Abstract

A C^*-algebra satisfies the Universal Coefficient Theorem (UCT) of Rosenberg and Schochet if it is equivalent in Kasparov's KK-theory to a commutative C^*-algebra. This paper is motivated by the problem of establishing the range of validity of the UCT, and in particular, whether the UCT holds for all nuclear C^*-algebras.

We introduce the idea of a C^*-algebra that "decomposes" over a class $\mathcal{C}$ of C^*-algebras. Roughly, this means that locally there are approximately central elements that approximately cut the C^*-algebra into two C^*-subalgebras from $\mathcal{C}$ that have well-behaved intersection. We show that if a C^*-algebra decomposes over the class of nuclear, UCT C^*-algebras, then it satisfies the UCT. The argument is based on a Mayer–Vietoris principle in the framework of controlled KK-theory; the latter was introduced by the authors in an earlier work. Nuclearity is used via Kasparov's Hilbert module version of Voiculescu's theorem, and Haagerup's theorem that nuclear C^*-algebras are amenable.

We say that a C^*-algebra has finite complexity if it is in the smallest class of C^*-algebras containing the finite-dimensional C^*-algebras, and closed under decomposability; our main result implies that all C^*-algebras in this class satisfy the UCT. The class of C^*-algebras with finite complexity is large, and comes with an ordinal-number invariant measuring the complexity level. We conjecture that a C^*-algebra of finite nuclear dimension and real rank zero has finite complexity; this (and several other related conjectures) would imply the UCT for all separable nuclear C^*-algebras. We also give new local formulations of the UCT, and some other necessary and sufficient conditions for the UCT to hold for all nuclear C^*-algebras.

Keywords. Amenable C^*-algebra, decomposition of a C^*-algebra, controlled KK-theory, Universal Coefficient Theorem

Mathematics Subject Classification (2020). Primary 19K35; Secondary 19K33, 46L05, 46L80, 46L85

Acknowledgements. We thank Wilhelm Winter for some helpful comments on a preliminary draft. We thank Claude Schochet for pointing out the work of Gray [28] on inverse limits; this reference allowed us to significantly improve the results from an earlier circulated draft of this memoir. Finally, we thank the anonymous referee for useful comments.

Funding. The authors gratefully acknowledge support from the US NSF (DMS 1564281, DMS 1700021, DMS 1901522, DMS 2000082, DMS 2247313, DMS 2247968), and the Simons Fellow Program.