Chapter 5

Reformulating the UCT II

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. For each n, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to reformulate the vanishing results on the UCT of Chapter 2 in terms of the groups $KK^i_{\kappa,\varepsilon}(X, B)$ of Chapter 3. We look at the even ($i = 0$) and odd ($i = 1$) cases separately.

5.1 The even case

Lemma 5.1. *Let $\kappa \geq 1$ and $\varepsilon > 0$. Let B be a separable C^*-algebra, and let X be a self-adjoint subset of the unit ball of $\mathcal{L}_B$. Then, there is a homomorphism $\psi_* : KK^*_{\kappa,\varepsilon/4}(X, B) \to KK^*_{1,\varepsilon}(X, B)$ such that the diagrams*

$$\begin{array}{ccc} KK^0_{1,\varepsilon}(X,B) & & \\ \uparrow{\scriptstyle \psi_*} & \searrow & \\ KK^0_{\kappa,\varepsilon/4}(X,B) & \longrightarrow & KK^0_{\kappa,\varepsilon}(X,B) \end{array} \tag{5.1}$$

and

$$\begin{array}{ccc} KK^0_{1,\varepsilon/4}(X,B) & \longrightarrow & KK^0_{1,\varepsilon}(X,B) \\ & \searrow & \uparrow{\scriptstyle \psi_*} \\ & & KK^0_{\kappa,\varepsilon/4}(X,B) \end{array} \tag{5.2}$$

commute, where the unlabeled arrows are the forget control maps of Definition 3.4.

Proof. Let (p, q) be an element of $\mathcal{P}_{n,\kappa,\varepsilon/4}(X, B)$. Let r and s be the projections associated to p and q respectively as in Definition 4.3. Using Lemma 4.5 parts (i) and (iii) we may define a map

$$\psi : \mathcal{P}_{n,\kappa,\varepsilon/4}(X, B) \to \mathcal{P}_{n,1,\varepsilon}(X, B), \quad (p, q) \mapsto (r, s).$$

Allowing n to vary, and noting that the process of taking associated projections takes homotopies to homotopies (by part (iv) of Lemma 4.5) and block sums to block sums, we get a well-defined homomorphism

$$\psi_* : KK^0_{\kappa,\varepsilon/4}(X, B) \to KK^0_{1,\varepsilon}(X, B).$$

To check commutativity of the diagram in line (5.1), it suffices to show that if $(r, s) \in \mathcal{P}_{n,1,\varepsilon}(X, B)$ is the pair of projections associated to $(p, q) \in \mathcal{P}_{n,\kappa,\varepsilon/4}(X, B)$ as above, then (r, s) and (p, q) are in the same path component of $\mathcal{P}_{n,\kappa,\varepsilon}(X, B)$. This follows from parts (ii) and (iii) of Lemma 4.5. Commutativity of the diagram in line (5.2) is immediate; if (p, q) is in $\mathcal{P}_{n,1,\varepsilon}(X, B)$ for some n, then p and q are themselves projections, so equal their associated projections. ∎

The following lemma records some results from [68, Section A.3] that we will need. For the statement, recall the notion of a unitally strongly absorbing representation from Definition 2.5 above.

Lemma 5.2. *In the statement of this lemma, all unlabeled arrows are forget control maps as in Definitions 2.11 and 3.4. Let A be a separable unital C^*-algebra, and let B be a separable C^*-algebra. Let $\pi : A \to \mathcal{L}_B$ be a strongly unitally absorbing representation of A, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_B$.*

Let $\varepsilon > 0$, and let X be a finite subset of A_1. Then, there exist homomorphisms

$$\alpha : KK^0_{1,\varepsilon}(X, B) \to KK_{5\varepsilon}(X, B)$$

and

$$\beta : KK_\varepsilon(X, B) \to KK^0_{1,\varepsilon}(X, B)$$

that are natural with respect to forget control maps; more precisely if $(X, \varepsilon) \leq (Y, \delta)$ in $\mathcal{X}_A$ as in Definition 2.10 then the diagrams

$$\begin{array}{ccc} KK^0_{1,\delta}(Y, B) & \longrightarrow & KK^0_{1,\varepsilon}(X, B) \\ \downarrow \beta & & \downarrow \beta \\ KK^0_{\delta}(Y, B) & \longrightarrow & KK^0_{\varepsilon}(X, B) \end{array} \quad \text{and} \quad \begin{array}{ccc} KK^0_{1,5\delta}(Y, B) & \longrightarrow & KK^0_{1,5\varepsilon}(X, B) \\ \alpha \uparrow & & \alpha \uparrow \\ KK^0_{\delta}(Y, B) & \longrightarrow & KK^0_{\varepsilon}(X, B) \end{array}$$

are defined and commute.

Moreover, the diagrams

$$\begin{array}{ccc} KK^0_{1,\varepsilon}(X, B) & \longrightarrow & KK^0_{1,5\varepsilon}(X, B) \\ \downarrow \alpha & \nearrow \beta & \\ KK_{5\varepsilon}(X, B) & & \end{array}$$

and

$$\begin{array}{ccc} KK_{\varepsilon}(X, B) & \longrightarrow & KK_{5\varepsilon}(X, B) \\ \downarrow \beta & \nearrow \alpha & \\ KK^0_{1,\varepsilon}(X, B) & & \end{array}$$

commute.

Proof. Let $\pi : A \to M_2(\mathcal{L}_B)$ be (the amplification of) our fixed representation. In the language of [68, Appendix A.2], the groups $KK_\varepsilon(X, B)$ are the same as the groups that are called there $KK_\varepsilon^{\pi,p}(X, B)$, while in the language of [68, Appendix A.3], the groups $KK^0_{1,\varepsilon}(X, B)$ would there be called $KK_\varepsilon^{\pi_0,m}(X, B)$. The lemma thus follows from the arguments of [68, Lemmas A.22, A.23, and A.24]. ∎

We are now able to deduce a version of Corollary 2.22 for the groups of Definition 3.1.

Corollary 5.3. *Let A be a separable, unital, nuclear C^*-algebra. The following are equivalent:*

(i) *A satisfies the UCT.*

(ii) *Let $\kappa \geq 1$ and $\varepsilon \in (0, 1)$. Let B be a separable C^*-algebra with $K_*(B) = 0$. Let $\pi : A \to \mathcal{L}_{SB}$ be a strongly unitally absorbing representation, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, for any finite subset X of A_1, there is a finite subset Z of A_1 such that*

$$(X, \kappa, \varepsilon) \leq (Z, \kappa, \varepsilon/160)$$

in the sense of Definition 3.4, and such that the forget control map

$$KK^0_{\kappa,\varepsilon/160}(Z, SB) \to KK^0_{\kappa,\varepsilon}(X, SB)$$

of Definition 3.4 is zero.

(iii) *There exist $\kappa \geq 1$ and $\nu \geq \kappa$ with the following property. Let $\gamma > 0$, let B be a separable C^*-algebra with $K_*(B) = 0$, and let X be a finite subset of A_1. Let*

$$\pi : A \to \mathcal{L}_{SB}$$

be a strongly unitally absorbing representation, which we use to identify A with a C^*-subalgebra of $\mathcal{L}_{SB}$. Then, there is $\varepsilon > 0$ and a finite subset Z of A_1 such that $(X, \nu, \gamma) \leq (Z, \kappa, \varepsilon)$ in the sense of Definition 3.4, and such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

of Definition 3.4 is zero.

Proof. In the following proof, all unlabeled arrows are forget control maps as in Definition 2.11, or Definition 3.4. Assume first that condition (i) from the statement holds, and let $\kappa \geq 1$ and $\varepsilon > 0$; we may assume moreover that $\varepsilon < 1$. Let a finite subset X be given as in condition (ii). Then, by the equivalence from Corollary 2.22, there is a finite subset Z of A_1 such that the forget control map

$$KK_{\varepsilon/8}(Z, SB) \to KK_\varepsilon(X, SB)$$

is zero. Replacing Z by $Z \cup Z^*$ if necessary, we may assume that Z is self-adjoint. Lemma 5.2 gives a commutative diagram

$$\begin{array}{ccc} KK_{\varepsilon/8}(Z, SB) & \xrightarrow{\;\;0\;\;} & KK_{\varepsilon}(X, SB) \\ {\scriptstyle\alpha}\uparrow & & \downarrow{\scriptstyle\beta} \\ KK^0_{1,\varepsilon/40}(Z, SB) & \longrightarrow & KK^0_{1,\varepsilon}(X, SB), \end{array}$$

whence the bottom horizontal map is zero. On the other hand, Lemma 5.1 (see in particular line (5.1)) gives a map ψ_* such that the bottom triangle in the diagram below

$$\begin{array}{ccc} KK^0_{1,\varepsilon/40}(Z, SB) & \xrightarrow{\;\;0\;\;} & KK^0_{1,\varepsilon}(X, SB) \\ {\scriptstyle\psi_*}\uparrow & \searrow & \downarrow \\ KK^0_{\kappa,\varepsilon/160}(Z, SB) & \longrightarrow & KK_{\kappa,\varepsilon}(X, SB) \end{array}$$

commutes. The top triangle also commutes as all the maps involved are forget control maps, whence the bottom horizontal map is zero. This gives us condition (ii) from the statement.

Condition (ii) clearly implies condition (iii), so it remains to show that condition (iii) implies condition (i). For this, it suffices to establish condition (ii) from Theorem 2.15, so let $\gamma > 0$ and a finite subset X of A_1 be given. Then, according to condition (iii) there are $\nu \geq \kappa \geq 1$, $\varepsilon > 0$ and a finite subset Z of A_1 such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma/20}(X, SB)$$

is defined and zero. Replacing Z with $Z \cup Z^*$ if necessary, we may assume Z is self-adjoint. Using Lemma 5.1 (see in particular line (5.2)) there is a map ψ_* such that the top right triangle in the diagram below commutes

$$\begin{array}{ccccc} KK^0_{1,\varepsilon}(Z, SB) & \longrightarrow & KK_{1,\gamma/20}(X, SB) & \longrightarrow & KK_{1,\gamma/5}(X, SB) \\ \downarrow & & \downarrow & \searrow & \uparrow{\scriptstyle\psi_*} \\ KK^0_{\kappa,\varepsilon}(Z, SB) & \xrightarrow[\;\;0\;\;]{} & KK_{\nu,\gamma/20}(X, SB) & \longrightarrow & KK^0_{\nu,\gamma/20}(X, SB). \end{array}$$

The rest of the diagram also commutes, as all the arrows are forget control maps, whence the composition

$$KK^0_{1,\varepsilon}(Z, SB) \to KK_{1,\gamma/20}(X, SB) \to KK_{1,\gamma/5}(X, SB)$$

of the two top horizontal maps is zero. Using Lemma 5.2, there is a commutative diagram

$$\begin{array}{ccc} KK_{\varepsilon}(Z, SB) & \longrightarrow & KK_{\gamma}(X, SB) \\ \beta \big\downarrow & & \big\uparrow \alpha \\ KK^{0}_{1,\varepsilon}(Z, SB) & \xrightarrow[0]{} & KK_{1,\gamma/5}(X, SB). \end{array}$$

The top horizontal map is therefore zero; this is the conclusion we need for Theorem 2.15, condition (ii), so we are done. ■

5.2 The odd case

For the statement of the next lemma, consider the Hilbert module $\ell^2 \otimes SB$ associated to the suspension $SB = C_0((0,1), B)$ of a separable C^*-algebra B. Let $C_{sb}(X, M(C))$ denote the C^*-algebra of bounded and strictly continuous functions from a locally compact space X to the multiplier algebra $M(C)$ of a C^*-algebra C. For any C^*-algebra C there are canonical identifications

$$\mathcal{L}_C = M(C \otimes \mathcal{K})$$

(see for example [45, Theorem 2.4]) and $M(C_0(X, C)) = C_{sb}(X, M(C))$ (see for example [1, Corollary 3.4]). Hence, there is a canonical identification

$$\mathcal{L}_{SB} = C_{sb}((0,1), \mathcal{L}_B). \tag{5.3}$$

We identify $\mathcal{L}_B = \mathcal{L}(\ell^2 \otimes B)$ with a C^*-subalgebra of $\mathcal{L}_{SB} = \mathcal{L}(\ell^2 \otimes B \otimes C_0(0,1))$ via the $*$-homomorphism $a \mapsto a \otimes 1_{C_0(0,1)}$. We recall also that $\mathcal{K}_B^+$ denotes the unitization of $\mathcal{K}_B$.

Lemma 5.4. *Let B be a separable C^*-algebra. Let $\kappa \geq 1$, $\varepsilon > 0$, and let X be a subset of the unit ball of $\mathcal{L}_B$. Then,*

(i) *Elements of $\mathcal{P}_{n,\kappa,\varepsilon}(X, SB)$ (see Definition 3.1) identify canonically with continuous paths $(p_t, q_t)_{t\in[0,1]}$ of idempotents in $M_n(\mathcal{K}_B^+) \oplus M_n(\mathcal{K}_B^+)$ satisfying the following conditions:*

   (a) *for all $t \in [0,1]$, $\|p_t\| \leq \kappa$ and $\|q_t\| \leq \kappa$;*

   (b) *for all $t \in [0,1]$ and all $x \in X$, $\|[p_t, x]\| < \varepsilon$ and $\|[q_t, x]\| < \varepsilon$;*

   (c) *there are $p, q \in M_n(\mathbb{C})$ such that $p_0 = p_1 = p$, $q_0 = q_1 = q$ and such that if $\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$ is the canonical quotient map then $\sigma(p_t) = p$ and $\sigma(q_t) = q$ for all $t \in [0,1]$.*

   Moreover, the element (p,q) is in the subset $\mathcal{P}^1_{n,\kappa,\varepsilon}(X, SB)$ of Definition 4.9 if and only if p and q are equal to 1_l for some $l \in \mathbb{N}$.

(ii) *Elements of* $\mathcal{U}_{n,\kappa,\varepsilon}(X, SB)$ *(see Definition 3.5) identify with continuous paths* $(u_t)_{t\in[0,1]}$ *of invertibles in* $M_n(\mathcal{K}_B^+)$ *satisfying the following conditions:*

(a) *for all* $t \in [0,1]$, $\|u_t\| \leq \kappa$ *and* $\|u_t^{-1}\| \leq \kappa$;

(b) *for all* $t \in [0,1]$ *and all* $x \in X$, $\|[u_t, x]\| < \varepsilon$ *and* $\|[u_t^{-1}, x]\| < \varepsilon$;

(c) *there is* $u \in M_n(\mathbb{C})$ *such that* $u_0 = u_1 = u$ *and such that if*

$$\sigma : M_n(\mathcal{K}_B^+) \to M_n(\mathbb{C})$$

is the canonical quotient map then $\sigma(u_t) = u$ *for all* $t \in [0,1]$.

Moreover, the element is in the subset $\mathcal{U}^1_{n,\kappa,\varepsilon}(X, SB)$ *of Definition 4.11 if and only if* u *is the identity.*

Proof. We have a canonical identification

$$\mathcal{K}_{SB}^+ = \{f \in C([0,1], \mathcal{K}_B^+) \mid \sigma(f(t)) = f(0) = f(1) \text{ for all } t \in [0,1]\}.$$

Part (i) follows directly by comparing this with Definitions 3.1 and 4.9; similarly, part (ii) follows from comparing this with Definitions 3.5 and 4.11. We leave the details to the reader. ■

Lemma 5.5. *For any* $\kappa \geq 1$ *there exists a positive constant* M_1 *with the following property. Let* $\varepsilon > 0$, *let* A *be a separable, unital, nuclear* C^**-algebra that satisfies the UCT, and let* B *be a separable* C^**-algebra with* $K_*(B) = 0$. *Let* $\pi : A \to \mathcal{L}_{SB}$ *be a strongly unitally absorbing representation that factors through the subalgebra* $\mathcal{B}(\ell^2)$ *(such exists by Lemma 2.6), and use this to identify* A *with a* C^**-subalgebra of* $\mathcal{L}_{SB}$.

Then, for any finite subset X *of* A_1 *there exists a finite subset* Z *of* A_1 *such that the forget control map*

$$KK^1_{\kappa,\varepsilon}(Z, SB) \to KK^1_{M_1,M_1\varepsilon}(X, SB)$$

of Definition 3.7 is defined and zero.

Proof. We claim $M_1 = 2^{(200\kappa^8)^3} \cdot 320\kappa^7$ works. Using Corollary 5.3 there is a finite subset Z of A_1 such that the forget control map

$$KK^0_{\kappa^8,2\kappa^6\varepsilon}(Z, SB) \to KK^0_{\kappa^8,320\kappa^6\varepsilon}(X, SB) \tag{5.4}$$

of Definition 3.4 is zero. We claim this set Z works.

Let u be an arbitrary element of $\mathcal{U}_{n,\kappa,\varepsilon}(Z, B)$. Using Proposition 4.13 (i), and with notation as in Definition 4.11, there is an element v of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(Z, B)$ in the same path component of $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(Z, B)$ as u. Define now a path $(v_t)_{t\in[0,1]}$ by

$$v_t := \begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) \\ -\sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

Note that each v_t is an element of $\mathcal{U}^1_{2n,\kappa^4,2\kappa^3\varepsilon}(Z,B)$. Define

$$p_t := v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1}.$$

Write $\underline{p}$ for the path (p_t), and note that according to Lemma 5.4 (i), we may identify the pair $(\underline{p}, 1_n \oplus 0_n)$ with (using the notation of Definition 4.9) an element of $\mathcal{P}^1_{2n,\kappa^8,2\kappa^7\varepsilon}(Z,SB)$, and therefore also a class $[\underline{p}, 1_n \oplus 0_n] \in KK^0_{\kappa^8,2\kappa^7\varepsilon}(Z,SB)$. By assumption, the forget control map in line (5.4) is zero, and therefore the image of $[\underline{p}, 1_n \oplus 0_n]$ in $KK^0_{\kappa^8,320\kappa^7\varepsilon}(X,SB)$ is zero. For notational simplicity, at this point let us define $\varepsilon_1 := 320\kappa^7\varepsilon$.

Now, Lemma 3.3 gives $m \in \mathbb{N}$ and $(s,s) \in \mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X,SB)$ such that $(\underline{p} \oplus 1_m \oplus 0_m, 1_n \oplus 0_n \oplus 1_m \oplus 0_m)$ and (s,s) are in the same path component of the set $\mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X,SB)$. Let x be a unitary matrix in $M_{2(n+m)}(\mathbb{C})$ such that $x(1_n \oplus 0_n \oplus 1_m \oplus 0_m)x^* = 1_{n+m} \oplus 0_{n+m}$. As x is connected to the identity through unitaries, the element $(x(\underline{p} \oplus 1_m \oplus 0_m)x^*, 1_{n+m} \oplus 0_{n+m})$ is also homotopic to (s,s) in $\mathcal{P}_{2(n+m),2\kappa^8,\varepsilon_1}(X,SB)$; moreover (with notation as in Definition 4.9), it is in $\mathcal{P}^1_{2(n+m),2\kappa^8,\varepsilon_1}(X,SB)$. We may now apply Proposition 4.17 to see that if

$$M = 2^{(200\kappa^8)^3}$$

then there is $k \in \mathbb{N}$ and an element $\underline{w}$ of $\mathcal{U}^1_{2(n+m+k),M,M\varepsilon_1}(X,SB)$ such that

$$\underline{w}(x(\underline{p} \oplus 1_m \oplus 0_m)x^* \oplus 1_k \oplus 0_k)\underline{w}^{-1} = 1_{n+m} \oplus 0_{n+m} \oplus 1_k \oplus 0_k.$$

Write $\underline{v}$ for the path defined in line (5.5) above, which naturally defines an element of $\mathcal{L}_{SB}$ using the identification in line (5.3). Then, if we define

$$\underline{y} := \underline{w}(x \oplus 1_{2k})(\underline{v} \oplus 1_{2(m+k)}) \in \mathcal{L}_{SB},$$

we have

$$\underline{y}(1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k)\underline{y}^{-1} = 1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k.$$

In other words, the element $\underline{y}$ commutes with $1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k$. Define

$$\underline{z} := (1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k)\underline{y}(1_n \oplus 0_n \oplus 1_m \oplus 0_m \oplus 1_k \oplus 0_k).$$

Using Lemma 5.4 (ii), we may think of $\underline{z}$ as a continuous path $(z_t)_{t\in[0,1]}$ in $\mathcal{U}_{n+m+k,M,M\varepsilon_1}(X,B)$. Now, write $\underline{w}$ as a path $(w_t)_{t\in[0,1]}$, and note that as $\underline{w}$ is in $\mathcal{U}^1_{2(n+m+k),M,M\varepsilon_1}(X,SB)$, then by Lemma 5.4 (ii), $w_0 = w_1 = 1_{2(n+m)}$. Moreover, $v_0 = 1_{2n}$ by definition. Hence, $z_0 = x \oplus 1_k$. On the other hand $v_1 = u \oplus u^{-1} \oplus 1_{2(m+k)}$ and so $z_1 = (x \oplus 1_k)(u \oplus 1_{m+k})$. Hence, $(x \oplus 1_k)^*\underline{z}$ defines a homotopy in $\mathcal{U}_{n+m+k,M,M\varepsilon_1}(X,B)$ between 1_{n+m+k} and $u \oplus 1_{m+k}$. This implies $[u]$ maps to zero in $KK^1_{M,M\varepsilon_1}(X,SB)$, which completes the proof. ∎