Chapter 3

Flexible models for controlled KK-theory

In this section (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to give flexible models for controlled KK-theory that will be useful for computations. Contrary to the usual conventions of C^*-algebra K-theory, we base our new even and odd groups on idempotents and invertibles rather than projections and unitaries. The extra flexibility this allows is very useful for computations. The main reason for not writing the whole paper using the more flexible model is that we previously established Theorem 2.13 in [68] using the version of controlled KK-theory from Definition 2.2 above, so need to use that model where we are directly applying Theorem 2.13. Moreover, we need the results from Chapter 4 in the current paper (which are also independently needed in Chapter 6) to relate the two models.

3.1 The even case

Our goal in this section is to define a variant of the controlled KK-theory groups of Chapter 2, but based on idempotents rather than projections. For the next definition, we recall that C^+ denotes the unitization of a C^*-algebra C, and that if $a \in M_n(C)$ and $b \in M_m(C)$ are matrices over a C^*-algebra, then $a \oplus b$ denotes the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$ in $M_{n+m}(C)$.

Definition 3.1. Let B be a separable C^*-algebra, let X be a subset[1] of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ to be the collection of pairs (p, q) of idempotents in $M_n(\mathcal{K}_B^+)$ satisfying the following conditions:

(i) $\|p\| \leq \kappa$ and $\|q\| \leq \kappa$;

(ii) $\|[p, x]\| < \varepsilon$ and $\|[q, x]\| < \varepsilon$ for all $x \in X$;

(iii) the classes $[\sigma(p)], [\sigma(q)] \in K_0(\mathbb{C})$ defined by the images of p and q under the canonical quotient map $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ are the same.

[1] Unlike Definition 2.2, we do not require X to be "large" in the sense of Definition 2.1. Essentially, largeness is needed to ensure that the sets $KK_\varepsilon(X, B)$ of Definition 2.2 are groups; we show the sets we define in Definition 3.1 are groups by using matrix arguments and a weaker equivalence relation in this definition.

Define

$$\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B) := \bigsqcup_{n=1}^{\infty} \mathcal{P}_{n,\kappa,\varepsilon}(X,B),$$

i.e., $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ is the *disjoint* union of all the sets $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$.

Equip each $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$ with the norm topology it inherits from $M_n(\mathcal{L}_B) \oplus M_n(\mathcal{L}_B)$, and equip $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ with the disjoint union topology. Let $\sim$ be the equivalence relation on $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ generated by the following relations:

(i) $(p,q) \sim (p \oplus r, q \oplus r)$ for any element $(r,r) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ with both components the same;

(ii) $(p_1,q_1) \sim (p_2,q_2)$ whenever these elements are in the same path component of $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$.[2]

Define $KK^0_{\kappa,\varepsilon}(X,B)$ to be equal as a set to $\mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)/\sim$, and provisionally define a binary operation $+$ on $KK^0_{\kappa,\varepsilon}(X,B)$ by

$$[p_1,q_1] + [p_2,q_2] := [p_1 \oplus q_1, p_2 \oplus q_2].$$

The next lemma is essentially the same as [68, Lemma A.21].

Lemma 3.2. *With notation as in Definition 3.1, $KK^0_{\kappa,\varepsilon}(X,B)$ is a well-defined abelian group with identity element the class $[0,0]$ of the zero idempotent.*

Proof. Checking directly from the definitions shows that $KK^0_{\kappa,\varepsilon}(X,B)$ is a well-defined (associative) monoid with identity element the class $[0,0]$. A standard rotation homotopy shows that $KK^0_{\kappa,\varepsilon}(X,B)$ is commutative. To complete the proof we need to show that any element $[p,q]$ has an inverse. We claim that this is given by $[q,p]$. Indeed, applying the rotation homotopy

$$\left(\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\right), \quad t \in [0,\pi/2]$$

shows that $(p \oplus q, q \oplus p) \sim (p \oplus q, p \oplus q)$, and the element $(p \oplus q, p \oplus q)$ is equivalent to $(0,0)$ by definition of the equivalence relation. ∎

The following lemma gives a useful description of cycles $(p,q) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X,B)$ that define the zero class in $KK^0_{\kappa,\varepsilon}(X,B)$.

Lemma 3.3. *With notation as in Definition 3.1, let $(p,q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X,B)$, and assume that $[p,q] = 0$ in $KK^0_{\kappa,\varepsilon}(X,B)$. Then, there is $m \in \mathbb{N}$ and an element (s,s) of $\mathcal{P}_{n+2m,\kappa,\varepsilon}(X,B)$ such that $(p \oplus 1_m \oplus 0_m, q \oplus 1_m \oplus 0_m)$ is in the same path component of $\mathcal{P}_{n+2m,2\kappa,\varepsilon}(X,B)$ as (s,s).*

[2] Equivalently, both are in the same $\mathcal{P}_{n,\kappa,\varepsilon}(X,B)$, and are in the same path component of this set.

Proof. For elements (p_1, q_1) and (p_2, q_2) in $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ let us write $(p_1, q_1) \to (p_2, q_2)$ if

$$(p_2, q_2) = (p_1 \oplus r, q_1 \oplus r)$$

for some $(r, r) \in \mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$; $(p_1, q_1) \stackrel{h}{\sim} (p_2, q_2)$ if there is a path connecting these elements; and $(p_1, q_1) \leftarrow (p_2, q_2)$ if $(p_2, q_2) \to (p_1, q_1)$. Then, $[p, q] = 0$ means that there is some sequence of moves from $\{\to, \leftarrow, \stackrel{h}{\sim}\}$ starting at (p, q) and finishing at $(0, 0)$. It is not difficult to see the following: any time a move from $\{\to, \leftarrow, \stackrel{h}{\sim}\}$ is consecutively repeated we may replace it by a single move of the same type; any occurrence of "$\stackrel{h}{\sim}\to$" may be replaced by an occurrence of "$\to\stackrel{h}{\sim}$"; any occurrence of "$\leftarrow\stackrel{h}{\sim}$" may be replaced by an occurrence of "$\stackrel{h}{\sim}\leftarrow$"; any occurrence of "$\leftarrow\to$" or "$\leftarrow\stackrel{h}{\sim}\to$" may be replaced by "$\to\stackrel{h}{\sim}\leftarrow$" (we leave the details to the reader in each case). Using these replacements, we see that our moves relating (p, q) to $(0, 0)$ may be assumed to be of the form

$$(p, q) \to\stackrel{h}{\sim}\leftarrow (0, 0),$$

or in other words that there are elements (r, r) and (t, t) in $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ such that $(p \oplus r, q \oplus r)$ is homotopic to (t, t).

To complete the proof, note then that $(p \oplus r \oplus 1 - r, q \oplus r \oplus 1 - r)$ is homotopic to $(t \oplus 1 - r, t \oplus 1 - r)$. For $t \in [0, \pi/2]$, define

$$r_t := \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 - r \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}$$

so $(r_t)_{t\in[0,\pi/2]}$ is a path connecting $r \oplus 1 - r$ and $1 \oplus 0$. One computes that $\|r_t\| \leq 1 + \kappa \leq 2\kappa$ for all t, and that $\|[r_t, x]\| < \varepsilon$ for all $x \in X$. Hence, with $s = t \oplus 1 - r$ we get the claimed result. ∎

We will need a more general variation of Definitions 2.10 and 2.11.

Definition 3.4. Let C be a C^*-algebra. Let $\mathcal{X}'_C$ consist of all triples of the form (X, κ, ε) where X is a finite subset of the unit ball of C, $\kappa \geq 1$, and $\varepsilon > 0$. Put a partial order on $\mathcal{X}'_C$ by $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ if $\delta \leq \varepsilon$, $\lambda \leq \kappa$ and if for all $x \in X$ there exists $y \in Y$ with $\|x - y\| \leq \frac{1}{2\lambda}(\varepsilon - \delta)$.

Let now B be a separable C^*-algebra. Then, if $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ in $\mathcal{X}'_{\mathcal{L}_B}$, one checks that for each n we have

$$\mathcal{P}_{n\lambda,\delta}(Y, B) \subseteq \mathcal{P}_{n,\kappa,\varepsilon}(X, B). \tag{3.1}$$

We call the canonical map

$$KK^0_{\lambda,\delta}(Y, B) \to KK^0_{\kappa,\varepsilon}(X, B)$$

induced by the inclusions in line (3.1) above a *forget control map*.

3.2 The odd case

Our goal in this section is to introduce an odd parity version of the controlled KK-theory groups of the previous section. For the statement, recall that C^+ denotes the unitization of a C^*-algebra C.

Definition 3.5. Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Define $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ to be the subset of those invertible elements u in $M_n(\mathcal{K}_B^+)$ satisfying the following conditions:

(i) $\|u\| \leq \kappa$ and $\|u^{-1}\| \leq \kappa$;

(ii) $\|[u, x]\| < \varepsilon$ and $\|[u^{-1}, x]\| < \varepsilon$ for all $x \in X$.

Define

$$\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B) := \bigsqcup_{n=1}^{\infty} \mathcal{U}_{n,\kappa,\varepsilon}(X, B),$$

i.e., $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ is the *disjoint* union of all the sets $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$.

Equip each $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ with the norm topology it inherits from $M_n(\mathcal{L}_B)$, and equip $\bigsqcup_{n=1}^{\infty} \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ with the disjoint union topology. Define an equivalence relation on $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ to be generated by the following relations:

(i) for any $k \in \mathbb{N}$, if $1_k \in \mathcal{U}_{k,\kappa,\varepsilon}(X, B)$ is the identity element, then

$$u \sim u \oplus 1_k;$$

(ii) $u_1 \sim u_2$ if both are elements of the same path component of $\mathcal{U}_{\infty,2\kappa,\varepsilon}(X,B)$.[3]

Define $KK^1_{\kappa,\varepsilon}(X, B)$ to be $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)/\sim$, and provisionally define a binary operation $+$ on $KK^1_{\kappa,\varepsilon}(X, B)$ by

$$[u_1] + [u_2] := [u_1 \oplus u_2].$$

Lemma 3.6. *With notation as in Definition 3.5, $KK^1_{\kappa,\varepsilon}(X,B)$ is a well-defined abelian group with identity element the class $[1_B]$ of the unit of B.*

Proof. It is straightforward to check that $KK^1_{\kappa,\varepsilon}(X, B)$ is a monoid, and the class $[1]$ is neutral by definition. A standard rotation homotopy shows that $KK^1_{\kappa,\varepsilon}(X, B)$ is commutative. It remains to show that inverses exist. We claim that for $u \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$, the inverse of the class $[u]$ is given by $[u^{-1}]$. Indeed, consider the homotopy

$$u_t := \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix}, \quad t \in [0, \pi/2].$$

[3] Equivalently, both are in the same $\mathcal{U}_{n,2\kappa,\varepsilon}(X, B)$, and are in the same path component of this set. Notice also the switch from κ to 2κ here, which is needed for our proof that $KK^1_{\kappa,\varepsilon}(X, B)$ is a group.

This connects $u \oplus u^{-1}$ and 1_{2k}, so it suffices to show that this passes through $\mathcal{U}_{2n,2\kappa,\varepsilon}(X, B)$. For the commutator condition, we compute that for $a \in X$ and $t \in [0, 2\pi]$

$$[a, u_t] = \begin{pmatrix} [a, u] & 0 \\ 0 & [u^{-1}, a] \end{pmatrix} \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) \\ \cos(t)\sin(t) & -\cos^2(t) \end{pmatrix}.$$

The scalar matrix on the right has norm $|\cos(t)|$, and the matrix on the left has norm at most $\max\{\|[a, u]\|, \|[a, u^{-1}]\|\} < \varepsilon$, so $\|[a, u_t]\| < \varepsilon$. For the norm condition, we compute that

$$u_t = \begin{pmatrix} u & 0 \\ 0 & -u^{-1} \end{pmatrix} \begin{pmatrix} \cos^2(t) & \cos(t)\sin(t) \\ \cos(t)\sin(t) & -\cos^2(t) \end{pmatrix} + \begin{pmatrix} \sin^2(t) & -\cos(t)\sin(t) \\ \cos(t)\sin(t) & \sin^2(t) \end{pmatrix}.$$

The first scalar matrix appearing above has norm $|\cos(t)|$, and the second has norm $|\sin(t)|$. We thus have that $\|u_t\| \leq \kappa|\cos(t)| + |\sin(t)|$, which is at most[4] 2κ as required. ■

Definition 3.7. Let C be a C^*-algebra, and let $\mathcal{X}'_C$ be the directed set of Definition 3.4 above. Let B be a separable C^*-algebra. Then, if $(X, \kappa, \varepsilon) \leq (Y, \lambda, \delta)$ in $\mathcal{X}'_{\mathcal{L}_B}$, one checks that for each n we have

$$\mathcal{U}_{n,\lambda,\delta}(Y, B) \subseteq \mathcal{U}_{n,\kappa,\varepsilon}(X, B) \tag{3.2}$$

for all n. We call the canonical map

$$KK^1_{\lambda,\delta}(Y, B) \subseteq KK^1_{\kappa,\varepsilon}(X, B)$$

induced by the inclusions in line (3.2) above a *forget control map*.

[4]We suspect the optimal estimate is κ – this is the case if u is normal, for example – but were unable to do better than $\sqrt{1 + \kappa^2}$ in general.