Chapter 2

Reformulating the UCT

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ are respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$.

Our goal in this chapter is to recall the definition of the controlled KK-theory groups, and then to reformulate the universal coefficient theorem in these terms.

We first recall the definition of the controlled KK-theory groups from [68]; to be precise, we need the version from [68, Sections A.1 and A.2] that is specific to unital C^*-algebras. We need a definition.

Definition 2.1. Let B be a separable C^*-algebra. Choose a unitary isomorphism $\ell^2 \cong \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2$, which induces a unitary isomorphism

$$\ell^2 \otimes B \cong (\mathbb{C}^2 \otimes \ell^2 \otimes \ell^2) \otimes B$$

of Hilbert B-modules. With respect to this isomorphism, let $e \in \mathcal{L}_B$ be the projection corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \otimes 1_{\ell^2 \otimes \ell^2 \otimes B}$. We call e the *neutral projection*. A subset X of $\mathcal{L}_B$ is called *large* if every $x \in X$ is of the form $1_{\mathbb{C}^2 \otimes \ell^2} \otimes y$ for some $y \in \mathcal{L}(\ell^2 \otimes B)$ with respect to this decomposition.

Definition 2.2. Let B be a separable C^*-algebra. Let $\varepsilon > 0$, let X be a finite, large, subset of the unit ball of $\mathcal{L}_B$ and let $e \in \mathcal{L}_B$ be the neutral projection as in Definition 2.1. Let $\mathcal{P}_\varepsilon(X, B)$ consist of those projections p in $\mathcal{L}_B$ such that

(i) $p - e \in \mathcal{K}_B$; and

(ii) $\|[p, x]\| < \varepsilon$ for all $x \in X$.

Define $KK_\varepsilon(X, B)$ to be the set $\pi_0(\mathcal{P}_\varepsilon(X, B))$ of path components of $\mathcal{P}_\varepsilon(X, B)$. We write $[p] \in KK_\varepsilon(X, B)$ for the class of $p \in \mathcal{P}_\varepsilon(X, B)$.

Choose now isometries $t_1, t_2 \in \mathcal{B}(\ell^2)$ satisfying the *Cuntz relation*

$$t_1 t_1^* + t_2 t_2^* = 1,$$

and define $s_i := 1_{\mathbb{C}^2} \otimes t_i \otimes 1_{\ell^2 \otimes B} \in \mathcal{L}_B$. Define an operation on $KK_\varepsilon(X, B)$ by the *Cuntz sum*

$$[p] + [q] := [s_1 p s_1^* + s_2 q s_2^*].$$

The same proof as [68, Lemma A.4] shows that $KK_\varepsilon(X, B)$ is an abelian group, with identity element given by the class $[e]$ of the neutral projection.

We finish this section with two ancillary lemmas. The first is extremely well-known; we include an argument for completeness as we do not know a convenient reference.

Lemma 2.3. *Let a and b be elements of a unital C^*-algebra with b normal. Then, any z in the spectrum of a is contained within distance $\|a - b\|$ of the spectrum of b.*

Proof. We need to show that if z is further than $\|a - b\|$ from the spectrum of b, then $a - z$ is invertible. Indeed, in this case the continuous functional calculus implies that $\|(b - z)^{-1}\| < \|a - b\|^{-1}$. Hence,

$$\|(a - z)(b - z)^{-1} - 1\| \leq \|(a - z) - (b - z)\|\|(b - z)^{-1}\| < 1,$$

whence $(a - z)(b - z)^{-1}$ is invertible, and so $a - z$ is invertible too. ∎

Lemma 2.4. *Let B be a separable C^*-algebra, let $\varepsilon > 0$, and let X be a finite, large, subset of the unit ball of $\mathcal{L}_B$. With notation as in Definition 2.2, the group $KK_\varepsilon(X, B)$ is countable.*

Proof. As B is separable $\mathcal{K}_B$ is separable, and so the set $\mathcal{P}_\varepsilon(X, B)$ is also separable. Let S be a countable dense subset of $\mathcal{P}_\varepsilon(X, B)$. It suffices to show that the map $S \to KK_\varepsilon(X, B)$ defined by $p \mapsto [p]$ is surjective.

Let $p \in \mathcal{P}_\varepsilon(X, B)$ be arbitrary, and define

$$\delta := \min\left\{\frac{1}{4}(\varepsilon - \max_{x \in X} \|[p, x]\|), \frac{1}{2}\right\}.$$

Let $q \in S$ be such that $\|p - q\| < \delta$, and let $p_t := (1 - t)p + tq$ for $t \in [0, 1]$. Then, for each $t \in [0, 1]$, $\|p_t - p\| < \delta$, so Lemma 2.3 and that p_t is a positive contraction implies that the spectrum p_t is contained in $[0, \delta) \cup (1 - \delta, 1]$. Let χ be the characteristic function of $(\frac{1}{2}, \infty)$. Then, $\|\chi(p_t) - p_t\| < \delta$ for all t, whence $\|\chi(p_t) - p\| < 2\delta$ for all t, from which it follows that $\|[\chi(p_t), x]\| < \varepsilon$ for all t and all $x \in X$. As $p_t - e \in \mathcal{K}_B$ for all t, it follows from the fact that $\mathcal{K}_B$ is an ideal in $\mathcal{L}_B$ that $\chi(p_t) - e \in \mathcal{K}_B$ too. Hence, $(\chi(p_t))_{t \in [0,1]}$ is a path connecting p and q within $\mathcal{P}_\varepsilon(X, B)$ so $[p] = [q]$, and we are done. ∎

2.1 The general case

We need a special class of representations on Hilbert B-modules, essentially taken from work of Thomsen [62, Definition 2.2] (see also [68, Definition A.11]). We do not need the details of the definition below, and only include it for completeness; all we really need are the facts about existence of such representations in Lemma 2.6 below.

Definition 2.5. Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. A representation $\sigma : A \to \mathcal{L}_B$ is *unitally absorbing* if for any unital completely positive map $\phi : A \to \mathcal{L}_B$ there exists a sequence of isometries (v_n) in $\mathcal{L}_B$

such that $\|v_n^*\sigma(a)v_n - \phi(a)\| \to 0$ as $n \to \infty$, and such that $v_n^*\sigma(a)v_n - \phi(a) \in \mathcal{K}_B$ for all $n \in \mathbb{N}$.

For a representation $\sigma : A \to \mathcal{L}_B = \mathcal{L}(H_B)$, let $\sigma^\infty : A \to \mathcal{L}(H_B^{\oplus\infty})$ be its infinite amplification, which we identify with a representation $\sigma^\infty : A \to \mathcal{L}_B$ via a choice of unitary isomorphism $(\ell^2)^{\oplus\infty} \cong \ell^2$ as in the string of identifications below

$$\mathcal{L}(H_B^{\oplus\infty}) = \mathcal{L}((\ell^2 \otimes B)^{\oplus\infty}) = \mathcal{L}((\ell^2)^{\oplus\infty} \otimes B) \cong \mathcal{L}(\ell^2 \otimes B) = \mathcal{L}_B$$

(all of the identifications labeled "=" are canonical). A unital representation $\pi : A \to \mathcal{L}_B$ is *strongly unitally absorbing* if there is a unitally absorbing representation

$$\sigma : A \to \mathcal{L}_B$$

such that $\pi = \sigma^{\oplus\infty}$.

Note that a (strongly) unitally absorbing representation is faithful. The following result is essentially due to Thomsen and Kasparov. Our main use of part (ii) occurs much later in the paper.

Lemma 2.6. *Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. Then,*

(i) *There exists a strongly unitally absorbing representation $\pi : A \to \mathcal{L}_B$.*

(ii) *Assume in addition that A or B is nuclear. Let $\sigma : A \to \mathcal{B}(\ell^2)$ be any faithful unital representation, let $\iota : \mathcal{B}(\ell^2) \to \mathcal{L}_B$ be the canonical inclusion arising from the decomposition $H_B = \ell^2 \otimes B$, and let $\pi : A \to \mathcal{L}_B$ be the infinite amplification of $\iota \circ \sigma$. Then, π is strongly unitally absorbing.*

Proof. For part (i), Thomsen shows in [62, Theorem 2.4] that a unitally absorbing representation $\sigma : A \to \mathcal{L}_B$ exists under the given hypotheses. Its infinite amplification π is then strongly unitally absorbing.

For part (ii), note first that identifying $(\iota \circ \sigma)^\infty$ with $(\iota \circ (\sigma^{\oplus\infty}))^\infty$ we may assume σ is the infinite amplification of some faithful unital representation $A \to \mathcal{B}(\ell^2)$. Having made this assumption, note that $\sigma(A) \cap \mathcal{K}(\ell^2) = \{0\}$. In [40, Theorem 5], Kasparov shows that if A is a separable, unital C^*-algebra and $\sigma : A \to \mathcal{B}(\ell^2)$ is a faithful representation such that $\sigma(A) \cap \mathcal{K}(\ell^2) = \{0\}$, and moreover if either A or B is nuclear, then the composition $\iota \circ \sigma$ satisfies the condition Thomsen gives in [62, Theorem 2.1, condition (4)]. Comparing [62, Theorem 2.1] and Definition 2.5, we see that $\iota \circ \sigma$ is unitally absorbing. Hence, $\pi = (\iota \circ \sigma)^{\oplus\infty}$ is strongly unitally absorbing. ∎

The following corollary is immediate from part (ii) of Lemma 2.6.

Corollary 2.7. *Let A be a separable, unital, nuclear C^*-algebra, and let B be a separable C^*-algebra. Then, there exists a strongly unitally absorbing representation*

$\pi : A \to \mathcal{L}_B$ such that the restriction of π to any unital, nuclear C^*-subalgebra of A is also strongly unitally absorbing. ■

Remark 2.8. Corollary 2.7 is one of the two places nuclearity is used in the proof of Theorem 1.2, so it would be interesting to establish the corollary under some weaker assumption than nuclearity. The following observation shows that the method we used to establish Corollary 2.7 cannot extend beyond the nuclear case, however.

Let A be a separable, unital C^*-algebra, and let $A = B$. Let $\sigma : A \to \mathcal{B}(\ell^2)$ be a unital representation, and let $\pi := \iota \circ \sigma : A \to \mathcal{L}_A$ be as in Lemma 2.6 (ii). We claim that if π is unitally absorbing, then A is nuclear[1]. Let $\phi : A \to \mathcal{L}_A$ be the $*$-homomorphism $a \mapsto 1_{\ell^2} \otimes a$. If π is unitally absorbing then for any ε and finite subset X of A there is an isometry $v \in \mathcal{L}_A$ such that $\|v^*\pi(a)v - \phi(a)\| < \varepsilon$ for all $a \in X$. For each n, let $p_n \in \mathcal{B}(\ell^2)$ be the orthogonal projection onto $\ell^2(\{1, \dots, n\})$, and let $q_n := p_n \otimes 1_A \in \mathcal{L}_A$. Note that $q_1\mathcal{L}_A q_1$ identifies canonically with A, and up to this identification $q_1\phi(a)q_1 = a$ for all $a \in A$, so in particular $\|q_1 v^*\pi(a)vq_1 - a\| < \varepsilon$ for all $a \in X$. As (q_n) converges strictly to the identity in $\mathcal{L}_A$, and as $q_1 v \in \mathcal{K}_A$, we have moreover that $q_1 v^* q_n \pi(a) q_n v q_1$ converges in norm to $q_1 v^* \pi(a) v^* q_1$, so there is n such that $\|q_1 v^* q_n \pi_n(a) q_n v q_1 - a\| < \varepsilon$ for all $a \in X$. We thus have ucp maps

$$A \xrightarrow{a \mapsto q_n \pi(a) q_n} q_n(\mathcal{B}(\ell^2) \otimes 1_A)q_n \cong M_n(\mathbb{C}) \xrightarrow{b \mapsto q_1 v^* b v q_1} A$$

whose composition agrees with the identity on X to within ε error. As X and ε were arbitrary, this implies nuclearity of A (see for example [10, Chapter 2]).

To state the main result of [68], we need some more definitions.

Definition 2.9. Let A be a separable, unital C^*-algebra, and let B be a separable C^*-algebra. A representation $\pi : A \to \mathcal{L}_B$ is *large* if there is a unitally absorbing representation $\sigma : A \to \mathcal{L}_B$ such that with respect to the choice of isomorphism

$$\ell^2 \otimes B \cong \mathbb{C}^2 \otimes \ell^2 \otimes \ell^2 \otimes B$$

of Definition 2.1, we have $\pi(a) = 1_{\mathbb{C}^2 \otimes \ell^2} \otimes \sigma(a)$ for all $a \in A$.

Lemma 2.6 (i) implies that large representations exist for any (separable) A and B. Note that if π is large in the sense of Definition 2.9 then for any $X \subseteq A$, the subset $\pi(X) \subseteq \mathcal{L}_B$ is large in the sense of Definition 2.1. In particular, if we identify X with $\pi(X)$, the group $KK_\varepsilon(X, B)$ of Definition 2.2 makes sense.

Definition 2.10. Let C be a C^*-algebra, and let $\mathcal{X}_C$ consist of all pairs of the form (X, ε) where X is a finite subset of C_1, and $\varepsilon > 0$. Put a partial order on $\mathcal{X}_C$ by

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[1] The following argument is inspired by [60, Théorème 1.5, Definition 1.6, and Remarque 1.7].

stipulating that $(X, \varepsilon) \leq (Y, \delta)$ if $\delta \leq \varepsilon$, and if for all $x \in X$ there exists $y \in Y$ with $\|x - y\| \leq \frac{1}{2}(\varepsilon - \delta)$.

A *good approximation* of C is a cofinal sequence[2] $((X_n, \varepsilon_n))_{n=1}^{\infty}$ of elements of $\mathcal{X}_C$.

Note that if $X \subseteq Y$ and $\delta \leq \varepsilon$, then $(X, \varepsilon) \leq (Y, \delta)$; in particular, this implies that $\mathcal{X}_C$ is a directed set. Note also that good approximations exist if and only if C is separable: if (ε_n) is a decreasing sequence that tends to zero, and (X_n) is an increasing sequence with dense union in C_1, then $((X_n, \varepsilon_n))_{n=1}^{\infty}$ is a good approximation; and if $((X_n, \varepsilon_n))_{n=1}^{\infty}$ is a good approximation, then $\bigcup_{n=1}^{\infty} X_n$ is a countable dense subset of C_1.

Definition 2.11. Let B be a separable C^*-algebra, and let $\mathcal{X}_{\mathcal{L}_B}$ be the directed set from Definition 2.10 above for the C^*-algebra $\mathcal{L}_B$. If $(X, \varepsilon) \leq (Y, \delta)$ and X and Y are both large in the sense of Definition 2.1, then with notation as in Definition 2.2 there is an inclusion

$$\mathcal{P}_\delta(Y, B) \subseteq \mathcal{P}_\varepsilon(X, B). \tag{2.1}$$

We call the canonical map

$$KK_\delta(Y, B) \to KK_\varepsilon(X, B)$$

induced by the inclusion in line (2.1) above a *forget control map*.

We now briefly recall some terminology from homological algebra; see for example [66, Section 3.5] or [58, Section 3] for more background on this material[3]. An *inverse system* of abelian groups consists of a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\phi_n} A_n \xrightarrow{\phi_{n-1}} A_{n-1} \xrightarrow{\phi_{n-2}} \cdots \xrightarrow{\phi_2} A_2 \xrightarrow{\phi_1} A_1.$$

Associated to such a system is a homomorphism

$$\phi : \prod_{n \in \mathbb{N}} A_n \to \prod_{n \in \mathbb{N}} A_n, \quad (a_n) \mapsto (\phi_n(a_{n+1})).$$

The *inverse limit*, denoted $\varprojlim A_n$, is defined to be the kernel of $\mathrm{id} - \phi$, and the $\varprojlim^1$-*group*, denoted $\varprojlim^1 A_n$, is defined to be the cokernel of $\mathrm{id} - \phi$. Note that if $m \geq n$, there is a canonical homomorphism $A_m \to A_n$ defined as $\phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{m-1}$. The inverse system satisfies the *Mittag–Leffler condition* if for any n there is $N \geq n$

[2]A sequence $(s_n)_{n=1}^{\infty}$ in a partially ordered set S is *cofinal* if $s_1 \leq s_2 \leq s_3 \leq \cdots$ and if for all $s \in S$ there is n such that $s \leq s_n$.

[3]Readers interested in a more sophisticated and general treatment can also see [38].

such that for all $m \geq N$, the image of the canonical map $A_m \to A_n$ equals the image of the canonical map $A_N \to A_n$.

Proposition 2.12. *Let (A_n) be an inverse system of abelian groups. If (A_n) satisfies the Mittag–Leffler condition, then $\varprojlim^1 A_n = 0$. Conversely, if $\varprojlim^1 A_n = 0$ and each A_n is countable, then the inverse system satisfies the Mittag–Leffler condition.*

Proof. It is well-known that the Mittag–Leffler condition implies vanishing of

$$\varprojlim{}^1 A_n = 0;$$

see for example [66, Proposition 3.5.7]. The converse in the case of countable groups follows from [28, Proposition on page 242]. ∎

Now, let A be a separable, unital C^*-algebra, let B be a separable C^*-algebra, and use a large representation $\pi : A \to \mathcal{L}_B$ (see Definition 2.9) to identify A with a C^*-subalgebra of $\mathcal{L}_B$. Let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a good approximation of A as in Definition 2.10, so the forget control maps of Definition 2.11 form an inverse system

$$\cdots \to KK_{\varepsilon_n}(X_n, B) \to KK_{\varepsilon_{n-1}}(X_{n-1}, B) \to \cdots \to KK_{\varepsilon_1}(X_1, B)$$

from which we define $\varprojlim KK_{\varepsilon_n}(X_n, B)$ and $\varprojlim^1 KK_{\varepsilon_n}(X_n, B)$ as above.

The following is [68, Proposition A.10].

Theorem 2.13. *Let A and B be separable C^*-algebras with A unital. Let*

$$\pi : A \to \mathcal{L}_B$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_B$. Let $((X_n, \varepsilon_n))_{n=1}^{\infty}$ be a good approximation for A. Then, there is a short exact sequence

$$0 \to \varprojlim{}^1 KK_{\varepsilon_n}(X_n, SB) \to KK(A, B) \to \varprojlim KK_{\varepsilon_n}(X_n, B) \to 0. \qquad ∎$$

We are now almost ready to state and prove our reformulation of the UCT. It will be convenient to use the following well-known reformulation of the UCT; see [55, p. 457] or [60, Proposition 5.3] for a proof.

Theorem 2.14. *A separable C^*-algebra A satisfies the UCT if and only if for any separable C^*-algebra B such that $K_*(B) = 0$ we have that*

$$KK(A, B) = 0. \qquad ∎$$

Theorem 2.15. *Let A be a separable C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let B be a separable C^*-algebra with $K_*(B) = 0$. Let*

$$\pi : A \to \mathcal{L}_{SB}$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, for any (X, γ) in the set $\mathcal{X}_A$ of Definition 2.10, there is $(Z, \varepsilon) \in \mathcal{X}_A$ with $(X, \gamma) \leq (Z, \varepsilon)$ and so that the forget control map

$$KK_\varepsilon(Z, SB) \to KK_\gamma(X, SB)$$

of Definition 2.11 is zero.

Proof. Assume first that A satisfies condition (i), and let X, ε, B and π be as in condition (ii). Let $((X_n, \varepsilon_n))_{n=1}^\infty$ be a good approximation of A with $X_1 = X$ and $\varepsilon_1 = \gamma$. As A satisfies the UCT and as $K_*(B) = 0$, we have $KK(A, B) = 0$. Hence, using Theorem 2.13, $\varprojlim^1 KK_{\varepsilon_n}(X_n, SB) = 0$. Lemma 2.4 implies that the groups $KK_{\varepsilon_n}(X_n, SB)$ are all countable, whence by Proposition 2.12, the inverse system $(KK_{\varepsilon_n}(X_n, SB))_{n=1}^\infty$ satisfies the Mittag–Leffler condition. On the other hand, as A satisfies the UCT and $K_*(SB) = 0$, we have $KK(A, SB) = 0$ by Theorem 2.14. Hence, by Theorem 2.13 again, $\varprojlim KK_{\varepsilon_n}(X_n, SB) = 0$, whence the definition of the inverse limit implies that for any n,

$$\bigcap_{m \geq n} \operatorname{Image}\big(KK_{\varepsilon_m}(X_m, SB) \to KK_{\varepsilon_n}(X_n, SB)\big) = 0.$$

The Mittag–Leffler condition implies that there is $N \geq n$ such that

$$\bigcap_{m \geq n} \operatorname{Image}\big(KK_{\varepsilon_m}(X_m, SB) \to KK_{\varepsilon_n}(X_n, SB)\big)$$
$$= \operatorname{Image}\big(KK_{\varepsilon_N}(X_N, SB) \to KK_{\varepsilon_n}(X_n, SB)\big)$$

so we may conclude that the forget control map

$$KK_{\varepsilon_N}(X_N, SB) \to KK_{\varepsilon_n}(X_n, SB)$$

is zero. In particular, such an N exists for $n = 1$, and we may set $Z = X_N$ and $\varepsilon = \varepsilon_N$.

Conversely, say A satisfies condition (ii). Using Theorem 2.14, it suffices to show that if B is a separable C^*-algebra with $K_*(B) = 0$, then $KK(A, B) = 0$. Let $\pi_2 : A \to \mathcal{L}_{S^2B}$ (respectively, $\pi_3 : A \to \mathcal{L}_{S^3B}$) be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{S^2B}$ (respectively, $\mathcal{L}_{S^3B}$). Using condition (ii) we may construct a good approximation $((X_n, \varepsilon_n))_{n=1}^\infty$ for A in the sense of Definition 2.10 such that for any n the maps

$$KK_{\varepsilon_{n+1}}(X_{n+1}, S^3B) \to KK_{\varepsilon_n}(X_n, S^3B) \tag{2.2}$$

and

$$KK_{\varepsilon_{n+1}}(X_{n+1}, S^2B) \to KK_{\varepsilon_n}(X_n, S^2B) \tag{2.3}$$

are zero. As the maps in line (2.2) are all zero, the inverse system

$$(KK_{\varepsilon_n}(X_n, S^3B))_{n=1}^{\infty}$$

satisfies the Mittag–Leffler condition, whence by Proposition 2.12 we have that

$$\underleftarrow{\lim}^1 KK_{\varepsilon_n}(X_n, S^3B) = 0.$$

On the other hand, the fact that the maps in line (2.3) are all zero and the definition of the inverse limit immediately imply that $\underleftarrow{\lim} KK_{\varepsilon_n}(X_n, S^2B) = 0$. Hence, in the short exact sequence

$$0 \to \underleftarrow{\lim}^1 KK_{\varepsilon_n}(X_n, S^3B) \to KK(A, S^2B) \to \underleftarrow{\lim}\, KK_{\varepsilon_n}(X_n, S^2B) \to 0$$

from Theorem 2.13 the left and right groups are zero, whence $KK(A, S^2B) = 0$. Hence, by Bott periodicity, $KK(A, B) = 0$ as desired. ■

We include the following remark as the comparison to the existing literature might help orient some readers; it also gives a sense of why Corollary 2.7 is useful (our main use of that corollary will come later in the paper).

Remark 2.16. Theorem 2.15 can be used to deduce a weak version of a theorem of Dadarlat [21, Theorem 1.1]. Dadarlat shows that if A is a separable nuclear C^*-algebra such for any finite subset X of A and any $\varepsilon > 0$, one has a UCT subalgebra C of A such that $x \in_{\varepsilon} C$ for all $x \in X$, then A satisfies the UCT. Theorem 1.2 implies the special case of Dadarlat's theorem where the subalgebras C can also be taken nuclear.

To see this, note first that as a C^*-algebra satisfies the UCT (respectively, is nuclear) if and only if its unitization satisfies the UCT (respectively, is nuclear) by [55, Proposition 2.3 (a)] (respectively, by [10, Exercise 2.3.5]), we may assume that A is unital. We aim to establish the condition in Theorem 2.15 (ii). Let then B be a separable C^*-algebra with $K_*(B) = 0$. Using Corollary 2.7, there exists a large representation $\pi : A \to \mathcal{L}_{SB}$ such that the restriction of π to any unital nuclear C^*-subalgebra of A is also large. Let X be a finite subset of A_1, and let $\varepsilon > 0$. Let C be a nuclear, unital, UCT C^*-subalgebra of A such that $x \in_{\varepsilon/5} C$ for all $x \in X$. Let X' be a finite subset of C_1 such that for each $x \in X$ there is $x' \in X'$ such that

$$\|x - x'\| < 2\varepsilon/5.$$

Then, the forget control map

$$KK_{\varepsilon/5}(X', SB) \to KK_{\varepsilon}(X, B) \tag{2.4}$$

of Definition 2.11 is defined. As C satisfies the UCT, and as the restriction of π to C is also large, condition (ii) from Theorem 2.15 gives a finite subset Y of C_1 and $\delta > 0$ such that the forget control map

$$KK_\delta(Y, SB) \to KK_{\varepsilon/5}(X', SB) \tag{2.5}$$

is defined and zero. Composing the forget control maps in lines (2.4) and (2.5), we have established the condition from Theorem 2.15 (ii) for A, and are done.

It would be interesting if one could use these techniques to recover Dadarlat's theorem without the extra nuclearity assumption on the UCT subalgebras. This would seem to require better control over the representations involved; however, compare Remark 2.8 above.

2.2 The nuclear case

In this section, we prove a stronger version of Theorem 2.15 in the special case that the C^*-algebra A is nuclear. The key ingredient for this is an averaging argument due to Christensen, Sinclair, Smith, White, and Winter [17, Section 3], which in turn relies on Haagerup's theorem [33] that nuclear C^*-algebras are amenable.

Let us recall some terminology about bimodules.

Definition 2.17. Let A be a unital C^*-algebra. An *A-bimodule* is a Banach space E equipped with left and right module actions of A such that $1_A e = e 1_A = e$ for all $e \in E$, and such that $\|ae\|_E \leq \|a\|_A \|e\|_E$ and $\|ea\|_E \leq \|e\|_E \|a\|_A$ for all $a \in A$ and $e \in E$.

The following reformulation of nuclearity is implicit in [17, Section 3]; the reader is encouraged to see that reference for further background.

Lemma 2.18. *Let A be a unital C^*-algebra. Then, the following are equivalent:*

(i) *A is nuclear;*

(ii) *for any $\varepsilon > 0$ and any finite subset X of A, there exist contractions*

$$a_1, \ldots, a_n \in A$$

and scalars $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^n t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^n t_i a_i a_i^* \right\|_A < \varepsilon,$$

and such that for any A-bimodule E, any $e \in E_1$, and any $x \in X$,

$$\left\| x \left(\sum_{i=1}^n t_i a_i e a_i^* \right) - \left(\sum_{i=1}^n t_i a_i e a_i^* \right) x \right\|_E < \varepsilon. \tag{2.6}$$

Proof. We will need to recall the projective tensor product of Banach spaces. Let E and F be (complex) Banach spaces, and let $E \odot F$ denote their algebraic tensor product (over $\mathbb{C}$). The *projective norm* of $g \in E \odot F$ is defined by

$$\|g\| := \inf \sum_{i=1}^{n} \|e_i\|_E \|f_i\|_F, \tag{2.7}$$

where the infimum is taken over all ways of writing g as a sum $\sum_{i=1}^{n} e_i \otimes f_i$ of elementary tensors. The *projective tensor product* of E and F, denoted $E \mathbin{\hat{\otimes}} F$, is the completion of $E \odot F$ for the projective norm. If A is a C^*-algebra, we make $A \mathbin{\hat{\otimes}} A$ into an A-A-bimodule via the actions defined on elementary tensors by

$$a(b \otimes c) := ab \otimes c \quad \text{and} \quad (b \otimes c)a := b \otimes ca. \tag{2.8}$$

Now, it is shown in [17, Lemma 3.1][4] that a unital C^*-algebra is nuclear if and only if the following holds: "for any $\varepsilon > 0$ and any finite subset X of A, there exist contractions $a_1, \ldots, a_n \in A$ and scalars $t_1, \ldots, t_n \in [0, 1]$ such that $\sum_{i=1}^{n} t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^{n} t_i a_i a_i^* \right\|_A < \varepsilon,$$

and such that

$$\left\| x \left(\sum_{i=1}^{n} t_i a_i \otimes a_i^* \right) - \left(\sum_{i=1}^{n} t_i a_i \otimes a_i^* \right) x \right\|_{A \hat{\otimes} A} < \varepsilon \tag{2.9}$$

for all $x \in X$." For the sake of this proof, let us call this the "CSSWW" condition. It suffices for us to show that condition (ii) is equivalent to the CSSWW condition.

First assume A satisfies condition (ii) above. Then, taking $E = A \mathbin{\hat{\otimes}} A$ and $e = 1_A \otimes 1_A$ shows that A satisfies the CSSWW condition. Conversely, say A satisfies the CSSWW condition. Let X be a finite subset of A and let $\varepsilon > 0$, and let $a_1, \ldots, a_n$ and $t_1, \ldots, t_n$ satisfy the properties in the CSSWW condition with respect to this X and ε. Let E be an A-bimodule, and $e \in E_1$. Consider the map

$$\pi : A \odot A \to E, \quad a \otimes b \mapsto aeb$$

from the algebraic tensor product (over $\mathbb{C}$) of A with itself to E. Using the definition of the projective tensor norm (line (2.7) above), it is straightforward to check that π

[4] This is based on several deep ingredients: the key points are the result of Connes [20, Corollary 2] that amenability for a C^*-algebra implies nuclearity; the converse to this due to Haagerup [33, Theorem 3.1]; and Johnson's foundational work on amenability and virtual diagonals [39, Section 1].

is contractive for that norm, whence it extends to a contractive linear map

$$\pi : A \widehat{\otimes} A \to E.$$

Moreover, the extended map π is clearly an A-bimodule map for the bimodule structure on $A \widehat{\otimes} A$ defined in line (2.8). Applying π to the expression inside the norm in line (2.9) therefore implies the inequality in line (2.6), so we are done. ■

Remark 2.19. We will only need to apply Lemma 2.18 in the special case that the bimodule E in part (ii) is a C^*-algebra containing A as a unital C^*-subalgebra, with the bimodule actions defined by left and right multiplication. The corresponding, formally weaker, variant of condition (ii) still implies nuclearity, as we now sketch[5]. Let A be a unital C^*-algebra satisfying the variant of condition (ii) from Lemma 2.18, where E is a C^*-algebra containing A as a unital C^*-subalgebra. Let $\pi : A \to \mathcal{B}(H)$ be an arbitrary unital representation, which we use to make $\mathcal{B}(H)$ an A-bimodule. Let I be the directed set consisting of all pairs $i = (X, \varepsilon)$ where X is a finite subset of A, and $\varepsilon > 0$, and where $(X, \varepsilon) \leq (Y, \delta)$ if $X \subseteq Y$ and $\delta \leq \varepsilon$. For each $i = (X, \varepsilon) \in I$, let $a_1^{(i)}, \dots, a_{n_i}^{(i)}$ and $t_1^{(i)}, \dots, t_{n_i}^{(i)}$ have the properties in Lemma 2.18 (ii). For each i, define a ccp map

$$\phi_i : \mathcal{B}(H) \to \mathcal{B}(H), \quad b \mapsto \sum_{j=1}^{n_i} t_j^{(i)} \pi(a_j^{(i)}) b \pi(a_j^{(i)})^*,$$

and let $\phi : \mathcal{B}(H) \to \mathcal{B}(H)$ be any point-ultraweak limit point of the net (ϕ_i) (such exists by [10, Theorem 1.3.7], for example). Then, one checks that ϕ is a conditional expectation from $\mathcal{B}(H)$ onto $\pi(A)'$, whence the latter is injective. As π was arbitrary, this implies that A is nuclear; indeed, applying this to the universal representation π implies that $\pi(A)'$ is injective, whence

$$A^{**} = \pi(A)''$$

is injective by [8, Theorem IV.2.2.7], whence A is nuclear by the main result of [16].

Variants of the next lemma we need are well-known; see for example the lemma on page 332 of [3], which we could have used for a purely qualitative version. For the sake of concreteness, we give a quantitative[6] version.

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[5]This also gives an approach to the theorem of Connes that amenable C^*-algebras are nuclear that is maybe slightly more direct than the original argument from [20, Corollary 2]. However, it still factors through the theorem that injective von Neumann algebras are semi-discrete (see [19, Theorem 6] for the case of factors, and [65] for the general case), so cannot really be said to be genuinely simpler.

[6]The estimate it gives is optimal in some sense; to see this consider $C = M_2(\mathbb{C})$, $x = \left(\begin{smallmatrix} \delta & 0 \\ 0 & 1-\delta \end{smallmatrix}\right)$, and $c = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Lemma 2.20. *Let $\delta \in [0, \frac{1}{2})$, and let x be a self-adjoint element in a C^*-algebra C with spectrum that does not intersect the interval $(\delta, 1-\delta)$. Let χ be the characteristic function of $(\frac{1}{2}, \infty)$. Then, for any $c \in C$,*

$$\|[\chi(x), c]\| \leq \frac{1}{1-2\delta}\|[x, c]\|.$$

Proof. Let $N > \|x\|$. Let γ be the positively oriented rectangular contour in the complex plane with vertices at $\frac{1}{2} \pm iN$, and $2N \pm iN$. Then, by the holomorphic functional calculus, $\chi(x) = \frac{1}{2\pi i}\int_\gamma (z-x)^{-1}dz$. Hence, for any $c \in C$, $[\chi(x), c] = \frac{1}{2\pi i}\int_\gamma [(z-x)^{-1}, c]dz$. Applying the formula

$$[(z-x)^{-1}, c] = (z-x)^{-1}[c, x](z-x)^{-1}$$

and estimating gives

$$\|[\chi(x), c]\| \leq \frac{\|[c, x]\|}{2\pi}\int_\gamma \|(z-x)^{-1}\|^2 d|z|. \tag{2.10}$$

Let γ_1 be the side of γ described by $\{\frac{1}{2} + it \mid -N \leq t \leq N\}$, and let γ_2 be the union of the other three sides. Then, for z in the image of γ_2, the continuous functional calculus implies that

$$\|(z-x)^{-1}\| \leq (N - \|x\|)^{-1}.$$

As the length of γ_2 is $4N$, we thus see that

$$\int_{\gamma_2} \|(z-x)^{-1}\|^2 \|d|z| \leq \frac{4N}{(N-\|x\|)^2}. \tag{2.11}$$

On the other hand, for $z = \frac{1}{2} + it$ in the image of γ_1, the continuous functional calculus gives $\|(z-x)^{-1}\| \leq ((\frac{1}{2}-\delta)^2 + t^2)^{-1/2}$, whence

$$\begin{aligned}\int_{\gamma_2} \|(z-x)^{-1}\|^2 d|z| &\leq \int_{-N}^{N} \frac{1}{(\frac{1}{2}-\delta)^2 + t^2}dt \\ &\leq \int_{-\infty}^{\infty} \frac{1}{(\frac{1}{2}-\delta)^2 + t^2}dt = \frac{\pi}{\frac{1}{2}-\delta}.\end{aligned} \tag{2.12}$$

Combining lines (2.10), (2.11), and (2.12) we get

$$\|[\chi(x), c]\| \leq \frac{\|[c, x]\|}{2\pi}\left(\frac{4N}{(N-\|x\|)^2} + \frac{\pi}{\frac{1}{2}-\delta}\right).$$

Letting $N \to \infty$ gives $\|[\chi(x), c]\| \leq \frac{\|[c,x]\|}{1-2\delta}$, which is the claimed estimate. ∎

The following lemma is our key application of Lemma 2.18.

Lemma 2.21. *Let $\varepsilon \in (0,1)$. Let B be a separable C^*-algebra, and let A be a separable, unital, nuclear C^*-algebra. Let $\pi : A \to \mathcal{L}_{SB}$ be a large representation (see Definition 2.9), and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$.*

Let X be a finite subset of A_1, and let (Y,δ) be an element of the set $\mathcal{X}_A$ of Definition 2.10 such that $(X,\varepsilon) \leq (Y,\delta)$. Then, there exists a finite subset Z of A_1 containing X and a homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z,B) \to KK_\delta(Y,B)$$

such that the following diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z,B) & & \\ {\scriptstyle\phi_*}\downarrow & \searrow & \\ KK_\delta(Y,B) & \longrightarrow & KK_\varepsilon(X,B) \end{array}$$

(where the unlabeled maps are forget control maps as in Definition 2.11) commutes.

Proof. Let X, Y, and δ be as in the statement. If $\delta \geq \varepsilon/8$, we may just take $Z = Y$ and ϕ_* the forget control map. Assume then that $\delta < \varepsilon/8$. According to Lemma 2.18 there exists contractions $a_1,\dots,a_n \in A$ and $t_1,\dots,t_n \in [0,1]$ such that $\sum_{i=1}^n t_i = 1$, such that

$$\left\| 1_A - \sum_{i=1}^n t_i a_i a_i^* \right\|_A < \delta/4,$$

and such that for all $y \in Y$ and b in the unit ball of $\mathcal{L}_B$,

$$\left\| y\left(\sum_{i=1}^n t_i a_i b a_i^*\right) - \left(\sum_{i=1}^n t_i a_i b a_i^*\right) y \right\|_{\mathcal{L}_{SB}} < \delta/4. \tag{2.13}$$

We set $Z := X \cup \{a_1^*,\dots,a_n^*\}$, and claim this works.

Let $p \in \mathcal{P}_{\varepsilon/8}(Z,B)$, let $e \in \mathcal{L}_B$ be the neutral projection (see Definition 2.1), and define

$$\alpha(p) := \sum_{i=1}^n t_i a_i p a_i^* + \left(e - \sum_{i=1}^n t_i a_i e a_i^*\right) \in \mathcal{L}_B.$$

As the representation is large, we may use the fixed isomorphism $\ell^2 \otimes B \cong \mathbb{C}^2 \otimes \ell^2 \otimes B$ to identify $\mathcal{L}_B$ with $M_2(\mathcal{L}_B)$ and have that with respect to this identification, operators in A are diagonal matrices, and $e = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. In particular, e commutes with all the a_i, and so we have

$$\begin{aligned} \|p - \alpha(p)\| &\leq \left\| \left(1 - \sum_{i=1}^n t_i a_i a_i^*\right) p \right\| + \sum_{i=1}^n t_i \|a_i [p, a_i^*]\| + \left\| \left(1 - \sum_{i=1}^n t_i a_i a_i^*\right) e \right\| \\ &< \frac{\delta}{4} + \frac{\varepsilon}{8} + \frac{\delta}{4}. \end{aligned} \tag{2.14}$$

As $\delta < \varepsilon/8$ and as $\varepsilon < 1$, we see that $\|p - \alpha(p)\| < \frac{1}{4}$. As p is a projection, Lemma 2.3 implies that

$$\text{spectrum}(\alpha(p)) \cap (1/4, 3/4) = \varnothing. \tag{2.15}$$

Let χ be the characteristic function of $(\frac{1}{2}, \infty)$, so χ is continuous on the spectrum of $\alpha(p)$ and we may define $\phi(p) := \chi(\alpha(p))$. The rest of the proof will be spent showing that the formula $[p] \mapsto [\phi(p)]$ defines a homomorphism

$$\phi_* : KK_{\varepsilon/6}(Z, B) \to KK_\delta(Y, B)$$

with the claimed properties.

We first claim that if $p \in \mathcal{P}_{\varepsilon/8}(Z, B)$, then $\phi(p)$ is in $\mathcal{P}_\delta(Y, B)$. Note first that

$$\alpha(p) - e = \sum_{i=1}^{n} t_i a_i (p - e) a_i^*,$$

which is in $\mathcal{K}_B$. As $\mathcal{K}_B$ is an ideal in $\mathcal{L}_B$, it follows $f(\alpha(p)) - f(e)$ is in $\mathcal{K}_B$ for any polynomial f. Letting (f_n) be a sequence of polynomials that converges uniform to χ on the spectrum of $\alpha(p)$ and letting $n \to \infty$, we see that $\chi(\alpha(p)) - e$ is in $\mathcal{K}_B$. Let now $y \in Y$ and apply the inequality in line (2.13) once with $b = p$ and once with $b = e$ (and use that $[e, y] = 0$) to deduce that

$$\|[\alpha(p), y]\| < \delta/2. \tag{2.16}$$

Lines (2.16), (2.15), and Lemma 2.20 imply that $\|[\chi(\alpha(p)), y]\| < \delta$, completing the proof that $\phi(p)$ is an element of $\mathcal{P}_\delta(Y, B)$. Moreover, it is straightforward to see that the assignment

$$\mathcal{P}_{\varepsilon/8}(Z, B) \to \mathcal{P}_\delta(Y, B), \quad p \mapsto \phi(p)$$

takes homotopies to homotopies and Cuntz sums to Cuntz sums. Hence, we do indeed get a well-defined homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z, B) \to KK_\delta(Y, B), \quad [p] \mapsto [\phi(p)]$$

as claimed.

It remains to show that the diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z, B) & & \\ {\scriptstyle \phi_*}\downarrow & \searrow & \\ KK_\delta(Y, B) & \longrightarrow & KK_\varepsilon(X, B) \end{array}$$

commutes. For this, let $p \in \mathcal{P}_{\varepsilon/8}(Z, B)$ represent a class in $KK_{\varepsilon/8}(Z, B)$, and for $t \in [0, 1]$, define

$$p_t := (1 - t)p + t\alpha(p).$$

Then, by line (2.14), we have that $\|p_t - p\| < \frac{\varepsilon}{8} + \frac{\delta}{2} < \frac{1}{4}$ for all $t \in [0,1]$, so in particular

$$\text{spectrum}(p_t) \cap (1/3, 3/4) = \varnothing \quad \text{for all } t \in [0,1]. \tag{2.17}$$

Hence, $\chi(p_t)$ is a well-defined projection for all $t \in [0,1]$. We claim that $\chi(p_t)$ is an element of $\mathcal{P}_\varepsilon(X, B)$ for all $t \in [0,1]$; as $\chi(p_1) = \chi(\alpha(p))$ and $\chi(p_0) = p$, this will complete the proof.

For this last claim, note first that $p_t - e \in \mathcal{K}_B$ for all $t \in [0,1]$, whence (analogously to the case of $\chi(\alpha(p))$ argued above) $\chi(p_t) - e \in \mathcal{K}_B$ for all $t \in [0,1]$. Moreover, for all $z \in Z$,

$$\|[p_t, z]\| \leq \|[p_t - p, z]\| + \|[p, z]\| < 2\Big(\frac{\varepsilon}{8} + \frac{\delta}{2}\Big) + \frac{\varepsilon}{8} < \frac{\varepsilon}{2},$$

where the last inequality used that $\delta < \varepsilon/8$. Hence, by line (2.17) and Lemma 2.20, $\|[\chi(p_t), z]\| < \varepsilon$ for all $z \in Z$, and so in particular for all $z \in X$. This completes the proof that $\chi(p_t) \in \mathcal{P}_\varepsilon(X, B)$ for all $t \in [0,1]$, so we are done. ■

Corollary 2.22. *Let A be a separable, unital, nuclear C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let $\varepsilon \in (0,1)$, and let B be a separable C^*-algebra B with $K_*(B) = 0$. Let*

$$\pi : A \to \mathcal{L}_{SB}$$

be a large representation, and use this to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, for any finite subset X of A_1 there is a finite subset Z of A_1 such that $(X, \varepsilon) \leq (Z, \varepsilon/8)$ in the sense of Definition 2.10, and so that the forget control map

$$KK_{\varepsilon/8}(Z, SB) \to KK_\varepsilon(X, SB)$$

of Definition 2.11 is zero.

Proof. Using Theorem 2.15, it suffices to show that condition (ii) from that theorem implies condition (ii) from the current corollary (the converse is immediate). Let then ε, B, π, and X be as in the statement. Then, condition (ii) from Theorem 2.15 gives $(Y, \delta) \geq (X, \varepsilon)$ in the sense of Definition 2.10 such that the associated forget control map

$$KK_\delta(Y, SB) \to KK_\varepsilon(X, SB)$$

of Definition 2.11 is zero. Lemma 2.21 then gives a finite subset Z of A_1 containing X and a homomorphism

$$\phi_* : KK_{\varepsilon/8}(Z, SB) \to KK_\delta(Y, SB), \quad [p] \mapsto [\phi(p)]$$

such that the following diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z, SB) & & \\ \downarrow{\scriptstyle \phi_*} & \searrow & \\ KK_{\delta}(Y, SB) & \longrightarrow & KK_{\varepsilon}(X, SB) \end{array}$$

commutes (the unlabeled arrows are forget control maps). Hence, the diagonal forget control map in the above diagram is zero, which is what we wanted to show. ■