Chapter 4

Homotopies, similarities, and normalization

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal is to establish some technical lemmas about the controlled KK-groups $KK^0_{\kappa,\varepsilon}(X, B)$ and $KK^1_{\kappa,\varepsilon}(X, B)$ and the underlying sets of cycles $\mathcal{P}_{\infty,\kappa,\varepsilon}(X, B)$ and $\mathcal{U}_{\infty,\kappa,\varepsilon}(X, B)$ from Definitions 3.1 and 3.5 respectively. These are all variants of standard facts from C^*-algebra K-theory, but the arguments are more involved as we need to do extra work to control commutator estimates. Some of the material is adapted from the foundational work of Oyono-Oyono and the second author on controlled K-theory [47]; those authors work in the "dual" setting to us in some sense, and similar techniques are often useful.

Most of the results in this chapter come with explicit estimates. We have generally not tried to get optimal estimates, but as it might be useful for future work we have tried to point out where one might expect the estimates to be optimal where this is simple to do.

4.1 Background on idempotents

In this section we look at idempotents in C^*-algebras and their relationship to projections. Most of this is well-known; nonetheless, we give proofs for the sake of completeness where we could not find a good reference.

To establish notation, let us first note that if $p \in \mathcal{B}(H)$ is an idempotent, then with respect to the decomposition $H = \text{Image}(p) \oplus \text{Image}(p)^\perp$, p has a matrix representation

$$p = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

for some $a \in \mathcal{B}(\text{Image}(p)^\perp, \text{Image}(p))$; conversely, any operator admitting a matrix of this form with respect to some orthogonal direct sum decomposition of the underlying Hilbert space defines an idempotent.

Lemma 4.1. *If p is an idempotent bounded operator on a Hilbert space that is neither zero nor the identity, then*

$$\|1 - p\| = \|p\| \quad \textit{and} \quad \|p - p^*\| \le \|p\|.$$

Proof. Writing p as in line (4.1) (and using that neither Image(p) nor Image$(p)^{\perp}$ are the zero subspace), we compute that

$$\|p\|^2 = \|pp^*\| = \|1 + aa^*\| = 1 + \|a\|^2 \tag{4.2}$$

and moreover that

$$\|1 - p\|^2 = \|(1-p)^*(1-p)\| = \|1 + a^*a\| = 1 + \|a\|^2 = \|p\|^2.$$

Looking now at $p - p^*$, we see that

$$(p - p^*)(p - p^*)^* = \begin{pmatrix} 0 & a \\ -a^* & 0 \end{pmatrix} \begin{pmatrix} 0 & -a \\ a^* & 0 \end{pmatrix} = \begin{pmatrix} aa^* & 0 \\ 0 & a^*a \end{pmatrix},$$

whence $\|p - p^*\|^2 = \|a\|^2 \leq \|p\|^2$. ∎

Corollary 4.2. *If $\kappa \geq 1$, and p is any idempotent in a C^*-algebra with $\|p\| \leq \kappa$, then $\|1 - p\| \leq \kappa$, $\|p - p^*\| \leq \kappa$, and $\|2p - 1\| \leq 2\kappa$.*

Proof. The estimates for $\|1 - p\|$ and $\|p - p^*\|$ are immediate from Lemma 4.1 (and direct checks for the degenerate cases $p = 0$ and $p = 1$). The estimate for $2p - 1$ follows as

$$2p - 1 = p - (1 - p).$$ ∎

It will be convenient to formalize a standard construction in C^*-algebra K-theory for turning idempotents into projections (compare for example [7, Proposition 4.6.2]).

Definition 4.3. Let p be an idempotent in a C^*-algebra C. Define

$$z := 1 + (p - p^*)(p^* - p) \in C^+,$$

and note that $z \geq 1_{C^+}$ so z is invertible. Define

$$r := pp^*z^{-1},$$

which is an element of C. We call r the *projection*[1] *associated to* p.

Remark 4.4. If C is a concrete C^*-algebra and p is an idempotent with matrix representation as in line (4.1), then one computes that the associated projection has matrix representation

$$r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.3}$$

with respect to the same decomposition of the underlying Hilbert space. In particular, r is the projection with the same image as the idempotent p.

[1] It will be shown to be a projection in the next lemma.

Lemma 4.5. *Let p be an idempotent in a C^*-algebra C, and assume that $\|p\| \leq \kappa$ for some $\kappa \geq 1$. Let r be the projection associated to p as in Definition 4.3, and for $t \in [0,1]$ define $r_t := (1-t)p + tr$. Then, the following hold:*

(i) *The element r is a projection in C, and there is an invertible $u \in C^+$ such that $upu^{-1} = r$. Moreover, u and its inverse have norm at most $1 + \|p\|$, and u is connected to the identity through a path of invertibles such that all the invertibles in the path and all of their inverses have norm at most $1 + \|p\|$.*

(ii) *Each r_t is an idempotent such that $\|r_t\| \leq \kappa$ for all t, and the map $t \mapsto r_t$ is κ-Lipschitz.*

(iii) *For any $c \in C$ and $t \in [0,1]$ we have*

$$\|[r_t, c]\| \leq (1+2t)\|[p,c]\| + t\|[p,c^*]\|.$$

(iv) *The map*

$$\{p \in C \mid p = p^2\} \to \{p \in C \mid p = p^2 = p^*\}$$

that takes an idempotent to its associated projection is 1-Lipschitz.

Proof. Part (i) as in line (4.1), we may write $p = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}$, and note as in line (4.2) that $\|p\| = \sqrt{1 + \|a\|^2}$, so in particular $\|a\| \leq \|p\|$. Using the discussion in Remark 4.4 we see that $u = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ satisfies $upu^{-1} = r$, and that the path $u_t = \begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix}$ connects u to the identity through invertibles of norm at most $1 + \|ta\| \leq 1 + \|p\|$. The claims on the norms of the inverses follow as $\begin{pmatrix} 1 & ta \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -ta \\ 0 & 1 \end{pmatrix}$.

(Or see for example the proof of [7, Proposition 4.6.2]).

For part (ii), we write p as in line (4.1), note that $\|a\| \leq \kappa$, and also that r has the matrix representation as in line (4.3). This implies the claimed properties.

For part (iii), we again write p as a matrix as in line (4.1). Let $c \in C$, and with respect to the same decomposition of the underlying Hilbert space, let us write

$$c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

Then, one computes that

$$[p,c] = \begin{pmatrix} ac_{21} & c_{12} + ac_{22} - c_{11}a \\ -c_{21} & -c_{21}a \end{pmatrix}. \tag{4.4}$$

As the conditional expectation that sends a matrix to its diagonal is contractive, we have

$$\left\| \begin{pmatrix} ac_{21} & 0 \\ 0 & -c_{21}a \end{pmatrix} \right\| \leq \|[p,c]\|$$

and combining this with line (4.4) gives

$$\left\| \begin{pmatrix} 0 & c_{12} + ac_{22} - c_{11}a \\ -c_{21} & 0 \end{pmatrix} \right\| \leq 2\|[p,c]\|. \tag{4.5}$$

One computes that the top right entry of $[p - p^*, c]$ is $ac_{22} - c_{11}a$, whence

$$\|ac_{22} - c_{11}a\| \leq \|[p - p^*, c]\| \leq \|[p,c]\| + \|[p,c^*]\|.$$

This and line (4.5) together imply that

$$\left\| \begin{pmatrix} 0 & c_{12} \\ -c_{21} & 0 \end{pmatrix} \right\| \leq 3\|[p,c]\| + \|[p,c^*]\|. \tag{4.6}$$

As r has the matrix representation from line (4.3), the left-hand side of the inequality in line (4.6) equals $\|[r,c]\|$, and so line (4.6) can be rewritten as the inequality $\|[r,c]\| \leq 3\|[p,c]\| + \|[p,c^*]\|$. As $r_t = (1-t)p + tr$, this implies the claimed estimate.

For part (iv) we may assume that C is a concrete C^*-algebra. As noted in Remark 4.4, the projection r associated to an idempotent p is then simply the orthogonal projection with the same image as p. In this language, part (iv) is [41, Chapter One, Theorem 6.35]. ■

4.2 From similarities to homotopies

Our goal in this short section is to establish an analogue of the standard K-theoretic fact that similar idempotents are homotopic, at least up to increasing matrix sizes. Compare for example [7, Proposition 4.4.1].

Proposition 4.6. *Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, and let $\kappa \geq 1$ and $\varepsilon > 0$. Let (p_0, q) and (p_1, q) be elements of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$, and let $u \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ be such that $up_0u^{-1} = p_1$. Then, the elements $(p_0 \oplus 0_n, q \oplus 0_n)$ and $(p_1 \oplus 0_n, q \oplus 0_n)$ are in the same path component of $\mathcal{P}_{2n,\kappa^3,3\kappa^2\varepsilon}(X, B)$, and in particular, (p_0, q) and (p_1, q) define the same class in $KK^0_{\kappa^3,3\kappa^2\varepsilon}(X, B)$.*

The analogous statement holds with the roles of the first ("p") and second ("q") components reversed.

Proof. Define

$$v_t := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \in M_{2n}(\mathcal{K}_B^+).$$

Then, the path

$$t \mapsto (v_t(p_0 \oplus 0_n)v_t^{-1}, q \oplus 0_n), \quad t \in [0, \pi/2]$$

connects $(p_0 \oplus 0_n, q \oplus 0_n)$ to $(p_1 \oplus 0_n, q \oplus 0_n)$ through $\mathcal{P}_{2n,\kappa^3,3\kappa^2\varepsilon}(X, B)$. We leave the direct checks involved to the reader. ∎

4.3 Normalization

Our goal in this section is to show that cycles for $KK^0_{\kappa,\varepsilon}(X, B)$ and $KK^1_{\kappa,\varepsilon}(X, B)$ can be assumed to have prescribed "scalar part", at least up to some deterioration of κ and ε.

The following lemma is well-known without the Lipschitz condition[2]; see for example [7, Theorem 4.6.7] or [36, Corollary 4.1.8].

Lemma 4.7. *Let $L > 0$. Then, if $(p_t)_{t\in[0,1]}$ is an L-Lipschitz path of projections in a unital C^*-algebra C, there is a $(3L)$-Lipschitz path $(u_t)_{t\in[0,1]}$ of unitaries in C such that $u_0 = 1$, and such that $p_t = u_t p_0 u_t^*$ for all $t \in [0, 1]$.*

We need a preliminary lemma.

Lemma 4.8. *Let $\eta \geq 1$, and let C be a unital C^*-algebra. Then, the map*

$$\{c \in C \mid c \geq \eta^{-1}\} \to C, \quad c \mapsto c^{-1/2}$$

is $\frac{1}{2}\eta^{3/2}$-Lipschitz[3].

Proof. For any positive real number t, one has

$$t^{-1/2} = \frac{2}{\pi}\int_0^\infty (\lambda^2 + t)^{-1} d\lambda,$$

whence for any positive invertible elements $c, d \in C$

$$c^{-1/2} - d^{-1/2} = \frac{2}{\pi}\int_0^\infty \big((\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1}\big) d\lambda. \tag{4.7}$$

Using the formula

$$(\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1} = (\lambda^2 + c)^{-1}(d - c)(\lambda^2 + d)^{-1}$$

and assuming that $c \geq \eta^{-1}$ and $d \geq \eta^{-1}$, the continuous functional calculus implies that

$$\|(\lambda^2 + c)^{-1} - (\lambda^2 + d)^{-1}\| \leq \|c - d\|(\lambda^2 + \eta^{-1})^{-2}.$$

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[2] The constant 3 appearing in the statement is not optimal; one can see from the proof that 3 can be replaced with $2 + \varepsilon$, for any $\varepsilon > 0$. We do not know what the optimal constant is.

[3] The constant is optimal in some sense; this follows as the absolute value if the derivative of the function $t \mapsto t^{-1/2}$ on $[\eta^{-1}, \infty)$ has maximum value $\frac{1}{2}\eta^{3/2}$.

This inequality and line (4.7) imply that

$$\|c^{-1/2} - d^{-1/2}\| \leq \frac{2\|c-d\|}{\pi} \int_0^\infty (\lambda^2 + \eta^{-1})^{-2} d\lambda.$$

The integral on the right-hand side equals $(\pi\eta^{3/2})/4$, whence the result. ∎

Proof of Lemma 4.7. We first claim that it suffices to show we can choose a $\delta > 0$ such that if $[t_1, t_2]$ is a sub-interval of $[0,1]$ of length at most δ, and $t \mapsto p_t$ is a projection-valued L-Lipschitz function on $[t_1, t_2]$, then there is a unitary-valued $(3L)$-Lipschitz function $t \mapsto u_t$ on $[t_1, t_2]$ such that $u_0 = 1$ and $p_t = u_t p_0 u_t^*$ for all $t \in [t_1, t_2]$. Indeed, if we can do this, then let $0 = t_0 < t_1 < \cdots < t_N = 1$ be a partition of the interval $[0,1]$ such that each subinterval has length at most δ, and for each $i \in \{0, \ldots, N-1\}$ choose a unitary-valued $(3L)$-Lipschitz function $t \mapsto u_t^{(i)}$ on $[t_i, t_{i+1}]$ such that $u_{t_i}^{(i)} = 1$ and $p_t = u_t^{(i)} p_{t_i} (u_t^{(i)})^*$ for all $t \in [t_i, t_{i+1}]$. The function on $[0,1]$ defined on each subinterval $[t_i, t_{i+1}]$ by

$$t \mapsto u_t^{(i)} u_{t_i}^{(i-1)} u_{t_{i-1}}^{(i-2)} \cdots u_{t_1}^{(0)}$$

then has the right properties to establish the lemma.

Let us then establish the statement in the claim. Let $\varepsilon > 0$ be small enough that

$$(1-(2+\varepsilon)\varepsilon)^{-1/2} + (1+\varepsilon)^2(1-(2+\varepsilon)\varepsilon)^{-3/2} \leq 3,$$

and let $\delta > 0$ be such that if $t, s \in [0,1]$ satisfy $|t-s| \leq \delta$, then $\|p_s - p_t\| < \varepsilon$. Let $[t_1, t_2]$ be an interval of length at most δ. For $t \in [t_1, t_2]$, define

$$x_t := p_t p_{t_1} + (1-p_t)(1-p_{t_1})$$

and note that

$$\|x_t - 1\| = \|(2p_t - 1)(p_{t_1} - p_t)\| \leq \|2p_t - 1\| \|p_{t_1} - p_t\| < \varepsilon,$$

and so each x_t is invertible, $\|x_t\| < 1 + \varepsilon$, and also $\|x_t^{-1}\| < (1-\varepsilon)^{-1}$ by the Neumann series formula for the inverse. One computes that $x_t p_{t_1} = p_t p_{t_1} = p_t x_t$, and so $x_t p_{t_1} x_t^{-1} = p_t$. Moreover, $p_{t_1} x_t^* = x_t^* p_t$, and so $p_{t_1} x_t^* x_t = x_t^* p_t x_t = x_t^* x_t p_{t_1}$, i.e., $x_t^* x_t$ commutes with p_{t_1}. If we define $w_t := x_t (x_t^* x_t)^{-1/2}$, we have that w_t is unitary and moreover

$$w_t p_{t_1} w_t^{-1} = x_t (x_t^* x_t)^{-1/2} p_{t_1} (x_t^* x_t)^{1/2} x_t^{-1} = x_t p_{t_1} x_t^{-1} = p_t.$$

It remains to show that the path defined on $[t_1, t_2]$ by $t \mapsto w_t$ is $(3L)$-Lipschitz.

We first note that for $s, t \in [t_1, t_2]$, we have that

$$\|x_s - x_t\| = \|(p_t - p_s)(2p_{t_1} - 1)\| \leq \|p_t - p_s\| \leq L|s-t| \tag{4.8}$$

by assumption that (p_t) is L-Lipschitz. Using that $\|x_t\| < 1 + \varepsilon$, this implies that for any $s, t \in [t_1, t_2]$

$$\|x_t^* x_t - x_s^* x_s\| \leq \|x_t^* - x_s^*\| \|x_t\| + \|x_s^*\| \|x_t - x_s\| < 2(1+\varepsilon)L|s-t|.$$

Moreover, $\|1 - x_t^* x_t\| < (2+\varepsilon)\varepsilon$, whence $1 - (2+\varepsilon)\varepsilon \leq x_t^* x_t$ and so in particular

$$\|(x_t^* x_t)^{-1/2}\| \leq (1-(2+\varepsilon)\varepsilon)^{-1/2} \quad \text{for all } t \in [t_1, t_2]. \tag{4.9}$$

Hence, moreover Lemma 4.8 (with $\eta = (1-(2+\varepsilon))^{-1}$) implies that for any $s, t \in [t_1, t_2]$

$$\|(x_t^* x_t)^{-1/2} - (x_s^* x_s)^{-1/2}\| \leq (1-(2+\varepsilon)\varepsilon)^{-3/2}(1+\varepsilon)L|s-t|. \tag{4.10}$$

Lines (4.8), (4.10), and (4.9) combined with the fact that

$$\|x_t\| < 1 + \varepsilon$$

for all $t \in [t_1, t_2]$ implies that for any $s, t \in [t_1, t_2]$

$$\begin{aligned} \|w_t - w_s\| &\leq \|x_t - x_s\| \|(x_t^* x_t)^{-1/2}\| + \|x_s\| \|(x_t^* x_t)^{-1/2} - (x_s^* x_s)^{-1/2}\| \\ &\leq (1-(2+\varepsilon)\varepsilon)^{-1/2} L|s-t| + (1+\varepsilon)^2 (1-(2+\varepsilon)\varepsilon)^{-3/2} L|s-t| \end{aligned}$$

which implies the desired estimate by choice of ε. ■

For the statement of the next definition, recall that for $l \in \{1, \ldots, n\}$, we let $1_l \in M_n(\mathbb{C})$ be the rank l projection with l ones in the top-left part of the diagonal and zeros elsewhere.

Definition 4.9. With notation as in Definition 3.1, define

$$\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B) := \big\{(p,q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X, B) \mid \exists l \in \mathbb{N} \text{ such that } (p,q) - (1_l, 1_l) \text{ is in } M_n(\mathcal{K}_B) \oplus M_n(\mathcal{K}_B)\big\}.$$

Define $\mathcal{P}^1_{\infty,\kappa,\varepsilon}(X, B)$ to be the disjoint union of these sets as n ranges over $\mathbb{N}$.

Here, is the first of our main goals for this section; it allows control of the "scalar part" of cycles for $KK^0_{\kappa,\varepsilon}(X, B)$.

Proposition 4.10. *Let B be a separable C^*-algebra. Let X be a self-adjoint*[4] *subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, let $\kappa \geq 1$, and let $n \in \mathbb{N}$.*

(i) *Any element $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ is in the same path component of $\mathcal{P}_{n,4\kappa^3,\varepsilon}(X, B)$ as an element of $\mathcal{P}^1_{n,4\kappa^3,\varepsilon}(X, B)$*[5].

[4] We mean here that $X = X^*$, not the stronger assumption that every $x \in X$ is self-adjoint.

[5] If $\kappa = 1$, one can replace $4\kappa^3$ with 1 in the statement; we leave the details to the reader.

(ii) *If two elements (p_0, q_0) and (p_1, q_1) of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ are connected by a path in $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$, then they are connected by a path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$. Moreover, if $L \geq 1$ is such that there is an L-Lipschitz path in $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ connecting (p_0, q_0) and (p_1, q_1), then there is a $(20\kappa L)$-Lipschitz path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$ connecting (p_0, q_0) and (p_1, q_1).*

Proof of Proposition 4.10. For part (i), we assume that (p, q) is an element of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. Hence, by definition of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$, if $\mathcal{K}_B^+$ is the unitization of $\mathcal{K}_B$ and $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ is the canonical quotient map then the classes $[\sigma(p)]$ and $[\sigma(q)]$ in $K_0(\mathbb{C})$ are the same, so in particular the idempotents $\sigma(p)$ and $\sigma(q)$ have the same rank. Using Lemma 4.5 (i), there are paths of invertibles $(u_t)_{t\in[0,1]}$ and $(v_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ and projections r, s such that $u_1 = v_1$ is the identity, such that $u_0 r u_0^{-1} = \sigma(p)$, such that $v_0 s v_0^{-1} = \sigma(q)$, and such that the norms of all the u_t, all the v_t and their inverses are all at most $1 + \kappa \leq 2\kappa$. On the other hand, r and s have the same rank, whence there are paths of unitaries $(u_t)_{t\in[1,2]}$ and $(v_t)_{t\in[0,1]}$ in $M_n(\mathbb{C})$ such that $u_1 = v_1$ is the identity, and such that $u_2 r u_2^* = 1_l$, and $v_2 s v_2^* = 1_l$. As scalar matrices commute with X, the path $((u_t p u_t^{-1}, v_t q v_t^{-1}))_{t\in[0,2]}$ passes through $\mathcal{P}_{n,4\kappa^3,\varepsilon}(X, B)$, and connects (p, q) to an element of $\mathcal{P}^1_{n,4\kappa^3,\varepsilon}(X, B)$ as required.

For part (ii), we just look at the statement involving Lipschitz paths; the case of general continuous paths follows (in a simpler way) from the same arguments, and is left to the reader. Assume that (p_0, q_0) and (p_1, q_1) are elements of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ that are connected by an L-Lipschitz path that passes through $\mathcal{P}_{n,\kappa\cdot\varepsilon}(X, B)$. In particular, there exists $l \in \mathbb{N}$ such that $\sigma(p_0) = \sigma(q_0) = 1_l = \sigma(p_1) = \sigma(q_1)$. Let r_0 be the projection associated to p_0 as in Definition 4.3. As in Lemma 4.5 (ii), the path defined for $t \in [0, 1]$ by $t \mapsto (1 - t)p_0 + t r_0$ is κ-Lipschitz and connects p_0 and r_0 through idempotents of norm at most κ. Moreover, Lemma 4.5 (iii) implies that for all $x \in X$ and all $t \in [0, 1]$

$$\|[(1 - t)p_0 + t r_0, x]\| \leq (1 + 2t)\|[p_0, x]\| + t\|[p_0, x^*]\|.$$

As $X = X^*$, this implies that $\|[(1 - t)p_0 + t r_0, x]\| < 4\varepsilon$ for all $x \in X$, and all $t \in [0, 1]$. Note also that $\sigma((1 - t)p_0 + t r_0) = 1_l$ for all t. Similarly, we get s_0 which has the same properties with respect to q_0. We have thus shown that (p_0, q_0) is connected to the element (r_0, s_0) via a κ-Lipschitz path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$. Completely analogously, (p_1, q_1) is connected to its associated projection (r_1, s_1) via a κ-Lipschitz path in $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X, B)$. Moreover, using Lemma 4.5 (iv), we have that (r_0, s_0) and (r_1, s_1) are connected by an L-Lipschitz path of projections in $\mathcal{P}_{n,1,4\varepsilon}(X, B)$, say $((r_t, s_t))_{t\in[0,1]}$.

Now, consider the path $(\sigma(r_t), \sigma(s_t))_{t\in[0,1]}$ in $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, which is also L-Lipschitz. Lemma 4.7 gives $(3L)$-Lipschitz paths $(u_t)_{t\in[0,1]}$ and $(v_t)_{t\in[0,1]}$ of unitaries in $M_n(\mathbb{C})$ such that $\sigma(r_t) = u_t \sigma(r_0) u_t^*$ and $\sigma(s_t) = v_t \sigma(s_0) v_t^*$ for all

$t \in [0,1]$. The path $((u_t^* r_t u_t, v_t^* s_t v_t))_{t\in[0,1]}$ then passes through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, is $(6L)$-Lipschitz, and connects (r_0, s_0) to $(u_1^* r_1 u_1, v_1^* s_1 v_1)$.

Summarizing where we are, we have the following paths:

(i) A κ-Lipschitz path through $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (p_0, q_0) and (r_0, s_0).

(ii) A $(6L)$-Lipschitz path through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (r_0, s_0) and $(u_1^* r_1 u_1, v_1^* s_1 v_1)$.

(iii) A κ-Lipschitz path through $\mathcal{P}^1_{n,\kappa,4\varepsilon}(X,B)$, parametrized by $[0,1]$, and that connects (p_1, q_1) and (r_1, s_1).

We claim that there is a 2π-Lipschitz path passing through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, parametrized by $[0,1]$ and connecting $(u_1^* r_1 u_1, v_1^* s_1 v_1)$ and (r_1, s_1). Concatenating this new path with the three paths above (and using that $\kappa \geq 1$ and that $L \geq 1$), and rescaling the two κ-Lipschitz paths by $1/12$, the $6L$-Lipschitz path by $4/12$, and the 6π-Lipschitz by $6/12$, this will give us a $(20\kappa L)$-Lipschitz path connecting (p_0, q_0) and (p_1, q_1) through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, which will complete the proof.

To establish the claim note that u_1 commutes with 1_l, and is therefore connected to the identity in $M_n(\mathbb{C})$ via a π-Lipschitz path of unitaries that all commute with 1_l, say $(u_t)_{t\in[1,2]}$. Similarly, we get a π-Lipschitz path $(v_t)_{t\in[1,2]}$ with the same properties with respect to v_1. The path $((u_t^* r_1 u_t, v_t^* s_1 v_t))_{t\in[1,2]}$ then passes through $\mathcal{P}^1_{n,1,4\varepsilon}(X,B)$, is 2π-Lipschitz, and connects $(u_1^* r_1 u_1, v_1^* s_1 v_1)$ to (r_1, s_1), so we are done. ∎

We now move on to results that let us prescribe the “scalar part” of cycles for KK^1, which is much simpler.

Definition 4.11. With notation as in Definition 3.5, define

$$\mathcal{U}^1_{n,\kappa,\varepsilon}(X,B) := \{u \in \mathcal{U}_{n,\kappa,\varepsilon}(X,B) \mid u - 1 \in M_n(\mathcal{K}_B)\}.$$

Define $\mathcal{U}^1_{\infty,\kappa,\varepsilon}(X,B)$ to be the disjoint union of these sets as n ranges over $\mathbb{N}$.

We need a slight variant of the well-known fact that the group of invertibles in a C^*-algebra deform retracts onto the group of unitaries.

Lemma 4.12. *Let $\kappa \geq 1$, let C be a unital C^*-algebra, and let C_κ^{-1} be the set of invertible elements $u \in C$ such that $\|u\| \leq \kappa$ and $\|u^{-1}\| \leq \kappa$. Then, the unitary group of C is a deformation retract of C_κ^{-1}. In particular, $M_n(\mathbb{C})_\kappa^{-1}$ is connected.*

Proof. Let $u \in C_\kappa^{-1}$, and for $t \in [0, \frac{1}{2}]$ define $u_t := u(u^*u)^{-t}$. This is a homotopy between the identity $u \mapsto u_0$ on C_κ^{-1} and the map $u \mapsto u_{1/2}$; the latter is a retraction of C_κ^{-1} onto the unitary group of C, giving the first part. In particular, it follows that C_κ^{-1} is connected if and only if C_1^{-1} is connected; as the unitary group of $M_n(\mathbb{C})$ is connected, this gives the last statement. ∎

Proposition 4.13. *Let B be a separable C^*-space, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, let $\kappa \geq 1$, and let $n \in \mathbb{N}$.*

(i) *Any element $v \in \mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ is connected to an element of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$ by a path in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(X, B)$.*

(ii) *If two elements $v_0, v_1 \in \mathcal{U}^1_{n,\kappa,\varepsilon}(X, B)$ are in the same path component of $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$, then they are in the same path component of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$.*

Proof. For part (i), let $\mathcal{K}_B^+$ be the unitization of $\mathcal{K}_B$, let $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ be the canonical quotient map, and set $w = \sigma(u^{-1})$. Using Lemma 4.12, there is a path $(w_t)_{t\in[0,1]}$ of invertibles connecting $w = w_1$ to the identity and all with norm at most κ. Then, the path $(w_t v)_{t\in[0,1]}$ is in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(X, B)$ and connects v to the element $u := w_1 v$, which satisfies $\sigma(u) = 1$, and so $1 - u \in M_n(\mathcal{K}_B)$.

For part (ii), let $(v_t)_{t\in[0,1]}$ be a path in $\mathcal{U}_{n,\kappa,\varepsilon}(X, B)$ connecting v_0 and v_1. Let $w_t = \sigma(v_t^{-1})$, and note that $w_0 = w_1 = 1$. Moreover, $\|w_t\| \leq \kappa$ for all t. Then, $u_t := w_t v_t$ is a path connecting v_0 and v_1 in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(X, B)$ as required. ∎

4.4 From homotopies to similarities

Our goal in this section is to establish a controlled variant of the fact that homotopic idempotents are similar; compare for example [7, Proposition 4.3.2]. This requires some work, as we need to control the "speed" of the homotopy in order to control the commutator estimates for the invertible element appearing in the similarity. The final target is Proposition 4.17 below; the other results build up to it.

Lemma 4.14. *Let $\kappa \geq 1$, and let p_0 and p_1 be idempotents in a C^*-algebra C with norm at most κ, and such that $\|p_0 - p_1\| \leq 1/(12\kappa^2)$. Then, there is a path $(p_t)_{t\in[0,1]}$ of idempotents connecting p_0 and p_1, and with the following properties:*

(i) *each p_t is an idempotent in C of norm at most 2κ;*

(ii) *for all $c \in C$ and $t \in [0, 1]$,*

$$\|[c, p_t]\| \leq 21\kappa^2 \max_{i=0,1} \|[c, p_i]\|;$$

(iii) *the function $t \mapsto p_t$ is 1-Lipschitz.*

Proof. For each $t \in [0, 1]$, define $r_t := (1 - t)p_0 + tp_1 \in C$, and define

$$u_t := (1 - r_t)(1 - p_0) + r_t p_0 \in C^+.$$

Corollary 4.2 implies that $\|2p_0 - 1\| \leq 2\kappa$, whence

$$\|1 - u_t\| = \|(2p_0 - 1)(p_0 - r_t)\| \leq 2\kappa\|p_0 - p_1\| \leq 1/6.$$

In particular, u_t is invertible, $\|u_t\| \leq 7/6$, and $\|u_t^{-1}\| \leq 6/5$ by the Neumann series formula of the inverse. Define $p_t := u_t p_0 u_t^{-1}$, which is an idempotent in C. We claim that the path $(p_t)_{t\in[0,1]}$ has the desired properties. Note first that $r_0 = p_0$, whence $u_0 = 1$, and so the path $(p_t)_{t\in[0,1]}$ does start at the original p_0. On the other hand, $u_1 p_0 = r_1 p_0 = p_1 p_0 = p_1 u_1$, whence $u_1 p_0 u_1^{-1} = p_1$. Thus, the path (p_t) does connect p_0 and p_1.

For part (i), note that as $u_t p_0 = r_t p_0$, we get

$$\|p_t\| = \|r_t p_0 u_t^{-1}\| \leq \|(r_t - p_0)p_0 u_t^{-1}\| + \|p_0 u_t^{-1}\| \leq \frac{1}{12\kappa^2}\kappa\frac{6}{5} + \kappa\frac{6}{5} \leq 2\kappa.$$

For part (ii), let $\delta = \max_{i=0,1}\|[c, p_i]\|$. We compute using the identity $1 - u_t = (2p_0 - 1)(p_0 - r_t)$ that

$$\begin{aligned}\|[u_t, c]\| = \|[1-u_t, c]\| &\leq \|[2p_0 - 1, c]\|\|p_0 - r_t\| + \|2p_0 - 1\|\|[p_0 - r_t, c]\| \\ &\leq 2\|[p_0, c]\|\|p_0 - r_t\| + \|2p_0 - 1\|(\|[p_0, c]\| + \|[r_t, c]\|).\end{aligned}$$

Using that $\|2p_0 - 1\| \leq 2\kappa$ again, this implies that

$$\|[u_t, c]\| \leq 2\delta\frac{1}{12\kappa^2} + 2\kappa\cdot 2\delta = \Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta.$$

Hence, also

$$\|[u_t^{-1}, c]\| = \|u_t^{-1}[c, u_t]u_t^{-1}\| \leq \frac{36}{25}\Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta\|c\|$$

and so

$$\begin{aligned}\|[p_t, c]\| &= \|[u_t p_0 u_t^{-1}, c]\| \\ &\leq \|[u_t, c]\|\|p_0\|\|u_t^{-1}\| + \|u_t\|\|[p_0, c]\|\|u_t^{-1}\| + \|u_t\|\|p_0\|\|[u_t^{-1}, c]\| \\ &\leq \Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta\kappa\frac{6}{5} + \frac{7}{5}\delta + \frac{7}{6}\kappa\frac{36}{25}\Big(4\kappa + \frac{1}{6\kappa^2}\Big)\delta \\ &\leq 21\kappa^2\delta\end{aligned}$$

as claimed. Finally, for part (iii), we again use that $\|2p_0 - 1\| \leq 2\kappa$ to compute that for any $s, t \in [0, 1]$,

$$\begin{aligned}\|u_s - u_t\| &= \|(2p_0 - 1)(r_s - r_t)\| \leq \|2p_0 - 1\||s - t|\|p_0 - p_1\| \leq 2\kappa|s-t|\frac{1}{12\kappa^2} \\ &= \frac{1}{6\kappa}|s - t|\end{aligned}$$

and so

$$\|u_s^{-1} - u_t^{-1}\| = \|u_t^{-1}(u_t - u_s)u_s^{-1}\| \leq \frac{36}{25}\frac{1}{6\kappa}|s - t| = \frac{6}{25\kappa}|s - t|.$$

Hence,

$$\begin{aligned}\|p_t - p_s\| &\leq \|(u_t - u_s)p_0u_t^{-1}\| + \|u_s p_0(u_t^{-1} - u_s^{-1})\| \\ &\leq \frac{1}{6\kappa}|s-t|\kappa\frac{6}{5} + \frac{7}{6}\kappa\frac{6}{25\kappa}|s-t| \\ &\leq |s-t|\end{aligned}$$

as claimed. ∎

The next lemma gives universal control over the "speed" of a homotopy between idempotents (at the price of moving to larger matrices). The basic idea is not new; see for example [47, Proposition 1.31]. We give a complete proof, however, as we need to incorporate commutator estimates and work with idempotents rather than projections.

Lemma 4.15. *Let B be a separable C^*-algebra, let X be a subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, and let $n \in \mathbb{N}$. Let (p_0, q_0) and (p_1, q_1) be elements of the same path component of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. Then, there is $k \in \mathbb{N}$ and a homotopy $((r_t, s_t))_{t\in[0,1]}$ in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$ such that $(r_i, s_i) = (p_i \oplus 1_{nk} \oplus 0_{nk}, q_i \oplus 1_{nk} \oplus 0_{nk})$ for $i \in \{0, 1\}$, and such that the map $t \mapsto (r_t, s_t)$ is (16κ)-Lipschitz.*

Proof. Let $((p_t, q_t))_{t\in[0,1]}$ be an arbitrary homotopy in $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$ connecting (p_0, q_0) and (p_1, q_1). Let $\delta > 0$ be such that if $s, t \in [0, 1]$ satisfy $|s - t| \leq \delta$, then $\|p_s - p_t\| \leq 1/(12\kappa^2)$ and $\|q_s - q_t\| \leq 1/(12\kappa^2)$. Let $0 = t_0 < t_1 < \cdots < t_k = 1$ be a sequence of points in $[0, 1]$ such that $t_{i+1} - t_i \leq \delta$ for all i. We claim that this k works, and to show this we build an appropriate homotopy by concatenating the various steps below.

(i) Connect $(p_0 \oplus 1_{nk} \oplus 0_{nk}, q_0 \oplus 1_{nk} \oplus 0_{nk})$ to

$$\Big(p_0 \oplus \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}},\ q_0 \oplus \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}}\Big)$$

via a 2-Lipschitz rotation homotopy parametrized by $[0, \pi/2]$ and passing through $\mathcal{P}_{(2k+1)n,\kappa,\varepsilon}(X, B)$.

(ii) In the ith "block" $1_n \oplus 0_n$, use the homotopy

$$\begin{pmatrix}1-p_{t_i} & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}\cos(t) & -\sin(t)\\ \sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & p_{t_i}\end{pmatrix}\begin{pmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{pmatrix}$$

(parametrized by $t \in [0, \pi/2]$) to connect $1_n \oplus 0_n$ to $1 - p_{t_i} \oplus p_{t_i}$, and similarly for q. In order to compute commutator estimates, note that rearranging gives that the homotopy above is the same as

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \begin{pmatrix}p_{t_i} & 0\\ 0 & p_{t_i}\end{pmatrix}\begin{pmatrix}-\cos^2(t) & -\sin(t)\cos(t)\\ -\sin(t)\cos(t) & \cos^2(t)\end{pmatrix}, \quad t \in [0, \pi/2].$$

The scalar matrix appearing on the right above has norm $|\cos(t)|$, whence every element in this homotopy has norm at most 2κ. Hence, our homotopy connects the result of the previous stage to

$$(p_0 \oplus 1 - p_{t_1} \oplus p_{t_1} \oplus \cdots \oplus 1 - p_{t_k} \oplus p_{t_k}, q_0 \oplus 1 - q_{t_1} \oplus q_{t_1} \oplus \cdots \oplus 1 - q_{t_k} \oplus q_{t_k})$$

through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$, and is 2κ-Lipschitz.

(iii) From Corollary 4.2, each idempotent $1 - p_{t_i}$ has norm at most κ. For each $i \in \{1, \ldots, k\}$, using that $\|(1 - p_{t_i}) - (1 - p_{t_{i-1}})\| \leq 1/(12\kappa^2)$, Lemma 4.14 gives a path of idempotents connecting $1 - p_{t_i}$ and $1 - p_{t_{i-1}}$ and with the following properties: it is 1-Lipschitz; it consists of idempotents of norm at most 2κ; each idempotent r in the path satisfies $\|[r, x]\| \leq 21\kappa^2\varepsilon$ for all $x \in X$. We get similar paths with respect to the elements $1 - q_{t_i}$, and use these paths to connect the result of the previous stage to

$$(p_0 \oplus 1 - p_{t_0} \oplus p_{t_1} \oplus \cdots \oplus 1 - p_{t_{k-1}} \oplus p_{t_k}, q_0 \oplus 1 - q_{t_0} \oplus q_{t_1} \oplus \cdots \oplus 1 - q_{t_{k-1}} \oplus q_{t_k})$$

via a 1-Lipschitz path in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$.

(iv) Use an analog of the homotopy in step (ii) in each block of the form $p_{t_i} \oplus 1 - p_{t_i}$ (and similarly for q) to connect the result of the previous stage to

$$\Big(\underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}} \oplus p_{t_k}, \underbrace{(1_n \oplus 0_n) \oplus \cdots \oplus (1_n \oplus 0_n)}_{k \text{ times}} \oplus q_{t_k}\Big).$$

This passes through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$, and is 2κ-Lipschitz.

(v) Finally, noting that $p_{t_k} = p_1$ and $q_{t_k} = q_1$, use a rotation homotopy parametrized by $[0, \pi/2]$ to connect the result of the previous stage to $(p_1 \oplus 1_{nk} \oplus 0_{nk}, q_1 \oplus 1_{nk} \oplus 0_{nk})$. This passes through $\mathcal{P}_{(2k+1)n,\kappa,\varepsilon}(X, B)$ and is 2κ-Lipschitz.

Concatenating the five homotopies above gives a 2κ-Lipschitz homotopy, parametrized by $[0, 2\pi + 1]$, that passes through $\mathcal{P}_{(2k+1)n,2\kappa,\varepsilon}(X, B)$ and connects $(p_0 \oplus 1_{nk} \oplus 0_{nk}, q_0 \oplus 1_{nk} \oplus 0_{nk})$ and $(p_1 \oplus 1_{nk} \oplus 0_{nk}, q_1 \oplus 1_{nk} \oplus 0_{nk})$. Reparametrizing by $[0, 1]$, we get a (16κ)-Lipschitz homotopy as required. ■

Before we get to the main result of this section, we give one more elementary lemma; we record it as it will be used multiple times below.

Lemma 4.16. *Say x and $y_1, \ldots, y_n$ are elements of a C^*-algebra such that*

$$\|[x, y_i]\| \leq \delta \quad \text{and} \quad \|y_i\| \leq m$$

for all i. Then, if $y := y_1 y_2 \cdots y_n$, we have $\|[x, y]\| \leq n m^{n-1} \delta$.

Proof. This follows from the formula

$$[x, y] = \sum_{i=1}^{n} \Big(\prod_{1 \leq j < i} y_j \Big) [x, y_i] \Big(\prod_{i < j \leq n} y_j \Big),$$

which itself follows from induction on n and the usual Leibniz formula $[x, y_1 y_2] = y_1[x, y_2] + [x, y_1]y_2$. ∎

Here, is the main result of this section. The basic idea of the proof is contained in [47, Corollary 1.32], but as usual we need to do more work in order to get our estimates.

Proposition 4.17. *Let B be a separable C^*-algebra, let X be a self-adjoint subset of the unit ball of $\mathcal{L}_B$, let $\kappa \geq 1$, and let $\varepsilon > 0$. Let $M = 2^{(100\kappa)^3}$. With notation as in Definition 4.9, let $n \in \mathbb{N}$, and let (p, q) be in the same path component of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, B)$ as an element (r, r) with both entries the same. Then, there is $m \in \mathbb{N}$ and (with notation as in Definition 4.11) an element $u \in \mathcal{U}^1_{n+2m,M,M\varepsilon}(X, B)$ such that*

$$u(p \oplus 1_m \oplus 0_m)u^{-1} = q \oplus 1_m \oplus 0_m.$$

Proof. Let $k \in \mathbb{N}$ be as in the conclusion of Lemma 4.15, so there exists a (16κ)-Lipschitz homotopy in $\mathcal{P}_{(2k+1)n,2\kappa,21\kappa^2\varepsilon}(X, B)$ between $(p \oplus 1_{nk} \oplus 0_{nk}, q \oplus 1_{nk} \oplus 0_{nk})$ and $(r \oplus 1_{nk} \oplus 0_{nk}, r \oplus 1_{nk} \oplus 0_{nk})$. Set $m = kn$. Proposition 4.10 gives a $(20\kappa \cdot 16\kappa)$-Lipschitz path $((p_t, q_t))_{t\in[0,1]}$ passing through $\mathcal{P}^1_{n+2m,2\kappa,84\kappa^2\varepsilon}(X, B)$ that connects $(p \oplus 1_{nk} \oplus 0_{nk}, q \oplus 1_{nk} \oplus 0_{nk})$ and $(r \oplus 1_{nk} \oplus 0_{nk}, r \oplus 1_{nk} \oplus 0_{nk})$. To simplify notation, note this path is $(2^9\kappa^2)$-Lipschitz, and that it passes through $\mathcal{P}^1_{n+2m,2\kappa,2^7\kappa^2\varepsilon}(X, B)$.

Define $N := \lceil 2^{13}\kappa^3 \rceil$ (where $\lceil y \rceil$ is the least integer at least as large as y), and define $t_i = i/N$ for $i \in \{0, \dots, N\}$. As the path $((p_t, q_t))_{t\in[0,1]}$ is $(2^9\kappa^2)$-Lipschitz, for any $i \in \{1, \dots, N\}$, $\|p_{t_i} - p_{t_{i-1}}\| \leq (16\kappa)^{-1}$. For $i \in \{1, \dots, N\}$, define

$$v_i := p_{t_{i-1}} p_{t_i} + (1 - p_{t_{i-1}})(1 - p_{t_i}).$$

As $\|p_{t_i}\| \leq 2\kappa$ for all i, Corollary 4.2 implies that

$$\|2p_{t_i} - 1\| \leq 4\kappa \tag{4.11}$$

for all i, and so

$$\|1 - v_i\| = \|(2p_{t_{i-1}} - 1)(p_{t_{i-1}} - p_{t_i})\| \leq 4\kappa \cdot (16\kappa)^{-1} \leq \frac{1}{2}.$$

It follows that each v_i is invertible, $\|v_i\| \leq 2$, and (by the Neumann series formula for the inverse) $\|v_i^{-1}\| \leq 2$. Note also that as the homotopy $((p_t, q_t))_{t\in[0,1]}$ passes through $\mathcal{P}^1_{(2k+1)n,2\kappa,2^7\kappa^2\varepsilon}(X, B)$ all the elements p_{t_i} must have the same "scalar part" (i.e., the same image under the canonical map $M_{n+2m}(\mathcal{K}_B^+) \to M_{n+2m}(\mathbb{C})$), and so the elements v_i must satisfy

$$1 - v_i \in M_{n+2m}(\mathcal{K}_B).$$

Moreover, for $x \in X$, using line (4.11) again we see that

$$\begin{aligned}\|[v_i, x]\| &= \|[v_i - 1, x]\| \\ &= \|[(2p_{t_{i-1}} - 1)(p_{t_{i-1}} - p_{t_i}), x]\| \\ &\leq 2\|[p_{t_{i-1}}, x]\|(\|p_{t_{i-1}}\| + \|p_{t_i}\|) + \|2p_{t_{i-1}} - 1\|(\|[p_{t_{i-1}}, x]\| + \|[p_{t_i}, x]\|) \\ &\leq 12\kappa \cdot 2^7\kappa^2\varepsilon.\end{aligned}$$

Hence, moreover

$$\|[v_i^{-1}, x]\| = \|v_i^{-1}[x, v_i]v_i^{-1}\| \leq 4 \cdot 12\kappa \cdot 2^7\kappa^2\varepsilon \leq 2^{13}\kappa^3\varepsilon.$$

At this point we have that each v_i is an element of $\mathcal{U}^1_{n+2m,2,2^{13}\kappa^3\varepsilon}$.

Note also that $v_i p_{t_i} = p_{t_{i-1}} p_{t_i} = p_{t_{i-1}} v_i$, and so $v_i p_{t_i} v_i^{-1} = p_{t_{i-1}}$ for each i. Define v to be the product $v_1 v_2 \cdots v_N$, so v satisfies $v^{-1} p_0 v = p_1$, or in other words

$$v^{-1}(p \oplus 1_m \oplus 0_m)v = r \oplus 1_m \oplus 0_m.$$

Note that $1 - v \in M_{n+2m}(\mathcal{K}_B)$. As $\|v_i\| \leq 2$ and $\|v_i^{-1}\| \leq 2$ for each i, we have that $\|v\| \leq 2^N$ and similarly $\|v^{-1}\| \leq 2^N$. Moreover, for any $x \in X$, Lemma 4.16 gives $\|[v, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ and similarly $\|[v^{-1}, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$. Applying the same construction with (q_t) in place of (p_t), we get an invertible element w such that $w^{-1}(q \oplus 1_m \oplus 0_m)w = r \oplus 1_m \oplus 0_m$, such that $1 - w \in M_{n+2m}(\mathcal{K}_B)$, such that $\|w\| \leq 2^N$, $\|w^{-1}\| \leq 2^N$, and such that $\|[w, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ and $\|[w^{-1}, x]\| \leq N2^{N-1} \cdot 2^{13}\kappa^3\varepsilon$ for all $x \in X$. Define $u = wv^{-1}$. As $N = \lceil 2^{13}\kappa^3 \rceil$, this has the claimed properties. ∎