the terms "story" and "useful."

Within this present context, the use of the colloquial word "story" may make some of you feel uneasy. I claim, however, that the word is in place in describing not only mathematics, but also all other domains of research, such as physics, biology, or history. Thus, for instance, a story about living organisms, such as "Plants convert light energy into chemical energy in the process of photosynthesis," is a typical output of research in biology, whereas the formula "$S = 1/2gt^2$" is among stories about bodies in motion told by physicists. Yes, also this last string of symbols, as unlikely as it may seem at the first glance as an example of a story, does turn into a narrative once we decode it and write it in words rather than symbols: "The distance S traveled by a free falling object is equal to half of the gravitational acceleration, g, multiplied by the square of the time of the travel, t." Similarly, mathematical equality $(x^2)' = 2x$ can be seen as a narrative about a function and its derivative. Of course, the three propositions I brought here as examples of scientific or mathematical stories present these stories in a highly condensed form. For elaboration, one needs to consult academic literature.

Let me complete my explanations by clarifying how the term "story" is to be understood in the present context. From now on, I will be using the expression *story about X*, where X is a noun, as referring to *a coherent sequence of utterances* (*propositions*) *that, when taken together, can be said to be "about X"* (or "on X" or "of X"). The "aboutness" means that X is the grammatical object or subject of some of the utterances in the sequence, and the sequence in its entirety is consistent and cohesive. The term "consistent" says that the sequence does not logically imply both a proposition and its negation. This term "cohesive" indicates the presence of lexico-grammatical links that hold the sequence together, that is, connect its successive utterances thematically. The connection may be chronological, as is the case when the successive utterances are linked with words such as "before," "after," or "next"; it can be logical, attained by the use of connectors such as "therefore," "it follows," "and," "or"; and it can be causal, expressing itself in the presence of words such as "because." Most current uses of the term "story" or "narrative" imply chronological interconnection of the different parts, and thus our present definition leads to a wider than the common application of the term.

The last question that needs to be answered in the attempt to complete the definition of mathematics as "the activity of telling potentially useful stories about mathematical objects" regards the term "useful." What does this adjective mean and why should it be preferred to "true," which is mathematicians' favorite? In the preceding section, I have already stressed that mathematical objects are useful in their roles of "compressors" of mathematical prose and of action-mediating devices. With their help, we are able to perform tasks that would not be workable otherwise. For the operations on mathematical objects to be truly helpful, we have to draw on what we have learned about their properties. In other words, stories about mathematical objects are those that guide our decisions about how to use these entities in problem-solving and in practical action.

Some of you may shrug at my mention of usefulness as the required feature of mathematical stories. Some mathematicians may share D. H. Hardy's conviction that their activities have nothing to do with "anything useful" [8, P. 150]. Yet, the majority of mathematicians seem to be of one mind with Andrew Forsyth [4, P. 35], who famously claimed that almost any mathematical story would eventually turn useful beyond mathematics itself, provided we have the patience to wait for a "real-life" problem that can be solved with its help. The message about potential practical usefulness of even most abstract mathematical ideas can also be heard in Alfred North Whitehead's disclaimer: "It is no paradox to say that in our most theoretical moods we may be nearest to our most practical applications" [25, P. 100].

Of course, not all stories come equal and not all of them can serve as reliable mediators of practical actions. Only those mathematical narratives are endorsed as reliable and potentially useful that have been constructed and shown to be endorsable with the help of well-defined communicational tools, that is, within a special discourse. This latter word, *discourse*, may be defined as referring to a communicational game that determines a community. Its game-like nature expresses itself in its being rules-regulated activity, similar in this respect to, say, the game of chess. It determines a community in that, like chess, it splits the humanity into those who are able to participate in this activity and those who are not (of course, the split is never clear-cut, but the idea of the "community of discourse" is useful nevertheless). It is important to remember that discourse may be in words, but more often than not, it is multimodal. Sometimes, mathematical conversation may take place just in sounds other than words, in body movement, gestures, facial expressions, pictures – any of these or all of them together. Mathematical discourse, as any other, can be practiced with partners or with oneself. In this latter case, the discursive activity it is called "thinking."

Different discourses are created for different types of mathematical objects, and they differ among them along four dimensions. The first and most obvious of the distinctive features is the set of *keywords* pertaining to the discourse's characteristic objects, such as the words "number," "one," "eleven," "sum," "product" in arithmetic, "figure" and "triangle" in geometry, and "function" in mathematical analysis. Most of these keywords come with explicit rules for use, known as definitions. Second, there are the characteristic *visual mediators*, that is, visual means with which one makes clear what it is she or he is talking about. Thus, in mathematical analysis we use algebraic expressions and curves known as graphs, and in geometric discourse we help ourselves with drawings of different shapes. Third, each of mathematical discourses has a well-defined set of communicational *routines*, the patterned, recurrent ways of doing things. Some of these routines are common to all mathematical discourses, whereas some others are discourse-specific. Among them, there may be routines for reading mathematical notations and for operating on symbols, those to be applied in constructing stories about mathematical objects, and some others, to be performed in testing stories already created or in showing whether they can be endorsed. The routine used in this latter task is known as "proof." Finally, the discourse on X comes with a small set of endorsed narratives on X, known as axioms, on the basis of which other endorsed narratives on X will gradually be constructed. Together, all these endorsed narratives will constitute the *theory of X*. In natural sciences, a collection of narratives, to count as a theory, must be unambiguous, consistent with experience, general rather than specific, and this is only the beginning of the long list of requirements. In mathematics, on the other hand, at least in principle, consistency and cohesiveness are all that is necessary to ensure that a story be seen as a part of theory. Mathematicians strive to make their theories as complete as possible, hoping that for every proposition about X, either this sentence or its negation will turn out to be a part of the theory of X.

Viewing research, at large, and mathematics in particular, as communicational activities has an implication that goes against one widespread belief about mathematics, engender by its Platonic version: it is now clear that many seemingly competing theories, not just one, may be developed about the same X.[9] The phenomenon is well known from science – think, for instance, about Aristotelian, Newtonian, and Einsteinian theories of motion. To see that it occurs also in mathematics, one may consider the Euclidean and non-Euclidean geometries, each of which tells its own story of the construct called "space."[10] The different stories may sometimes appear to be contradicting each other, as is the case for the Euclidean, Bolyai–Lobachevskian (hyperbolic), and Riemannian (spherical) narratives about the sum of angles in a triangle. Here, the apparent contradiction stems from the fact that, in each of the discourses, the use of the basic keywords is defined with the help of a slightly different set of axioms. Some other examples that could be given here are much less obvious, simply because mathematicians agreed to opt for just one version that became canonic, with the others forgotten. This is what happened when integers were extended to rational numbers, and then when unsigned numbers were broadened to signed, or from real to complex. Within the nondualist approach to mathematics, therefore, unlike in the world of Platonic ideas, *the decision to label a narrative as "true" becomes relative to the discourse in which this narrative is told*. It is for this reason that the adjective "useful" may be a more appropriate descriptor for the basic criterion for endorsability than is the word "true" which, whether we want it or not, brings the connotation of universality. To forestall possible protests, let me immediately add that what has been said in this paragraph does not imply that mathematical

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9 Keep in mind that X is a *noun* that points us to a certain phenomenon, rather than the phenomenon as such. The different discourses on X are likely to use this noun differently, and this entails differing narratives about X.

10 The fact that these three theories can be subsumed under a common metadiscourse may give rise to the assertion that they are parts of a single higher-level theory; this, however, does not contradict the claim that when taken separately, they constitute different theories of the same X, and that these different theories pertain to, or are useful for, different interpretations of the X.

"truth" (or endorsability) is arbitrary. Whereas we are free to opt for any properly constructed mathematical discourse,[11] once we make our choice, we lose our freedom to decide what can count as true. Within the boundaries of the chosen discourse, the veracity of narratives we are going to create will be uniquely determined by the rules and routines of this discourse.

Before concluding this brief introduction to commognition, it is important to stress that this approach, and more generally, our conversion from covert Platonists to overt nondualists did not come out of nowhere. It was inspired by many recent developments in several seemingly unrelated domains, with philosophy of science and learning sciences among them. On the one hand, we followed in the footsteps of leading thinkers of the 20th century who turned to communication as the key to understanding human uniqueness. The word "knowledge," signifying one of the hallmarks of humanity, has been interpreted by Rorty as referring to the "conversation of mankind" **[15, P. 389]**. In a similar vein, Foucault claimed that discourses are "things said... those familiar yet enigmatic groups of statements that are known as medicine, political economy, and biology" (see the blurb on the cover of **[5]**; mathematics can now be added to this list). This nondualist position with regard to knowledge, as observed at the level of humanity as a whole, paralleled the work of psychologists whose observations on individual human beings and on their cognitive activities was inspired by the ideas of the Austrian–British philosopher Ludwig Wittgenstein and of the Russian thinker Lev Vygotsky. In tune with Vygotsky's claims on the inseparability of word and its meaning, the writers who called themselves "discursive psychologists" started questioning the ontological split between thinking and communication **[3, 9]**. We have been encouraged by all these thinkers when we decided to view mathematical thinking as a self-dialogue involving the discourse known as mathematics. The unity of these hitherto separate ontological categories, cognition and communicating, is reflected in the portmanteau *commognition* **[19]**.

3. HOW COMMOGNITIVE INSIGHTS ABOUT LEARNING HELPED TO SOLVE THE INITIAL CONUNDRUMS

Having introduced the nondualist way of thinking about mathematics and its object, I now have to convince you that the result was worth the effort. More specifically, I need to show that commognition is a powerful tool for making sense of what people do in their encounters with mathematics, and that it is more successful in this role than any dualist approach so far. I will do this by showing how the discursive conceptualization of mathematics helps us resolve the three conundrums that initiated us on our way toward commognition. I will now attend these conundrums in the order reverse to that in which they are presented above. On my way, I will discuss some of the more general changes brought by commognition to our understanding of what people do when they learn mathematics, what obstacles

11 We choose discourse according to the criterion of prospective usefulness, as it is measured by either its practical applications or by its power to generate a rich mathematical theory or, preferably, according to both these considerations.

they need to tackle on their way, and what may help or obstruct their efforts to overcome the hurdles.

3.1. Seeing as the same what so far appeared as different: the paradoxical conditions for objectification

Just to remind, the heroin of this formative event was the 4-year old Roni who, when faced with two boxes with a pair of marbles each, was unable to say what her father desperately wanted to hear: that there was "the same" number of marbles in the two boxes. This was puzzling because while opening each box, Roni could be heard saying the word "two" and then claiming that there is more "in none." Already when introducing this conundrum, I have raised an explanatory conjecture: for the 4-year old, there was nothing in the two boxes that could be called "the same." Now I can say that at this point, the young child evidently did not yet create for herself any abstract objects, mathematical or otherwise, that could be seen as being present in both boxes with two marbles and described as "the same."

This brief story gives rise to a much more general, and some may say quite unorthodox conclusion about sources of numerical thinking. According to cognitivist theories, produced in the mainstream psychological research, this kind of thinking is an inborn property of humans, with the first signs of "number sense" detectable already in newborns. Commognitive researchers do agree that some special human abilities, rarely found in other species, are necessary to make numerical thinking possible. As a good example, let me mention one ability that may well appear already at birth – the ability to distinguish between small sets of different cardinalities. Yet, once mathematical thinking is conceptualized as a *discursive* activity, the mere recognition of quantitative difference does not yet count as a case of mathematical thinking. According to commognition, mathematical thinking, *by definition*, does not exist before the child developed some uniquely human communicational skills. Note that this disagreement between the dualist and non-dualist visions of mathematical thinking is not just a matter of semantics. Indeed, the difference of opinion on the ontology of numbers has far-reaching consequences for our understanding of how this thinking emerges and how it develops later. Eventually, it is bound to affect our ideas about the ways in which children may be helped – or hindered – on their way toward numeracy.

To give just one example, let me consider yet another conundrum, one that has been challenging cognitive psychologists ever since the seminal studies by Jean Piaget. To put it in their own words, these psychologists have been puzzling over the fact that "children who know how to count may not use counting to compare sets with respect to number" [13, P. 35]. In this sentence, the authors summarized the phenomenon that has been observed time and time again: When presented with two sets of, say, marbles and asked "In which of them are there more marbles?", 4- or 5-year old children would not count even if they could. This, indeed, may seem puzzling to a person who considers numbers as self-sustained things which, like spoons or bicycles, can be experienced by children long before they are able to act with these objects themselves. And the puzzle may go, more or less, like this: The fact the children can count indicates that they are already familiar with the entities called numbers. Of course, they need some time to develop the routine of comparing-by-counting. But even when they are already adept in this latter routine, why do they stay away from it when asked such question as "Where are there more marbles"? In our long conversations with Roni and Eynat, we observed this phenomenon many times [11,21]. It was puzzling indeed, but only as long as it was described in this cognitivist language, which we too used at that time. The effect of puzzle disappeared when we began seeing number as but a reification of the discursive action of counting. The commognitive vision reversed the order of learning: the routine of comparing-by-counting, with counting understood at this stage as but an incantation (reciting number words in a constant order) comes first, and the idea of number as an abstract object emerges form it much later. Thus, as long as the child cannot actually do things with number words, there is simply no such thing as number. And even when she gains some mastery over the discursive operations of counting and comparing-by-counting, it must still take time until she reifies counting and stops seeing it as merely the favorite game of the grownups. All this seemed to solve, or rather resolve, the cognitivist conundrum: as long as the process of counting has not been reified, which seems to be the common state of affairs in 4- or 5-year olds, saying that children are trying to "compare sets with respect to numbers" makes no sense – and the puzzling disappears.

All that has been said here evokes also one important metacommognitive reflection. Our studies taught us quite a lesson about ourselves as observers of others. Events such as the latter one opened our eyes to the fact that one's own view of mathematics serves as a highly selective lens for seeing and understanding other people. We realized that unless we take precautions, we tend, as teachers or researchers, to attribute our own numerical way of thinking to those whom we observe, while also assuming that in the learner this thinking may be not as well developed as in an expert. This tendency comes to the fore when the dualistically-minded observer takes for granted that the questions she asked has been interpreted by the young participants according to her intention ("children compared sets *with regard to number*"). In result, when the child's performance does not meet her expectations, the observer tends to put the blame on procedural insufficiencies. She says to herself, "The child did try to do this, but she erred in the procedure." While stressing what is missing in children's actions, the cognitivist observer remains blind to what is actually there. In research, she does not even record the "strange" things children are actually doing in the attempt to cope. This oversight leaves her ignorant of the fact that children could be trying to perform a task quite different from that she had in mind. This is how the observer who thinks in dualist terms is misled by her own language and misses the opportunity to get a deeper insight into the meandering route the children travel before they become skillful participants of the canonic mathematical discourse.

3.2. The complexity of complex numbers: the need to reconcile yourself with the incommensurability between the old and the new discourses of numbers

Another puzzle left us with the question about difficulties students experience while learning about complex numbers. Why, we asked, in order to turn the learner into a skillful, competent participant of the discourse on complex numbers, does it not suffice to provide the definition of these numbers and then ask the learners to practice the well-defined operations?

Within the commognitive approach, one possible answer offers itself immediately: as in the previous case, we are talking here about the introduction of a new mathematical object, and as already stated, processes of objectification take time. Yet, although this statement sounds like answering our question, it leaves us with a new one: Why is the process of objectifying so demanding in the case of complex numbers? And more generally, what obstacles must the learner overcome on his/her way toward a new mathematical object?

Admittedly, not everybody experiences the task of objectifying as an uphill struggle. In some cases, the birth of a mathematical objects is recalled as an exhilarating event, an epiphany. Here is, for example, the story told by the topologist William Thurston:

> *I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is 134 over 29... What a tremendous labor-saving device! To me, "134 divided by 29" meant a certain tedious chore, while 134 over 29 was an object with no implicit work.* **[23, P. 4]**

And Thurston continues: "I went excitedly to my father to explain my discovery. He told me that of course this is so, '*a over b'* and 'a divided by b' are just synonyms. To him, it was just a small variation in notation" (ibid). Yet, as demonstrated in our examples, not every mathematics learner is as fortunate as Thurston. A closer look shows that the learners' difficulties may have several sources.

First, there is a certain circularity of requirements. If mathematical objects, such as numbers, whether natural or complex, are discursive constructions, then in order to build such an object one needs to talk about it. But to talk about it, the person must have already brought this object into being. And there is also another, slightly different circularity: the learner is unlikely to make the necessary effort without understanding its prospective gains. Indeed, she needs to be aware of the usefulness of the object she is trying to construct. But how can she comprehend its usefulness before she actually uses it?

Another objectification-hindering circumstance is the fact that what happens in the process of reifying may appear counterintuitive. Indeed, when you reify a mathematical process, such as that of extracting a square root from a number, and you write $\sqrt{-1} = i$, you claim that there is a product to the operation that has been considered so far as giving no result and was described as "forbidden." And now, who can say where and why this new number came from? It appeared with the introduction of the new signifier, "i." This new signifier reified the process of subtracting, but it did not add anything. This unlikely act of conjuring something out of nothing seems as counterintuitive (and difficult to digest!) as would be reifying a recipe for a cake and claiming that it constitutes the cake itself.

Objectification may have yet another counterintuitive aspect. To reify, a revolution in the rules of the game is sometimes required. This dramatic change may express itself in adopting a new way of building and endorsing new narratives, in changing how we think about familiar objects, and in disqualifying some of hitherto unquestioned truths. Thus, when complex numbers are to be introduced, some defining features of the object known as "number" may have to be abandoned. So far, numbers have been understood as what answers such questions as "How many?" or "How much?" Each of them had a magnitude, and for any two of them it was clear which is "bigger." Not any longer. Also some previously endorsed stories must now be compromised. For instance, in the transition from the discourse of real numbers to that of complex ones, the narrative "Some polynomial equations have no solutions" is not true anymore. In spite of the apparent contradiction, the old truth and the new one are not mutually exclusive. They just belong to different discourses, because each one of them is using the word "number" in different way. Such two narratives are called "incommensurable" (as opposed to incompatible), and so do the discourses that produced them. Summing up, objectification projects back onto familiar discourses and transforms them, sometimes beyond recognition.

In the view of all this, it is not surprising that students may struggle to construct mathematical objects for themselves, and that they may take time to succeed. As long as the success refuses to come, they may have a considerable difficulty benefitting from what their teacher does or says. Obviously, the question now cries to be asked of how we can support the learners in their coping with all these hurdles. How to help them overcome the circularity and counterintuitiveness of objectification? A partial answer will be given below, when I show how commognition helped us tackle Poincaré's query. For now, let me just say that those who teach, having long forgotten their own past struggles, are mostly unaware of incommensurability between their own discourse and that of the learners. This was certainly so in the case of the mathematician with whom I discussed students' difficulties with complex numbers. The very awareness of the nature of the problem may take the teacher half way toward a solution.

3.3. The insufficiency of logic for understand mathematics? Some mathematical developments are a matter of choice, not of deductive reasoning

If mathematics "invokes only the rules of logic, those accepted by all well-formed minds, how does it happen that there are so many people who are entirely impervious to it?", wondered Poincaré while pondering on his own abilities as mathematician. As can already be seen from the former examples, commognition dissolves this puzzle by showing the falsity of its premise. Yes, according to commognition, the assumption that mathematics is the exclusive province of logic is untrue. Whereas logic wields the absolute power *inside* every mathematical discourse, the choice of the discourse is not a purely deductive act.

Let me elaborate. One of the implications of the commognitive vision of mathematics and its objects is that the growth of mathematics, whether historical or ontogenetic (in learning), involves two types of developments: adding ever-new stories about already existing objects and, from time to time, adding new objects and reforming the discourse. The first of these changes happens inside an existing discourse, whereas the other is metadiscursive: it is a transformation of the discourses themselves. We can thus speak about two types of learning that can be described, respectively, as object-level and meta-level. I will now argue that only the former kind of learning can be considered as just a matter of logic. Indeed, although mathematics is often described as a purely analytic discipline, that is, one whose narratives are constructed and endorsed exclusively on the basis of deduction, this feature holds only within the boundaries of a well-defined discourse. Once a discourse is chosen, its rules, combined with those of deduction, uniquely determine how new endorsed narratives are to be derived from those that have been endorsed before, axioms and definitions included. Thus, as long as a skillful participant stays within the confines of a particular mathematical discourse, he can, at least in principle, produce new narratives and test their endorsablity independently, without being helped by others. In school, this is the situation for the learner who is already well acquainted with, say, the discourse on functions and is now supposed to explore properties of different families of functions.

The situation changes, however, when the student faces the need for meta-level learning. Here, in order to proceed, he will have to make the transition to a discourse incommensurable with the one he is coming from. Historically, this kind of transition is an outcome of mathematicians' personal choices – of their assessment of how useful or beautiful would be the results of following in one direction or another. To develop new mathematical discourse, they often needed to revise their shared beliefs on what should count as useful, aesthetic, and as "mathematically permissible." Clearly, these choices were not *dictated* by logic – they were a matter of contingency and of personal preferences rather than of necessity. Making such decisions required the ability to see mathematics as a whole and to foresee the long-term effects of these decisions. Incapable of this kind of considerations, novice participants of mathematical discourse are unlikely to replicate these historical choices on their own, and must thus be ushered into the new incommensurable discourse by others.

The need for meta-level learning appears many times along the school and university curricula, with this need being often invisible even to the teachers. How can meta-level learning happen? It is unlikely to begin in any other way than with the learner's exposure to the new discourse, as practiced by experts. Such exposure is likely to create a communicational conflict between the learner and the teacher: coming from different discourses, the interlocutors will be using the same words in different ways, possibly remaining unaware of this latter difference. If the learner is to enter the new discourse, she needs to recognize the need for a change and must be willing to make it even if she does not yet have any independent rationale for doing this. She must, however, be confident that those who introduced the new discourse had good reasons for doing so, and that once she is better acquainted with how the new discourse works, these reasons will become clear to her. This means she has to start acting according to the rules of the new discourse before she can say what they are good for. Thus, the first stage in learning involves participating in the discourse by imitation. While performing what must appear at this time as a mere ritual, the learner has to engage in the sustained effort to figure out the rationale for implementing these unfamiliar discursive routines. In most cases, the student's persistence may be trusted to pay. In the end, the new discourse and its stories will combine into a sensible, logical whole, and what appeared so far as mere rituals will turn into the activity of genuine mathematical explorations. In short, meta-level learning begins with *emulation of expert activities*, accompanied by a constant attempt at *rationalization*. We call this procedure *reflective imitation*. The gradual objectification is a part and parcel of the process and it is the one that turns the learner from memorizer and rule-follower into an explorer of mathematical universe.

The relevant point in this story of meta-level learning is that rather than being dependent exclusively on the learner' logical thinking, the necessary meta-level developments are predominantly a matter of persistence. They also require suspense of old beliefs and preferences. Exactly as stated by Jourdain, the learner must be able to say to himself "Go on; faith will come to you." This principle, even if recognized by the student, is difficult to implement. Not everybody's confidence in her ability to eventually "see the light" would suffice to persist indefinitely in practices that may sometimes be quite frustrating. The "many people" whose evidently insufficient understanding of mathematics puzzled Poincaré are probably those individuals who, for one reason or another, gave up at a certain point – or perhaps did not ever begin this unending sense-making struggle in the first place.

4. POSTSCRIPT: MY PERSONAL TAKEAWAYS FROM THE JOURNEY

So, what is it that we achieved in our travel from thinking-as-mathematicians to thinking-as-mathematics-educators? To begin with, our vision of mathematics underwent an ontological upheaval. From the task of describing the independently existing world of ideal mathematical objects, it reincarnated into the activity of telling stories whose protagonists are being constructed on the go. As a result, also our vision of mathematics learning changed considerably. From the straightforward, even if at times challenging, activity of *cumulating* "mathematical knowledge" the learning of mathematics was converted into an obstacle-racing, with the obstacles imposing periodic changes of direction. In each resulting transition, a new discourse subsumed an old one, retroactively changing some of the old discourse's metarules and certain uses of its keywords.

Our own transition from crypto-Platonism to commognition was the case of meta-level learning. Indeed, this was a change in our stories about mathematics and in the ways they are told – and it was highly consequential. On the new onto-epistemological foundations, we started developing teaching practices that could now be theoretically justified and rigorously tested. This passage brought also some understandings about ourselves. We realized that because of deep-seated convictions about learning we inherited from our own teachers we were sometimes, unwittingly, teaching mathematics in ways that contributed to students' life-long failure. As researchers, we learned that our own well-developed mathematical discourse, which we once saw as developing by a mere accrual, could be blinding us to what is happening when people learn mathematics. We now know that what one sees from where her long mathematical journey takes her may be quite different from what she experienced in the point of departure. Moreover, we are also aware that by the time a person reaches a certain point in the development of her mathematical discourse, she has already forgotten the initial landscape, and does not even remember that it was once quite different! All this taught us that, as teachers and researchers, we have to be always mindful of this simple caveat: When you see people doing something that does not make any sense to you, do not assume that it is senseless for the actors. The odds are that they are just not doing what you think they are. And if you are aware of the abyss between the learners' present discourse and the discourse you wish them to reach, you no longer expect them to make it to your place in a leap, simply by hopping over the abyss. Instead, you join them in building a bridge that would take the novices safely to the other side of the dangerous gap. This technique, drawing heavily on insights earned in mathematics education research, can be trusted to save many mathematical lives.

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