THE HISTORY AND HISTORIOGRAPHY OF THE DISCOVERY OF CALCULUS IN INDIA

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ABSTRACT

Weaving through the emergence and convergence of various mathematical ideas that led towards the discovery of calculus in India provides an enthralling experience for aficionados of mathematics and its diverse history. This article attempts to briefly capture some of the milestones in the journey made by Indian mathematicians through two eras that paved the way for the discovery of infinite series for π and some of the trigonometric functions in India around the middle of the 14th century. In the first part we shall discuss the developments during what may be called the classical period, starting with the work of Āryabhaṭa (c. 499 CE) and extending up to the work Nārāyaṇa Paṇḍita (c. 1350). The work of the Kerala School starting with Mādhava of Saṅgamagrāma (c. 1340), which has a more direct bearing on calculus, will be dealt with in the second part. The third part recounts the story of the 19th century European discovery of infinite series in India which seems to have struck a wrong note among the targeted audience in Europe with a serious cascading effect.

MATHEMATICS SUBJECT CLASSIFICATION 2020

Primary 01A32; Secondary 01A85

KEYWORDS

Discovery of calculus, infinite series, Kerala school of mathematics, Mādhava, Āryabhaṭa, Charles Whish, historiography

INTERNATIONAL CONGRESS OF MATHEMATICIANS 2022 JULY 6–14

Proc. Int. Cong. Math. 2022, Vol. 7, pp. 5784–5812
DOI 10.4171/ICM2022/77

Published by EMS Press

1. INTRODUCTION

Couched in sublime poetry in a variety of rhythmic meters and codified in the classical Sanskrit language, a journey through the history of mathematics in India could be extremely fascinating and at the same time quite challenging too. The journey would indeed be highly enriching to those who have mastered the language and understood the subtlety of expressions and figures of speech employed in it. However, for those untrained in the nuances of such a knowledge system, it would be difficult to appreciate the beautiful blend of mathematics and poetry—usually characterized with brevity without sacrificing the perspicuity—that we find in most of the texts composed over the last two millennia. The distinct style adopted by the Indian mathematicians for practising (thinking, codifying, transmitting, etc.) mathematics, by directly plunging into results without much mathematical elaborations, has been succinctly and beautifully brought out by A. A. K. Ayyangar in his article [17, p. 4.101]:[1]

> *The Hindu mind has always shown peculiar aptitude for fundamental thinking, digging down into the depths of thought with the minimum of external equipment, while other minds are after heavy superstructures with complicated scaffolding, tools and machinery. One extra-ordinary illustration of this trait of the Hindu mind we have in Ramanujan.*

Perhaps being fascinated by this peculiar way of doing mathematics by Hindus, using poetic verses, and aphoristic expressions, some of the of European scholars who were serving the British establishment in various capacities—starting from the final decades of the 18th century—embarked on their journey to study the civilizational basis of India, and the route adopted by Indians to excel in mathematics and astronomy, besides arts, architecture, aesthetics, philosophy and other disciplines.[2]

One such European scholar who got deeply attracted towards the mathematics and astronomy of the Kerala School was the then civil servant of the East India Company, Charles M. Whish (1792–1833). Having been posted at the Malabar region of Kerala for more than a decade, Whish started interacting with the local pundits and gained proficiency in both the local language Malayalam and Sanskrit. He also began to communicate some of his fascinating findings concerning the breakthroughs made by the native astronomers of Kerala, by way of both authoring papers and sharing them with the Madras Literary Society. A remarkable paper of his carrying the details of signal contributions made by the Kerala School of mathematicians, which flourished during the medieval period (14–16 centuries CE), got published in the Transactions of Royal Asiatic Society of Great Britain and Ireland in 1834—unfortunately, only posthumously—due to his premature death in 1833.

1 Ayyangar, who came out in flying colors, with his Master's degree at the age of 18 years, has done remarkable research particularly with respect to second-order indeterminate equations.

2 See, for instance, [12].

It is this paper, which for the first time brings to the notice of European scholars the discovery of the infinite series for π, and some of the trigonometric functions by the Kerala mathematicians, almost three centuries before their advent in Europe. Strangely, this paper of Whish, instead of generating curiosity, discussion, and excitement among the European scholars, remained largely disregarded for almost a century. This deafening silence—along with the discount of its contents, among the historians of mathematics in the West—got broken only in the decades to follow from the 1940s, when some of the Indian mathematicians such as C. T. Rajagopal, Mukunda Marar, and others brought to fore the sophisticated mathematics produced by this school in the form of a series of articles [22, 23, 25, 26]. During the same period, Ramavarma Thampuran and Akhileswara Ayyar also brought out an edition of the first part (dealing with mathematics) of seminal text of Kerala astronomy and mathematics, *Yuktibhāṣā* (c. 1530), along with detailed explanations in Malayalam [28].

The Kerala School that we refer to in this article commences with Mādhava of Saṅgamagrāma (c. 1340–1420), the originator of this *guru-paraṃparā* or "lineage of teachers." His followers include Dāmodara, Parameśvara, Nīlakanṭha Somayājī, Jyesṭhadeva, Śaṅkara Vāriyar, and others. Though Mādhava's works containing the infinite series are not available to us, the later mathematicians in this tradition unanimously ascribe the series to Mādhava. In some of the recent studies, it has been convincingly argued by modern scholars that these series expansions for π and other trigonometric functions, and the evaluation of derivatives of various functions (while computing instantaneous velocities) rely indispensably on the central ideas of infinitesimal calculus, which include local approximation by linear function (see Section 3.4 of the present article).[3]

It is, however, important to understand that these breakthroughs achieved in the Kerala School of Mathematics cannot be narrowed to only the scope of work made in a span of two centuries. It is the continuum of mathematical ideas evolved by various Indian mathematicians spanning over nine centuries before—starting at least from the time of Āryabhaṭa (5th century)—till the dawn of the Kerala School that has led to the convergence point which has led Mādhava (14th century) to invent infinitesimal methods, thereby marking the advent of the discipline of calculus, though largely restricted to the consideration of the circular functions.[4]

This paper attempts to string the pearl of ideas and breakthroughs through the history of mathematics in India that led to this advent. The evolution of poignant ideas is traced in two parts. The first part, covered in Section 2, deals with precalculus breakthroughs and the germinating ideas for calculus that were intuitively apprehended in India well before Mādhava came on the scene. The second part, dealt with in Section 3, captures the discovery of calculus in the Kerala School. Section 4 of this paper recounts the story of how the revelations of the work of the Kerala School brought out by Whish seems to have struck a wrong note and alarmed some of the leading figures in the British academic establishment which led to the denigration and suppression of this work for almost a century.

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3 The reader is also referred to the articles [13, 14, 24].

4 For a detailed discussion on this evolution readers may refer to [15, 27].

2. DEVELOPMENTS IN THE CLASSICAL ERA OF INDIAN MATHEMATICS

In this section, we shall consider some of the ideas and methods developed in Indian mathematics, during the period 450–1350 CE, which have a bearing on the later work of the Kerala School. In particular, we shall focus on the following topics: the notion and mathematics of zero and infinity; iterative approximations for irrational numbers; summation of powers of natural numbers; the discrete form of the harmonic equation for the sine function given by Āryabhaṭa; and the emergence of the notion of instantaneous velocity of a planet in astronomy.

2.1. Notion of zero and infinity

2.1.1. Philosophical and cultural context of zero and infinity

Select passages in *Upaniṣads*, as well as contemporary Buddhist and Jaina philosophy, point to the philosophical and cultural context that has possibly led to the development of the fundamental and intriguing concepts such as void and the infinite which later got incorporated in mathematics as zero and infinity. In this section, we present quotes from different ancient literature in this regard.

The *śānti-mantra* of the *Īśāvāsyopaniṣad* refers to the ultimate absolute reality, the *Brahman*, as *pūrṇa*, the perfect, complete or full. Talking of how the universe emanates from the *Brahman*, it states:

> पूर्णमदः पूर्णमिदं पूर्णात्पूर्णमुदच्यते।
> पूर्णस्य पूर्णमादाय पूर्णमेवावशिष्यते॥
> *pūrṇamadaḥ pūrṇamidaṃ pūrṇātpūrṇamudacyate* |
> *pūrṇasya pūrṇamādāya pūrṇamevāvaśiṣyate* ||
>
> That (*Brahman*) is *pūrṇa*; this (the universe) is *pūrṇa*; [this] *pūrṇa* emanates from [that] *pūrṇa*; even when *pūrṇa* is drawn out of *pūrṇa*, what remains is also *pūrṇa.*

In the *Kṛṣṇa-Yajurveda Taittirīya-Brāhmaṇa* (*Kāṭhaka* 3.49), we have the word *śūnya* (generally employed to mean zero in mathematics) appearing in the form of a compound word with a negative particle (*nañ*) tagged to it. This is in the context of describing the glory of the sun:

> वेदैरशून्यस्त्रिभिरेति सूर्यः।
> *vedairaśūnyastribhireti sūryaḥ* |

Pāṇini's *Aṣṭādhyāyī* (c. 500 BCE) has the notion of *lopa* which functions as a null-morpheme. *Lopa* appears in several *sūtras*, starting with

> अदर्शनं लोपः। (1.1.60).
> *adarśanaṃ lopaḥ* |
>
> That which gets voided is [termed] *lopaḥ.*

The word *śūnya* also appears twice as a symbol in Piṅgala's *Chandaḥ-śūtra* (c. 300 BCE). In Chapter VIII, while enunciating an algorithm for evaluating any positive integral power of 2 in terms of an optional number of squaring and multiplication (duplication) operations, *śūnya* is used as a marker:

> रूपे शून्यम्। द्विः शून्ये। (8.29-30).
> *rūpe śūnyam* । *dviḥ śūnye* ।
>
> If you get one (*rūpe*) [as the remainder after doing modulo 2 arithmetic] place zero [as the marker]. If you get zero [as the remainder] place two.

Different schools of Indian philosophy have related notions such as the notion of absence (*abhāva*) in Nyāya School, and the *śūnyavāda* among the Bauddhas.

2.1.2. The mathematics of zero

The *Brāhmasphuṭa-siddhānta* (c. 628 CE) of Brahmagupta seems to be the first available text that thoroughly discusses the mathematics of zero. While describing arithmetic, the six operations with zero (*śūnya-parikarma*) are also discussed in Chapter XVIII on algebra (*kuṭṭakādhyāya*). While zero divided by zero is stated to be zero, any other quantity divided by zero is said to be *taccheda* (that with zero denominator). Of the six verses, two are presented below and the rest are paraphrased here [5, pp. 309–310]:

> धनयोर्धनमृणमृणयोः धनर्णयोरन्तरं समैक्यं खम्।
> ऋणमैक्यं च धनमृणधनशून्ययोः शून्यम्॥ ...
> खोद्धृतमृणं धनं वा तच्छेदं खमृणधनविभक्तं वा।
> ऋणधनयोर्वर्गः स्वं खं खस्य पदं कृतिर्यत् तत्॥
> *dhanayordhanamṛṇamṛṇayoḥ dhanarṇayorantaraṃ samaikyaṃ kham* ।
> *ṛṇamaikyaṃ ca dhanamṛṇadhanaśūnyayoḥ śūnyam* ॥ ...
> *khoddhṛtamṛṇaṃ dhanaṃ vā tacchedaṃ khamṛṇadhanavibhaktaṃ vā* ।
> *ṛṇadhanayorvargaḥ svaṃ khaṃ khasya padaṃ kṛtiryat tat* ॥
>
> … [The sum of] positive (*dhana*) and negative (*ṛṇa*), if they are equal, is zero (*kham*). The sum of a negative and zero is negative, of a positive and zero is positive and of two zeros, zero (*śūnya*). … Negative subtracted from zero is positive, and positive from zero is negative. Zero subtracted from negative is negative, from positive is positive, and from zero is zero (*ākāśa*).
>
> … The product of zero and a negative, of zero and a positive, or of two zeroes is zero. A zero divided by zero is zero. … A positive or a negative divided by zero is that with zero denominator (*taccheda*). The square (*kṛti*) of a positive or negative number is positive; the square and square-root (*padam*) of zero is zero.

Bhāskarācārya (c. 1150), while discussing the mathematics of zero in his work *Bījagaṇita*, explains that infinity (*ananta-rāśi*) which results when some number is divided by zero is called *khahara*. He also graphically describes [4, p. 6] the characteristic property of infinity that it is unaltered even if a huge quantity (*bahu*) is added to or taken away from it with a beautiful simile:[5]

> खहरो भवेत् खेन भक्तश्च राशिः॥ ...
> अस्मिन्विकारः खहरे न राशावपि प्रविष्टेष्वपि निःसृतेषु।
> बहुष्वपि स्याल्लयसृष्टिकालेऽनन्तेऽच्युते भूतगणेषु यद्वत्॥
> *khaharo bhavet khena bhaktaśca rāśiḥ* || ...
> *asminvikāraḥ khahare na rāśāvapi praviṣṭeṣvapi niḥsṛteṣu* |
> *bahuṣvapi syāllayasṛṣṭikāle'nante'cyute bhūtagaṇeṣu yadvat* ||
>
> A quantity divided by zero will be (called) *khahara* (an entity with zero as divisor). … In this quantity, *khahara*, there is no alteration even if many are added or taken out, just as there is no alteration in the Infinite (*ananta*), Infallible (*acyuta*) [Brahman] even though many groups of beings enter in or emanate from [It] at times of dissolution and creation.

From the above illustrations it is discernible that Indian mathematicians began dabbling with the notions of zero and infinity in varied mathematical contexts.

2.2. Irrationals and iterative approximations

2.2.1. Approximation for surds in *Śulbasūtras*

Śulbasūtras (c. 800 BCE) that form a part of *Kalpasūtras* (one of the six *Vedāṅgas*) are essentially manuals that contain systematic procedures (algorithms) for the exact construction of altars that were laid out on leveled ground by manipulating cords of various lengths tied to a gnomon. The manuals also contain certain other mathematical details that are relevant to the construction, and are composed in the form of short, cryptic phrases—usually prose, although sometimes including verses—called *sūtras* (literally "string" or "rule, instruction"). The term for the measuring-cords called *śulba* got associated with the name to this set of texts as the *Śulbasūtras* or "Rules of the cord." Starting with simple shapes involving symmetrical figures such as squares and rectangles, triangles, trapezia, rhomboids, and circles, the texts move on to discuss the construction of complex shaped figures such that of falcon. Frequently, one also finds problems pertaining to transformation of one shape into another. Hence, the *Śulbasūtra* rules often involve what we would call area-preserving transformations of plane figures, and thus include the earliest known Indian versions of certain geometric formulas and constants. More interestingly, *Baudhāyana-śulvasūtra* gives the following approximation for $\sqrt{2}$ [33, (1.61–2), p. 19]:

5 This simile can be better appreciated by those who are reasonably familiar with the fundamental tenets of Hinduism and its philosophy.

प्रमाणं तृतीयेन वर्धयेत्तच्च चतुर्थेनात्मचतुस्त्रिंशोनेन। सविशेषः।
pramāṇaṃ tṛtīyena vardhayettacca caturthenātmacatustriṃśonena । *saviśeṣaḥ* ।

The measure [of the side] is to be increased by its third and this [third] again by its own fourth less the thirty-fourth part [of the fourth]. That is the approximate diagonal (*saviśeṣa*).

$$\begin{aligned}\sqrt{2} &\approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \\ &= \frac{577}{408} \\ &\approx 1.4142156. \end{aligned} \tag{1}$$

The above approximation is accurate to 5 decimal places. From certain other prescriptions [33, (1.58), P. 19] given in this text, one could discern the approximation for π to be given as $\pi \approx 3.0883$.

2.2.2. Approximation for π by Āryabhaṭa

Āryabhaṭa (c. 499) gives the following approximate value for π: [6]

चतुरधिकं शतमष्टगुणं द्वाषष्टिस्तथा सहस्राणाम्।
अयुतद्वयविष्कम्भस्यासन्नो वृत्तपरिणाहः॥
caturadhikaṃ śatamaṣṭaguṇaṃ dvāṣaṣṭistathā sahasrāṇām ।
ayutadvayaviṣkambhasyāsanno vṛttapariṇāhaḥ ॥

One hundred plus four multiplied by eight and added to sixty-two thousand: This is the approximate measure of the circumference of a circle whose diameter is twenty-thousand.

Thus as per the above verse, $\pi \approx \frac{62832}{20000} = 3.1416$.

It appears that Indian mathematicians (at least in the Āryabhaṭan tradition) employed the method of successive doubling of the sides of a circumscribing polygon—starting from the circumscribing square leading to an octagon, etc.—to find successive approximations to the circumference of a circle. This method has been described in the later Kerala texts *Yukti-bhāṣā* (c. 1530) of *Jyeṣṭhadeva* and the *Kriyākramakarī* commentary (c. 1535) of Śaṅkara Vāriyar on the *Līlāvatī*, of Bhāskarācārya.

2.3. Summation of geometric series

The result obtained by summing the geometric series $1 + 2 + 2^2 + \cdots + 2^n$ is stated in Chapter VIII of Piṅgala's *Chandaḥ-sūtra* (c. 300 BCE). It is quite remarkable that Piṅgala also gives a systematic algorithm for evaluating any positive integral power of a number (2 in this context) in terms of an optimal number of squaring and multiplication operations.

6 [2, P. 45], *Gaṇitapāda*, verse 10.

Mahāvīrācārya (c. 850), in his *Gaṇita-sāra-saṅgraha* gives the sum of a geometric series and also explains Piṅgala's algorithm for finding the required power of the common ratio between the terms of the series **[16, PP. 28–29]**:

> पदमितगुणहतिगुणितप्रभवः स्याद्गुणधनं तदाद्यूनम्।
> एकोनगुणविभक्तं गुणसङ्कलितं विजानीयात्॥
> *padamitaguṇahatiguṇitaprabhavaḥ syādguṇadhanaṃ tadādyūnam* |
> *ekonaguṇavibhaktaṃ guṇasaṅkalitaṃ vijānīyāt* ||
>
> The first term when multiplied by the product of the common ratio (*guṇa*) taken as many times as the number of terms (*pada*) [in the series], gives rise to the *guṇadhana*. This *guṇadhana*,[7] when diminished by the first term and divided by the common ratio less one, is to be understood as the sum of the geometrical series (*guṇa-saṅkalita*).

If a is the first term and r the common ratio, then what is stated in the verse above may be expressed as

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{(r-1)}. \tag{2}$$

Vīrasena (c. 816), in his commentary *Dhavalā* on the *Ṣaṭkhaṇḍāgama*, has made use of the sum of the following infinite geometric series in his evaluation of the volume of the frustum of a right circular cone:[8]

$$1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \frac{4}{3}. \tag{3}$$

The proof of the above result is outlined by Nīlakaṇṭha Somayājī in his *Āryabhaṭīya-bhāṣya*. Nīlakaṇṭha presents this discussion in the context of deriving an approximation for a small arc in terms of the corresponding chord in a circle. More details are presented in Section 3.1 of the article.

2.4. Āryabhaṭa's computation of Rsine-differences

In the mathematical section of *Āryabhaṭīya* (c. 499), Āryabhaṭa presents two different methods for the computation of tabular Rsine values. While the first is the usual geometric method, the second is an ingenious method which is based on computing the Rsine-differences employing the important property that the second-order differences of Rsines are proportional to the Rsines themselves:[9]

> प्रथमाच्चापज्यार्धाद्यैरूनं खण्डितं द्वितीयार्धम्।
> तत्प्रथमज्यार्धांशैस्तैस्तैरूनानि शेषाणि॥

7 This is a technical term employed to refer to ar^n in (2).

8 See, for instance, **[29, PP. 203–205]**.

9 **[2, P. 51]**, *Gaṇitapāda*, verse 12.

prathamāccāpajyārdhādyairūnaṃ khaṇḍitaṃ dvitīyārdham |
tatprathamajyārdhāṃśaistaistairūnāni śeṣāṇi ||

The first Rsine divided by itself and then diminished by the quotient will give the second Rsine-difference. The same first Rsine, diminished by the quotients obtained by dividing each of the preceding Rsines by the first Rsine, gives the remaining Rsine-differences.

Let the quadrant be divided into 24 equal parts, and let J_i denote $R\sin(i\alpha)$ where $\alpha = 225'$ for $i = 1, 2, \ldots, 24$. Now $J_1 = R\sin(225')$, $J_2 = R\sin(450')$, $\ldots$, $J_{24} = R\sin(90°)$, are the 24 Rsines. Let $\Delta_1 = J_1$, $\Delta_2 = J_2 - J_1$, $\ldots$, $\Delta_k = J_k - J_{k-1}$, be the first-order Rsine-differences. Then, the prescription given in the above verse may be expressed as

$$\Delta_2 = J_1 - \frac{J_1}{J_1} \tag{4}$$

$$= \Delta_1 - \frac{J_1}{J_1}. \tag{5}$$

In general,

$$\Delta_{k+1} = \Delta_k - \frac{J_k}{J_1} \quad (k = 1, 2, \ldots, 23). \tag{6}$$

Since Āryabhaṭa also takes $\Delta_1 = J_1 = R\sin(225') \approx 225'$, the above relations reduce to

$$\Delta_2 = 224', \tag{7}$$

$$\Delta_{k+1} - \Delta_k = \frac{-J_k}{225'} \quad (k = 1, 2, \ldots, 23). \tag{8}$$

The renowned mathematician David Mumford refers to the above equation as "the differential equation for the sine function in its finite difference form" [24].

2.5. Instantaneous velocity of a planet (*tātkālika-gati*)

In Indian astronomy, the motion of a planet is computed by making use of two corrections: the *manda-saṃskāra* which essentially corresponds to the equation of center and the *śīghra-saṃskāra* which corresponds to the conversion of the heliocentric longitudes to geocentric longitudes.

In Figure 1, C is the center of a circle on which the mean planet P_0 is located; CU is the direction of the *ucca* (aphelion or apogee as the case may be); P is the true planet which lies on the epicycle of (variable) radius r centered at P_0, such that P_0P is parallel to CU. If M is the mean longitude of a planet, α the longitude of the *ucca*, then the correction (*manda-phala*) $\Delta\mu$ is given by

$$R\sin(\Delta\mu) = \left(\frac{r}{K}\right) R\sin(M - \alpha). \tag{9}$$

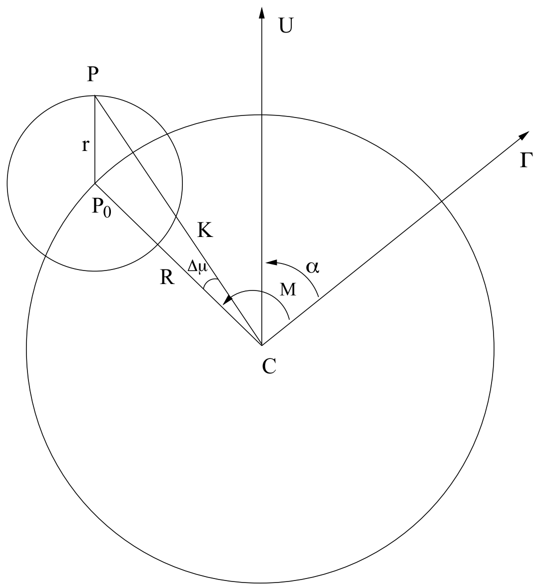

FIGURE 1
Manda correction.

Here K is the *karṇa* (hypotenuse) or the (variable) distance of the planet from the center of the concentric. The texts on Indian astronomy while giving the *manda-phala*, present the following formula:

$$R\sin(\Delta\mu) = \left(\frac{r_0}{R}\right) R\sin(M-\alpha), \tag{10}$$

where r_0 is the tabulated (or mean) radius of the epicycle in the measure of the concentric circle of radius R.

Thus there seems to have been an implicit understanding among the Indian astronomers in accepting this model that the true planet P moves on the variable epicycle of radius r in a way such that the following equation is satisfied:

$$\frac{r}{K} = \frac{r_0}{R}. \tag{11}$$

For small r, the left-hand side of (10) is usually approximated by the arc itself. Thus we have

$$\Delta\mu = \left(\frac{1}{R}\right)\left(\frac{r_0}{R}\right) R\sin(M-\alpha). \tag{12}$$

The *manda*-correction is to be applied to the mean longitude M, to obtain the true or *manda*-corrected longitude μ given by

$$\begin{aligned}\mu &= M - \Delta\mu \\ &= M - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R\sin(M-\alpha). \end{aligned}\tag{13}$$

If n_m and n_u are the mean daily motions of the planet and the *ucca*, then the true longitude of the planet on the next day may be expressed as

$$\mu + n = (M + n_m) - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R\sin(M + n_m - \alpha - n_u). \tag{14}$$

Thus the true daily motion (n), obtained by finding the difference of the two equations (13) and (14) is given by

$$n = n_m - \left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right)\big[R\sin\big\{(M-\alpha) + (n_m - n_u)\big\} - R\sin(M-\alpha)\big]. \tag{15}$$

The second term in the above is the correction to mean daily motion (*gati-phala*), which strictly involves evaluating the rate of change of the sine function. While an expression for this has been pursued by Bhāskara I (c. 629) in his *Mahābhāskarīya*, the correct formula for the true daily motion of a planet, employing the Rcosine as the "rate of change" of Rsine, seems to have been first given by Muñjāla (c. 932) in his short manual *Laghumānasa* [18, P. 125] and also by Āryabhaṭa II (c. 950) in his *Mahā-siddhānta* [20, P. 58]:

> कोटिफलघ्नी भुक्तिर्गज्याभक्ता कलादिफलम्॥
> *koṭiphalaghnī bhuktirgajyābhaktā kalādiphalam* ||
>
> The *koṭiphala* multiplied by the [mean] daily motion and divided by the radius gives the minutes of the correction [to the rate of the motion].

Essentially, the above verse gives the true daily motion in the form

$$n = n_m - (n_m - n_u)\left(\frac{r_0}{R}\right)\left(\frac{1}{R}\right) R\cos(M-\alpha). \tag{16}$$

Bhāskarācārya (c. 1150) in his *Siddhānta-śiromaṇi* clearly distinguishes the true daily motion from the instantaneous rate of motion [32]. And he gives the Rcosine correction to the mean rate of motion as the instantaneous rate of motion. He further emphasizes the fact that the velocity is changing every instant and this is particularly important in the case of the moon because of its rapid motion [27, PP. 225–227].

3. KERALA SCHOOL OF MATHEMATICS AND ASTRONOMY

The banks of the river Nīlā in the south Malabar region of Kerala witnessed for over 300 years, beginning from about the mid-14th century, what may arguably be considered the golden age of Indian mathematics. The Kerala School of Mathematics and Astronomy pioneered by Mādhava (c. 1340–1420) of Saṅgamagrāma, extended well into the 19th century as exemplified in the work of Śaṅkaravarman (c. 1830), *Rājā* of Kaḍattanāḍu. Only a couple of astronomical works of Mādhava (*Veṇvāroha*, *Lagnaprakaraṇa* and *Sphuṭacandrāpti*) seem to be extant now. Most of his celebrated mathematical discoveries—such as the infinite series for π and the sine and cosine functions—are available only in the form of citations in later works.

Mādhava's disciple Parameśvara (c. 1380–1460) of Vaṭasseri is reputed to have carried out detailed observations for around 55 years. Though a large number of original works and commentaries written by him have been published, one of his important works on mathematics, the commentary *Vivaraṇa* on *Līlāvatī* of Bhāskarācārya, is yet to be published. Nīlakaṇṭha Somayājī (c. 1444–1550) of Kuṇḍagrāma, disciple of Parameśvara's son Dāmodara (c. 1410–1520), is the most celebrated member of Kerala School after Mādhava. Nīlakaṇṭha has cited several important results of Mādhava in his various works, the most prominent of them being *Tantrasaṅgraha* (c. 1500) and *Āryabhaṭīya-bhāṣya*. In the latter work, while commenting on the *Gaṇitapāda* of *Āryabhaṭīya*, Nīlakaṇṭha has also provided ingenious demonstrations or proofs for various mathematical formulae [21].

However, the most detailed exposition of the work of the Kerala School, starting from Mādhava, and including the seminal contributions of Parameśvara, Dāmodara, and Nīlakaṇṭha, is to be found in the famous Malayalam work *Gaṇita-yuktibhāṣā* (henceforth simply *Yuktibhāṣā*) (c. 1530) of Jyeṣṭhadeva (c. 1500–1610), who was a junior contemporary of Nīlakaṇṭha. The direct lineage from Mādhava continued at least till Acyuta Piśāraṭi (c. 1550–1621), a disciple of Jyeṣṭhadeva, who wrote many important independent works in Sanskrit, as well as a couple of commentaries in the local language Malayalam.

In the following sections we shall present an overview of the contribution of the Kerala School to the development of calculus (during the period 1350–1500), following essentially the exposition given in *Yuktibhāṣā*. In order to indicate some of the concepts and methods developed by the Kerala astronomers, we first take up the summation of infinite geometric series as discussed by Nīlakaṇṭha Somayājī in his *Āryabhaṭīya-bhāṣya*, that was alluded to just before. We then consider the derivation of binomial series expansion and the estimation of the sum of integral powers of integers, $1^k + 2^k + \cdots + n^k$ for large n, as presented in *Yuktibhāṣā*. These results constitute the basis for the derivation of the infinite series for $\frac{\pi}{4}$ and its various fast convergents given by Mādhava. Following this, we shall outline another interesting work of Mādhava on the estimation of the end-correction terms called the *antya-saṃskāra*,[10] that had enabled him to arrive at the transformation of the π-series to fast convergent ones—whose multifarious forms may be noted from a citation in Section 4.3.

3.1. Discussion of the sum of an infinite geometric series

In his *Āryabhaṭīya-bhāṣya*, while explaining the *upapatti* (rationale) behind an interesting approximation for the arc of a circle in terms of the *jyā* (Rsine) and the *śara* (Rversine), Nīlakaṇṭha presents a detailed demonstration of how to sum an infinite geometric series. The context of this discussion is Nīlakaṇṭha's pursuit to approximate the arc of a circle in terms of *jyā* (sine) and *śara* (versine). The verse that succinctly presents this approximation is the following:

10 Interestingly, this term in common parlance refers to the last rites to be performed.

सत्र्यंशादिषुवर्गात् ज्यावर्गाढ्यात् पदं धनुः प्रायः।
satryaṃśādiṣuvargāt jyāvargāḍhyāt padaṃ dhanuḥ prāyaḥ ।

The arc is nearly (*prāyaḥ*) equal to the square root of the sum of the square of the *śara* added to one-third of it, and the square of the *jyā*.

In Figure 2, AB is the arc whose length (assumed to be small) is to be determined in terms of the chord lengths AD and BD. In the Indian mathematical literature, the arc AB, the semichord AD, and the segment BD are referred to as the *cāpa*, *jyārdha*, and *śara*, respectively. As can be easily seen from the figure, this terminology arises from the fact that these geometrical objects look like a bow, string, and arrow, respectively. Denoting them by c, j, and s, the expression for the arc given by Nīlakaṇṭha may be written as

$$c \approx \sqrt{\left(1 + \frac{1}{3}\right)s^2 + j^2}. \tag{17}$$

The proof of the above equation which has been discussed in detail by Sarasvati Amma [29, pp. 179–182] involves a summation of an infinite geometric series given by (19).

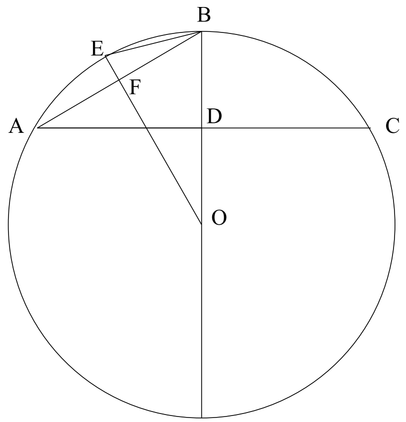

FIGURE 2
Arc-length in terms of *jyā* and *śara*.

The question that Nīlakaṇṭha poses as he commences his detailed discussion on the sum of geometric series is very important and pertinent to the current discussion. In fact, this is a general question that arises quite naturally whenever one encounters the sum of an infinite series [1, p. 106]:

कथं पुनः तावदेव वर्धते तावद्वर्धते च ?
kathaṃ punaḥ tāvadeva vardhate tāvadvardhate ca ?

How does one know that [the sum of the series] increases only up to that [limiting value] and that it certainly increases up to that [limiting value]?

Proceeding to answer the above question, Nīlakaṇṭha first states the general result

$$a\left[\left(\frac{1}{r}\right)+\left(\frac{1}{r}\right)^2+\left(\frac{1}{r}\right)^3+\cdots\right]=\frac{a}{r-1}. \tag{18}$$

Here, the left-hand side is an infinite geometric series with the successive terms being obtained by dividing by a common divisor, r, known as *cheda*, whose value is assumed to be greater than 1. He further notes that this result is best demonstrated by considering a particular case, say $a = 1$ and $r = 4$. In his own words **[1, PP. 106–107]**:

> उच्यते —एवं यः तुल्यच्छेदपरभागपरम्परायाः अनन्तायाः अपि संयोगः, तस्य अनन्तानामपि कल्प्यमानस्य योगस्य आद्यावयविनः परम्परांशच्छेदात् एकोनच्छेदांशसाम्यं सर्वत्र समानमेव। तद्यथा —चतुरंशपरम्परायामेव तावत् प्रथमं प्रतिपाद्यते।
>
> *ucyate — evaṃ yaḥ tulyacchedaparabhāgaparamparāyāḥ anantāyāḥ api saṃyogaḥ, tasya anantānāmapi kalpyamānasya yogasya ādyāvayavinaḥ paramparāṃśacchedāt ekonacchedāṃśasāmyaṃ sarvatra samānameva* । *tadyathā — caturaṃśaparamparāyāmeva tāvat prathamaṃ pratipādyate* ।
>
> It is being explained. Thus, in an infinite (*ananta*) geometrical series (*tulyaccheda-parabhāga-paramparā*)[11] the sum of all the infinite number of terms considered will always be equal to the value obtained by dividing by a factor which is one less than the common factor of the series. That this is so will be demonstrated by first considering the series obtained with one-fourth (*caturaṃśa-paramparā*).

What is intended to be demonstrated is

$$\left[\left(\frac{1}{4}\right)+\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right]=\frac{1}{3}. \tag{19}$$

It is noted that one-fourth and one-third are the only terms appearing in the above equation. Nīlakaṇṭha first defines these numbers in terms of one-twelfth of the multiplier a referred to by the word *rāśi*. For the sake of simplicity, we take the *rāśi* to be unity:

$$3\times\frac{1}{12}=\frac{1}{4};\quad 4\times\frac{1}{12}=\frac{1}{3}. \tag{20}$$

Having defined them, Nīlakaṇṭha first obtains the sequence of results:

$$\begin{aligned}\frac{1}{3}&=\frac{1}{4}+\frac{1}{(4\cdot 3)},\\ \frac{1}{(4\cdot 3)}&=\frac{1}{(4\cdot 4)}+\frac{1}{(4\cdot 4\cdot 3)},\\ \frac{1}{(4\cdot 4\cdot 3)}&=\frac{1}{(4\cdot 4\cdot 4)}+\frac{1}{(4\cdot 4\cdot 4\cdot 3)},\end{aligned}$$

11 This compound word that has been coined in Sanskrit for the geometric series is very cute and merits attention. It literally means “A series of terms (*paramparā*) in which the successive ones (*parabhāga*) are obtained by the same divisor (*tulyaccheda*) [as the previous].”

and so on, which leads to the general result

$$\frac{1}{3} - \left[\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n\right] = \left(\frac{1}{4}\right)^n \left(\frac{1}{3}\right). \tag{21}$$

Nīlakaṇṭha then goes on to present the following crucial argument to derive the sum of the infinite geometric series: As we sum more terms, the difference between $\frac{1}{3}$ and sum of powers of $\frac{1}{4}$ (as given by the right-hand side of the above equation) becomes extremely small, but never zero. Only when we take all the terms of the infinite series together, do we obtain the equality

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^n + \cdots = \frac{1}{3}. \tag{22}$$

3.2. Derivation of binomial series expansion

The text *Yuktibhāṣā* presents a very interesting derivation of the binomial series for $(1 + x)^{-1}$ by making iterative substitutions in a simple algebraic identity. The method given here may be summarized as follows:

Consider the product $a(\frac{c}{b})$, where some quantity a is multiplied by the multiplier c, and divided by the divisor b. Here, a is called *guṇya*, c the *guṇaka* and b the *hāra*, which are all assumed to be positive integers, with $b > c$. Now the above product can be rewritten as

$$a\left(\frac{c}{b}\right) = a - a\frac{(b-c)}{b}. \tag{23}$$

In the expression $a\frac{(b-c)}{b}$ of the equation above, if we want to replace the division by b (the divisor) by division by c (the multiplier), then we have to make a subtractive correction (called *śodhya-phala*) which amounts to the following equation:

$$a\frac{(b-c)}{b} = a\frac{(b-c)}{c} - \left(a\frac{(b-c)}{c} \times \frac{(b-c)}{b}\right). \tag{24}$$

Now, in the second term (inside parentheses) if we again replace the division by the divisor b by the multiplier c, then we have to make a subtractive-correction once again. Proceeding thus we obtain an alternating series:

$$a\frac{c}{b} = a - a\frac{(b-c)}{c} + a\left[\frac{(b-c)}{c}\right]^2 - \cdots + (-1)^{m-1}a\left[\frac{(b-c)}{c}\right]^{m-1} + (-1)^m a\left[\frac{(b-c)}{c}\right]^m + \cdots. \tag{25}$$

It may be noted that if we set $\frac{(b-c)}{c} = x$, then $\frac{c}{b} = \frac{1}{(1+x)}$. Hence, the series given by (25) is none other than the well-known binomial series

$$\frac{a}{1+x} = a - ax + ax^2 - \cdots + (-1)^m ax^m + \cdots,$$

which is known to be convergent for $-1 < x < 1$.

Regarding the question of termination of the process, both texts, *Yuktibhāṣā* and *Kriyākramakarī*, clearly mention that logically there is no end to the process of generating *śodhya-phalas*.

It is also noted that the process may be terminated after having obtained the desired accuracy by neglecting the subsequent *phalas* as their magnitudes become smaller and smaller. In fact, *Kriyākramakarī* explicitly mentions that $(b-c)$ should be smaller than c, so that the successive *phalas* become smaller and smaller. In other words, the text, besides presenting a technique to turn a simple algebraic expression into an infinite series, also states the condition that would ensure the convergence of the series.

3.3. Estimation of sums of integral powers of natural numbers

The word employed in the Indian mathematical literature for summation is *saṅkalita*. *Yuktibhāṣā* gives a general method of estimating the sums of integral powers of natural numbers or *samaghāta-saṅkalita*.[12] The detailed procedure given in the text, which is tantamount to providing a proof by induction may be outlined as follows. Before proceeding further with the discussion, a brief note on the notation employed may be useful. We employ S to denote the sum with a subscript and superscript. The subscript denotes the number of terms that are being summed and the superscript denotes the nature of the numbers that are being summed. For the sum of natural numbers, we use (1) as the superscript. For squares of natural numbers, we use (2), and so on. Now, the sum of the first n natural numbers may be written as:

$$\begin{aligned} S_n^{(1)} &= n + (n-1) + \cdots + 1 \\ &= n + [n-1] + [n-2] + \cdots + \big[n-(n-2)\big] + \big[n-(n-1)\big] \\ &= n \cdot n - \big[1 + 2 + \cdots + (n-1)\big]. \end{aligned} \tag{26}$$

When n is very large, the quantity to be subtracted from n^2 is practically (*prāyeṇa*) the same as $S_n^{(1)}$, thus leading to the estimate

$$S_n^{(1)} \approx n^2 - S_n^{(1)}, \quad \text{or} \quad S_n^{(1)} \approx \frac{n^2}{2}. \tag{27}$$

The sum of the squares of the natural numbers up to n may be written as

$$S_n^{(2)} = n^2 + (n-1)^2 + \cdots + 1^2. \tag{28}$$

It can also easily be shown that

$$n S_n^{(1)} - S_n^{(2)} = S_{n-1}^{(1)} + S_{n-2}^{(1)} + S_{n-3}^{(1)} + \cdots . \tag{29}$$

For large n, we have already estimated that $S_n^{(1)} \approx \frac{n^2}{2}$. Thus, for large n, the right-hand side of (29) can be written as

$$n S_n^{(1)} - S_n^{(2)} \approx \frac{(n-1)^2}{2} + \frac{(n-2)^2}{2} + \frac{(n-3)^2}{2} + \cdots . \tag{30}$$

Thus, the excess of $n S_n^{(1)}$ over $S_n^{(2)}$ is essentially $\frac{S_n^{(2)}}{2}$ for large n, so that we obtain

$$n S_n^{(1)} - S_n^{(2)} \approx \frac{S_n^{(2)}}{2}. \tag{31}$$

12 The compound *sama-ghāta* in this context means the product of a number with itself.

Again, using the earlier estimate for $S_n^{(1)}$, we obtain the result

$$S_n^{(2)} \approx \frac{n^3}{3}. \tag{32}$$

Proceeding along these lines, *Yuktibhāṣā* presents an argument essentially based on mathematical induction that the summation of the kth powers of natural numbers for a large n may be written as

$$S_n^{(k)} \approx \frac{n^{k+1}}{(k+1)}. \tag{33}$$

3.4. Mādhava's infinite series for π

The infinite series for π attributed to Mādhava is cited by Śaṅkara Vāriyar in his commentaries *Kriyākramakarī* and *Yuktidīpikā*. Mādhava's quoted verse runs as follows [19, p. 379]:

> व्यासे वारिधिनिहते रूपहृते व्याससागराभिहते।
> त्रिशरादिविषमसङ्ख्याभक्तमृणं स्वं पृथक् क्रमात् कुर्यात्॥
> *vyāse vāridhinihate rūpahṛte vyāsasāgarābhihate* |
> *triśarādiviṣamasaṅkhyābhaktamṛṇaṃ svaṃ pṛthak kramāt kuryāt* ||
>
> The diameter multiplied by four and divided by unity [is found and saved]. Again the products of the diameter and four are divided by the odd numbers (*viṣama-saṅkhyā*) three, five, etc., and the results are subtracted and added sequentially [to the earlier result saved].

The words *paridhi* and *vyāsa* in the above verse refer to the circumference (C) and diameter (D), respectively. Hence the content of the verse above, expressed in the form of an equation, becomes

$$C = \frac{4D}{1} - \frac{4D}{3} + \frac{4D}{5} - \frac{4D}{7} + \cdots. \tag{34}$$

Rearranging the terms and using the notation π, we get

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \tag{35}$$

We shall now present the derivation of the above result as outlined in *Yuktibhāṣā* of Jyeṣṭhadeva and *Kriyākramakarī* of Śaṅkara Vāriyar. For this purpose, let us consider the quadrant OP_0P_nA of the square circumscribing the given circle (see Figure 3). Let r be the radius of the circle. Divide the side $P_0P_n (= r)$ into n equal parts (n large). Then P_0P_i $(i = 1, 2, \ldots, n)$ are the *bhujās* (sides) and $k_i = OP_i$ are the *karṇas* (hypotenuses) of the triangle to be conceived of. The points of intersection of these *karṇas* and the circle are marked as A_is.

It is straightforward to see that the *bhujās* P_0P_i, the *karṇas* k_i, and the East–West line OP_0 form right-angled triangles. Hence we have the relation

$$k_i^2 = r^2 + \left(\frac{ir}{n}\right)^2. \tag{36}$$

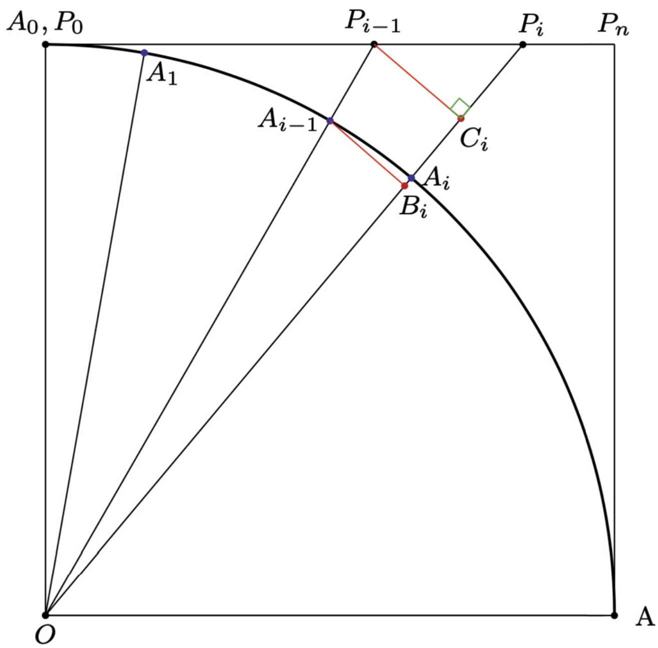

FIGURE 3
Geometrical construction used in the proof of the infinite series for π.

Considering two successive *karṇas*, and the pairs of similar triangles $OP_{i-1}C_i$ and $OA_{i-1}B_i$, and $P_{i-1}C_iP_i$ and OP_0P_i, it can be shown that the length of the segment $A_{i-1}B_i$ is given by

$$A_{i-1}B_i = \left(\frac{r}{n}\right)\left(\frac{r^2}{k_{i-1}k_i}\right). \tag{37}$$

Now the text presents the crucial argument that, when n is large, the Rsines $A_{i-1}B_i$ can be taken as the arc-bits $A_{i-1}A_i$ themselves.

> परिधिखण्डस्य अर्धज्या परिध्यंश एव।
> *paridhikhaṇḍasya ardhajyā paridhyaṃśa eva* ।
>
> The Rsines (*ardhajyā*) corresponding to the arc-bits (*paridhikhaṇḍa*) are essentially the arc-bits themselves.

Recalling that A_0 will merge with P_0, we can easily see that

$$\sum_{i=1}^{n} A_{i-1}A_i = \frac{C}{8}. \tag{38}$$

Thus, one-eighth of the circumference of the circle can be written as the sum of the contributions made by the individual segment $A_{i-1}B_i$ given by (37). That is,

$$\frac{C}{8} \approx \left(\frac{r}{n}\right)\left[\left(\frac{r^2}{k_0k_1}\right) + \left(\frac{r^2}{k_1k_2}\right) + \left(\frac{r^2}{k_2k_3}\right) + \cdots + \left(\frac{r^2}{k_{n-1}k_n}\right)\right]. \tag{39}$$

It is further argued that the denominators may be replaced by the square of either of the *karṇas* since the difference is negligible. Hence we obtain:

$$\begin{aligned}\frac{C}{8} &= \sum_{i=1}^{n} \frac{r}{n}\left(\frac{r^2}{k_i^2}\right)\\ &= \sum_{i=1}^{n}\left(\frac{r}{n}\right)\left(\frac{r^2}{r^2+(\frac{ir}{n})^2}\right)\\ &= \sum_{i=1}^{n}\left[\frac{r}{n} - \frac{r}{n}\left(\frac{(\frac{ir}{n})^2}{r^2}\right) + \frac{r}{n}\left(\frac{(\frac{ir}{n})^2}{r^2}\right)^2 - \cdots\right]. \end{aligned} \tag{40}$$

In the series expression for the circumference given above, factoring out powers of $\frac{r}{n}$, the sums involved are the even powers of the natural numbers. Now, recalling the estimates that were obtained earlier (33) for these sums when n is large, we arrive at the result (35), which was rediscovered by Gregory and Leibniz almost three centuries later.

3.5. Derivation of end-correction terms (*antya-saṃskāra*)

It is well known that the series given by (35) for $\frac{\pi}{4}$ is an extremely slowly converging series. Mādhava seems to have found an ingenious way to circumvent this problem with a technique known as *antya-saṃskāra*. The nomenclature stems from the fact that a correction (*saṃskāra*) is applied towards the end (*anta*) of the series after we terminate it, by considering only a certain number of terms from the beginning. We can, of course, terminate the series at any term we desire, provided we find a correction $\frac{1}{a_p}$ to be applied, that happens to be a good approximation for the rest of the truncated terms in the series. This seems to have been the thought process that has gone in in discovering this *antya-saṃskāra* technique.

Suppose we terminate the series after the term $\frac{1}{p}$ and consider applying the correction term $\frac{1}{a_p}$, then

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{\frac{p-3}{2}}\frac{1}{p-2} + (-1)^{\frac{p-1}{2}}\frac{1}{p} + (-1)^{\frac{p+1}{2}}\frac{1}{a_p}. \tag{41}$$

Three successive approximations to the correction divisor a_p given by Mādhava may be expressed as:

$$\begin{aligned} a_p(1) &= 2(p+2),\\ a_p(2) &= (2p+2) + \frac{4}{(2p+2)},\\ a_p(3) &= (2p+2) + \frac{4}{2p+2+\frac{16}{2p+2}}. \end{aligned} \tag{42}$$

Yuktibhāṣā contains a detailed discussion on how these correction terms of successive orders are arrived at. While the discussion in the text goes only up to the three terms as above, presumably because the expressions become increasingly cumbersome, the idea that the partial quotients of the continued fraction

$$(2p+2) + \cfrac{2^2}{(2p+2) + \cfrac{4^2}{(2p+2) + \cfrac{6^2}{(2p+2) + \cdots}}}$$

serve as correction factors to higher and higher orders is seen to be inherently present in the reasoning. A graph depicting the variation of error in the estimate of π using the three successive end-corrections by truncating the series at different values of p is shown in Figure 4. It may be noted that, when we use the third-order end-correction, by just considering about 25 terms in the series, we are able to obtain the π value correct to 10 decimal places.

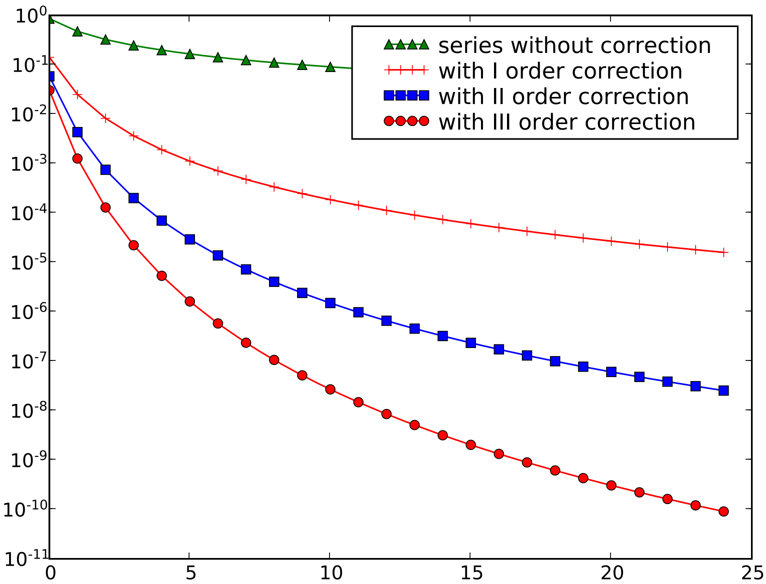

FIGURE 4
Graph depicting the accuracy that is obtained in estimating the value of π by truncating the series at different values of p and employing the three corrections given by (42).

The following accurate value of π (correct to 11 decimal places), given by Mādhava, has been cited by Nīlakaṇṭha in his *Āryabhaṭīya-bhāṣya* and by Śaṅkara Vāriyar in his *Kriyākramakarī*:[13]

13 [1, P. 42], comm. on *Gaṇitapāda* verse 10; [19, P. 377].

विबुधनेत्रगजाहिहुताशनत्रिगुणवेदभवारणबाहवः।
नवनिखर्वमिते वृतिविस्तरे परिधिमानमिदं जगदुर्बुधाः॥
vibudhanetragajāhihutāśanatriguṇavedabhavāraṇabāhavaḥ |
navanikharvamite vṛtivistare paridhimānamidaṃ jagadurbudhāḥ ||

The π value given above is

$$\pi \approx \frac{2827433388233}{9 \times 10^{11}} = 3.141592653592\ldots \tag{43}$$

The 13-digit number appearing in the numerator has been specified using object-numeral (*bhūta-saṅkhyā*) system, whereas the denominator is specified by word numerals.[14]

4. HISTORIOGRAPHY OF THE INCEPTION OF CALCULUS IN INDIA

4.1. Brief note on Charles Whish and his collections

Charles Matthew Whish (1794–1833), as noted earlier, was instrumental in first bringing to the notice of modern mathematical scholarship the achievements of the Kerala School through his historic paper that got posthumously published in TRAS (1934) [36]. The fact that Whish had discovered them more than a decade before the paper got published is evident from the correspondence between John Warren and George Hyne that has been noted down by the former in his *Kālasaṅkalita* [35]. It may also be mentioned here that the collection of manuscripts that Whish had made—which the author of this paper had an occasion to look at—amply demonstrates the fact that he was interested not only in astronomy and mathematics, but also in a wide variety of topics that includes vedic literature, *itihāsas* and *purāṇas*. Fortunately, these manuscripts were deposited in the Royal Asiatic Society of Great Britain and Ireland in July 1836 by his brother, and are still well preserved in the Royal Asiatic Society, London.

The personal notes (see Figure 5) found in various manuscripts in Whish's collection also reveal that during his stay in South Malabar, he had got in touch with several scholars, and read some of the Sanskrit and Malayalam texts with them. Given his abiding interest to acquire scholarship in a variety of fields by familiarizing with the culture, language, and knowledge systems of India—and also share it back with his counterparts in Europe—it is highly unfortunate that Charles Whish suffered a premature death in 1833 at the age of 38 years.[15]

14 In the *bhūta-saṅkhyā* system, *vibudha* = 33, *netra* = 2, *gaja* = 8, *ahi* = 8, *hutāśana* = 3, *tri* = 3, *guṇa* = 3, *veda* = 4, *bha* = 27, *vāraṇa* = 8, *bāhu* = 2. In word numerals, *nikharva* represents 10^{11}. Hence, *nava-nikharva* = 9×10^{11}.

15 The list of European tombs in the district of Cuddapah prepared by C. H. Mounsey in 1893 mentions: "Sacred to the memory of C. M. Whish, Esquire of the Civil Service, who departed this life on the 14th April 1833, aged 38 years."

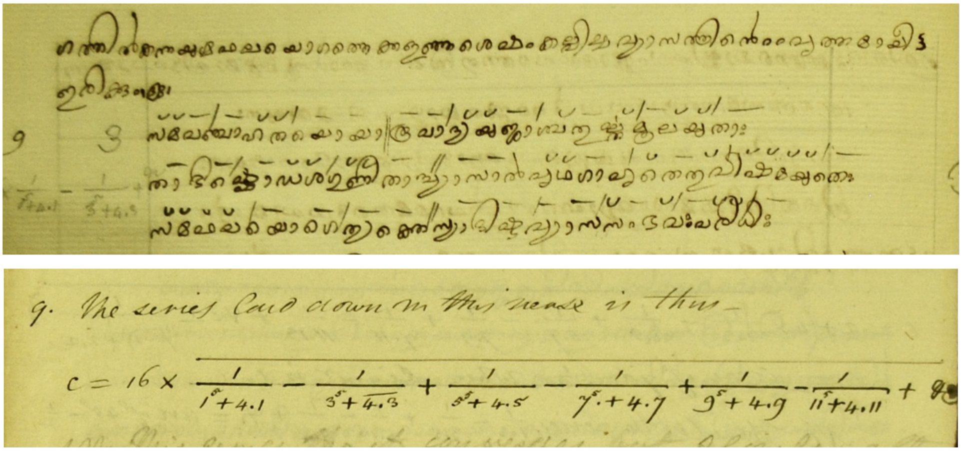

FIGURE 5
Excerpts from Whish's manuscript showing the verses in Malayalam along with his metrical markings and a portion from his mathematical notes in English (Courtesy: RAS, London).

4.2. About *Kālasaṅkalita*

Kālasaṅkalita, published in 1825 by John Warren who was the director of the Madras observatory for sometime, is a compendium of the different methods employed by the *pañcāṅga*-makers for reckoning time. The main purpose of preparing this text was to facilitate a comparison of the European and Indian chronologies, as is mentioned in the preface: "… their chief object being merely to explain the various modes according to which the Natives of India divide time, in these southern provinces, and to render their Kalendars intelligible. These may therefore be properly considered rather as instruments contrived for Chronological purposes, than as Astronomical Tracts."

It turns out that the text is useful in several other respects as well, especially from a historical perspective. Among other things, the one which is of particular interest to us in this paper is the exchange of ideas that took place among the three civil servants of the East India Company, namely, Warren, Whish, and Hyne, particularly with regard to the invention of the infinite series expansion by the "Natives."

4.3. Extracts from the exchanges between Whish, Hyne, and Warren

In the Second Memoir of *Kālasaṅkalita* on the Hindu Lunisolar year, before commencing his discussion on *śaṇku*[16] and the diurnal problems associated with it, John Warren notes:

16 The term *śaṇku* refers to a very simple contrivance, yet a powerful tool that has been extensively employed by Indian astronomers – right from the period of the *Śulvasūtras* (c. 800 BCE) – to carry out a variety of experiments related to shadow measurements.

Before entering into the resolution of the Problems which depend on the length of the Meridian shadow, it is proper to enquire …
Of their manner of resolving geometrically the ratio of the diameter to the circumference of a circle, I never saw any Indian demonstration: the common opinion, however, is that they approximate it in the manner of the ancients, by exhaustion; that is, by means of inscribed and circumscribed Polygons. However, a Native Astronomer who was a perfect stranger to European Geometry, gave me the well-known series $1 - \frac{1}{3} + \frac{1}{5} + \cdots$. *This person reduced the five first terms of the series before me, which he called Bagah Anoobanda, or Bagah Apovacha; to shew that he understood its use. This proves at least that the Hindus are not ignorant of the doctrine of series …*

This passage clearly indicates that John Warren is confronting a dilemma: on the one hand, he has met "a Native Astronomer who was a perfect stranger to European Geometry" giving the well-known series $1 - \frac{1}{3} + \frac{1}{5} + \cdots$ and, on the other hand, "he never saw any Indian demonstration" of the series. To the above passage, Warren appends a note where he mentions:

I owe the following note to Mr. Hyne's favor: "The Hindus never invented the series; it was communicated with many others, by Europeans, to some learned Natives in modern times. Mr. Whish sent a list of the various methods of demonstrating the ratio of the diameter and circumference of a Circle employed by the Hindus to the literary society, being impressed with the notion that they were the inventors. I requested him to make further inquiries, and his reply was that he had reasons to believe them entirely modern and derived from Europeans, observing that not one of those who used the Rules could demonstrate them. Indeed, the pretensions of the Hindus to such a knowledge of geometry, is too ridiculous to deserve refutation." I join in substance Mr. Hyne's opinion, but do not admit that the circumstance that none of the Sastras mentioned by Mr. Whish, who used the series could demonstrate them, would alone be conclusive.

John Warren returns to this issue in "Fragments II" attached at the end of his treatise *Kālasaṅkalita*, entitled *"On certain infinite Series collected in different parts of India, by various Gentlemen, from Native Astronomers."— Communicated by George Hyne, Esq. of the H. C.'s Medical Service*, which we reproduce below:

"My Dear Sir,
I have great pleasure in communicating the Series, to which I alluded …

$$C = 4D\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right), \tag{44}$$

$$C = \sqrt{12D^2} - \frac{\sqrt{12D^2}}{3 \cdot 3} + \frac{\sqrt{12D^2}}{3^2 \cdot 5} - \frac{\sqrt{12D^2}}{3^3 \cdot 7} + \cdots, \tag{45}$$

$$C = 2D + \frac{4D}{(2^2-1)} - \frac{4D}{(4^2-1)} + \frac{4D}{(6^2-1)} - \cdots, \tag{46}$$

$$C = 8D\left[\frac{1}{(2^2-1)} + \frac{1}{(6^2-1)} + \frac{1}{(10^2-1)} + \cdots\right], \tag{47}$$

$$C = 8D\left[\frac{1}{2} - \frac{1}{(4^2-1)} - \frac{1}{(8^2-1)} - \frac{1}{(12^2-1)} - \cdots\right], \tag{48}$$

$$C = 3D + \frac{4D}{(3^3-3)} - \frac{4D}{(5^3-5)} + \frac{4D}{(7^3-7)} - \cdots, \tag{49}$$

$$C = 16D\left(\frac{1}{1^5+4.1} - \frac{1}{3^5+4.3} + \frac{1}{5^5+4.5} - \cdots\right). \tag{50}$$

I am, my dear Sir, most sincerely, your's,

MADRAS, *17th August 1825.* *G. HYNE."*

Based on the nature of exchanges recorded by Warren in 1825, it is quite clear that:

1. Whish was convinced that the infinite series were discovered by the "Natives."
2. Hyne was convinced that the infinite series were NOT discovered by the "Natives" but was only borrowed, and that the Hindus were merely pretending as originators of the series.
3. Warren decides to go with the opinion of Hyne, though initially he felt that the latter's argument is not "conclusive."

Under such circumstances, with a lot of communication back and forth, one could only imagine how challenging it would have been for Whish[17] to swim against the current, and place on record his own understanding regarding the knowledge of the infinite series, or of their demonstration in the Indian astronomical tradition. The mere fact the paper authored by him in 1820s got accepted for publication in the 1830s posthumously, stands testimony to his courage, perseverance, assiduity, and tenacity with which he would wear down his opponents.

One of the remarkable statements in the paper of Whish that is of particular interest to us in the present context is: "A further account of the Yucti-Bhāshā, the demonstrations of the rules for the quadrature of the circle by infinite series, with the series for the sines, cosines, and their demonstrations, will be given in a separate paper." Unfortunately, Whish did not survive to publish this paper with demonstrations from *Yuktibhāṣā*, which could have silenced all those who doubted whether these series listed by them were discovered by Indians.

17 It may also be recalled that Whish was hardly 30-year old in 1825, whereas George Hyne and Warren were seniors. Warren was the director of Madras Observatory around 1805 and Hyne was a senior member of Madras Literary Society who was appointed as the first Secretary of the Committee of Public Instruction by the Madras Government.

More striking and intriguing development connected with Whish's paper than what is narrated above, is the kind of consensus that seems to have emerged among the European indologists and historians of mathematics and astronomy to undermine and suppress it for almost a hundred years since its publication in 1834. Either the work itself was not referenced in their writings, or even if it were, some of the well-established mathematicians, such as Augustus De Morgan and scholar administrators such as Charles P. Brown dismissed it—far more strongly than was done by Hyne—by castigating it as "hoax" and "forgery" [11], [6, PP. 48–49].

Not providing reference to this paper of Whish on the contributions of the Kerala School, or discussing its contents, is certainly not out of ignorance, which is perfectly understandable. But strangely it seems to be a volitional act! See, for instance, the scholarly monograph of Geroge Thibaut (in German) on Indian Astronomy, Astrology, and Mathematics [34, P. 2] which makes note of 1827 article of Whish, on the Greek origin of the Hindu Zodiac. However, it mysteriously fails to mention this 1934 paper of Whish, though the paper is germane to the subject of his discussion. We present below a clip (Figure 6) of the relevant section from Thibaut's volume, along with a concise translation (done with the help Google).

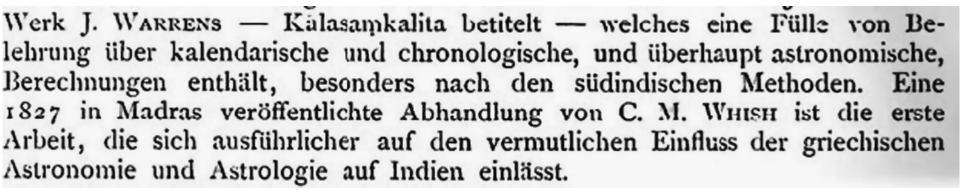
Werk J. WARRENS — Kālasaṃkalita betitelt — welches eine Fülle von Belehrung über kalendarische und chronologische, und überhaupt astronomische, Berechnungen enthält, besonders nach den südindischen Methoden. Eine 1827 in Madras veröffentlichte Abhandlung von C. M. WHISH ist die erste Arbeit, die sich ausführlicher auf den vermutlichen Einfluss der griechischen Astronomie und Astrologie auf Indien einlässt.

FIGURE 6
A clip of the relevant section from Thibaut's volume

> *J. WARREN'S work entitled Kālasaṅkalita, which contains a wealth of instruction on calendar and chronological, and generally astronomical, calculations, especially according to the South Indian methods. A treatise by C. M. WHISH, published in Madras in 1827, is the first to delve into the probable influence of Greek astronomy and astrology on India.*

Similarly, the popular translation of *Sūryasiddhānta* by Ebenezer Burgess [7, P. 174], and the review article by John Burgess of the European studies of Indian astronomy in the 18th and 19th centuries [8, PP. 746–750] do not refer to the 1934 paper of Whish while they take note of his other contributions.

Furthermore, David Eugene Smith (1860–1944), in his seminal two-volume history of mathematics completed in 1925, simply refers to the article of Whish, but does not touch upon its content except for noting that it deals with Indian values for π. Thus we find an interesting period of almost a century in European historiography where either both the title and the content, or at least the content of Whish's article remained an untouchable!

Fortunately, the references given by David Smith [31, P. 309] caught the attention of the renowned historian of Indian mathematics, Bibhutibhusan Datta, who drew attention to the various infinite series mentioned in Whish's article in an article published in 1926 [10]. This was followed a decade later by Datta's colleague, Avadesh Narayan Singh, who referred to the various manuscripts of the Kerala texts which discuss these infinite series [30]. And the next decade finally saw the publication of a series of articles by C. T. Rajagopal and his collaborators and the edition of the mathematics part of *Yuktibhāṣā* by Ramavarma Thampuran and Akhileswara Ayyar (for details, please, see [22,23,25,26,28]).

5. CONCLUDING REMARKS

It is quite evident from the above mathematical and historical discussions that the mathematicians of the Kerala School, around the 14th century, had clearly mastered the technique of handling the infinitesimal, the infinite and the notion of limit—the three pillars on which the edifice of calculus rests upon. The context and purpose for which the Kerala mathematicians developed these techniques are different from those in which they got developed in Europe a couple of centuries later. It must also be mentioned here that the Kerala mathematicians had restricted their discussions to the quadrature of a circle and certain trigonometric functions.[18] However, their mathematical formulation of the problem involving the "infinitesimally" small and summing up the "infinite" number of the resulting infinitesimal contributions, along with a clear understanding of the mathematical subtleties involved in it, are not in any way fundamentally different from the way it would be formulated or understood today.

While there were a number of European mathematicians and indologists who expressed their appreciations for the contributions made by Indians, the historiography captured in Section 4, in no uncertain terms reveals that there were many others who promulgated their views and tried to suppress the discovery of Kerala mathematicians, by brazenly discounting their work.[19] The cascading effect of it has resulted in some well-known authors producing books even in 1930s—almost a century after the publication of the Whish's historic paper—containing descriptions such as "… the Hindus may have inherited some of the bare facts of Greek science, but not the Greek critical acumen. Fools rush in where angels fear to tread [9][20] …" that are quite misleading, derailing, and damaging. It is perhaps a fitting tribute to Whish that today at least most historians of mathematics are aware of this "neglected chapter" in the history of mathematics. For this reason, the following statement by David Mumford is quite relevant [24]:

> *It is high time that the full story of Indian mathematics from Vedic times through 1600 became generally known. I am not minimizing the genius of the Greeks and*

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18 The mathematicians of Europe, however, took a different approach to the subject, by considering an arbitrary curve for analysis, and by providing formal definitions and generalized treatment to the topic.

19 The episode essentially reminds us of the important lesson: if we look through a malicious goggle, then even the genuine narratives may sound to be an elaborate hoax!

20 Quoted by A. A. K. Ayyangar in his article [3].

their wonderful invention of pure mathematics, but other peoples have been doing math in different ways, and they have often attained the same goals independently. Rigorous mathematics in the Greek style should not be seen as the only way to gain mathematical knowledge.

… the muse of mathematics can be wooed in many different ways and her secrets teased out of her. And so they were in India …

Apart from the topics discussed in the present article, several other ideas of calculus seem to have been employed by Indian astronomers in their studies related to planetary motion. For instance, one of the verses in the second chapter of *Tantrasaṅgraha* deals with the derivative of the inverse sine function.[21] We would also like to refer the reader to the literature for the very interesting proof of the sine and cosine series given in the *Yuktibhāṣā*. As has been remarked recently by Divakaran [15, p. 335] that, unlike the derivation that was given by Newton, which involved "guessing" successive terms "from their form," the *Yuktibhāṣā* approach of "integrating the difference/differential equation for sine and cosine is entirely different and very modern", which has also been briefly touched upon by Mumford in his article cited above.

For most of us who have got trained completely in the modern scheme of education, it may be hard to imagine doing mathematics without the "luxury" of expressing things "neatly" in symbolic forms. It is equally hard to think of expressing power series for trigonometric functions, derivatives of functions, and the like, purely in metrical forms. But that is how knowledge seems to have been preserved and handed down from generation to generation in India for millennia starting from Vedic age till the recent past. It only proves the point: equations may be handy but not essential; notations may be useful, but not indispensable. Formal definitions and structures are certainly valuable and helpful, but the absence of them does not inhibit or stagnate the birth and development of mathematical ideas. After all, mathematics is mathematics irrespective of how, where, and why it is practiced!

ACKNOWLEDGMENTS

The author would like to profusely thank Prof. M. D. Srinivas for useful discussions on the topic. He would like to place on record his sincere gratitude to the Ministry of Education, Government of India, for the generous support extended to carry out research activities on Indian science and technology by way of initiating the Science and Heritage Initiative (SandHI) at IIT Bombay. His thanks are also due to his doctoral student D. G. Sooryanarayan for readily and enthusiastically helping in the preparation of this article.

21 In fact, Nīlakaṇṭha ascribes this verse—dealing with the instantaneous velocity (*tātkālika-gati*) of the moon—to his teacher Dāmodara in his *Jyotirmīmāṃsā*. While the details of how Dāmodara, or someone else before him, arrived at this result is not evident to us, one thing is quite clear—the astronomers were adept at dealing with the derivatives of basic trigonometric functions.

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