Chapter 6

# A Mayer–Vietoris boundary map

In this chapter (as throughout), if $B$ is a separable $C^*$-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ denote respectively the adjointable and compact operators on the standard Hilbert $B$-module $\ell^2 \otimes B$. For each $n$, we consider $\mathcal{L}_B$ as a subalgebra of $M_n(\mathcal{L}_B)$ via the "diagonal inclusion" $\mathcal{L}_B = 1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$.

Our goal in this chapter is to construct and analyse a "Mayer–Vietoris boundary map" in controlled $KK$-theory. The main results of the chapter prove the existence of this boundary map (Proposition 6.1) and show it has an exactness property (Proposition 6.6). These results are the technical heart of the paper.

## 6.1 Existence

Here, is the construction of the boundary map.

**Proposition 6.1.** *Define an increasing function $N_0 : [1,\infty) \to [0,\infty)$ by the formula $N_0(\kappa) = 2^{27}\kappa^{24}$. This function has the following properties.*

*Let $\kappa \geq 1$, let $N_0 = N_0(\kappa)$, let $\varepsilon > 0$, let $B$ be a separable $C^*$-algebra, and let $X$ be a subset of the unit ball of $\mathcal{L}_B$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$. Then, there is a homomorphism*

$$\partial : KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0,N_0\varepsilon}(X \cup \{h\}, B)$$

*defined by applying the following process to a class from $KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$:*

(i) *Choose a representative $w \in \mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ for the class, and use Proposition 4.13 (i) to find an element*

$$u \in \mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

*that is in the same path component as $w$ in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$.*

(ii) *Define*

$$c = c(u,h) := hu + (1-h), \quad d = d(u,h) := hu^{-1} + (1-h) \tag{6.1}$$

*in $M_n(\mathcal{L}_B)$, and*

$$v = v(u,h) := \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -d & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_{2n}(\mathcal{L}_B). \tag{6.2}$$

(iii) *Define*

$$\partial[w] := \left[ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

*Moreover, the boundary map is "natural with respect to forget control maps"; precisely, if for some $\kappa \leq \lambda$ and $\varepsilon \leq \delta$, the boundary maps*

$$\partial : KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0(\kappa),N_0(\kappa)\varepsilon}(X \cup \{h\}, B)$$

*and*

$$\partial : KK^1_{\lambda,\delta}(h(1-h)X \cup \{h\}, B) \to KK^0_{N_0(\lambda),N_0(\lambda)\delta}(X \cup \{h\}, B)$$

*both exist, then the diagram*

$$\begin{array}{ccc} KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0(\kappa),N_0(\kappa)\varepsilon}(X \cup \{h\}, B) \\ \downarrow & & \downarrow \\ KK^1_{\lambda,\delta}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0(\lambda),N_0(\lambda)\delta}(X \cup \{h\}, B) \end{array}$$

*(with vertical maps the forget control maps of Definitions 3.4 and 3.7) commutes.*

In order to make the proof more palatable, we split off some computations as lemmas. The proofs of these lemmas are elementary, but the second one is quite lengthy. We record them for the sake of completeness, but recommend the reader skips the proofs.

**Lemma 6.2.** *Let $B$ be a separable $C^*$-algebra. Let $u \in M_n(\mathcal{L}_B)$ be an invertible element such that $1-u \in M_n(\mathcal{K}_B)$, and let $h \in \mathcal{L}_B$ be a positive contraction. Then, the elements $c = c(u,h)$ and $d = d(u,h)$ from line (6.1) above have the following properties.*

(i) *The elements $cd-1$ and $dc-1$ are in $M_n(\mathcal{K}_B)$.*

(ii) *If $\kappa \geq 1$ and $\varepsilon > 0$ are such that $\|u\| \leq \kappa$, $\|u^{-1}\| \leq \kappa$, $\|[h,u]\| < \varepsilon$, and $\|[h,u^{-1}]\| < \varepsilon$, then $cd-1$ and $dc-1$ are both closer than $(\kappa+1)\varepsilon$ to $h(1-h)(u+u^{-1}-2)$.*

*Proof.* We just look at the case of $cd-1$ for both parts (i) and (ii); the case of $dc-1$ is similar. Note first that because $1-u$ is in $M_n(\mathcal{K}_B)$ and $M_n(\mathcal{K}_B)$ is an ideal in $M_n(\mathcal{L}_B)$, we must have that $1-u^{-1}$ is in $M_n(\mathcal{K}_B)$ also. We compute that

$$\begin{aligned} cd-1 &= huhu^{-1} + (1-h)hu^{-1} + hu(1-h) - 2h + h^2 \\ &= h^2 + hu[h,u^{-1}] + h(1-h)u^{-1} \\ &\quad + h(1-h)u + [h,u](1-h) - 2h + h^2. \end{aligned} \tag{6.3}$$

Using that $u$ and $u^{-1}$ equal 1 modulo the ideal $M_n(\mathcal{K}_B)$, we compute that this equals 0 modulo $M_n(\mathcal{K}_B)$. Hence, $cd-1$ is in $M_n(\mathcal{K}_B)$

Looking at part (ii), note that the terms $hu[h,u^{-1}]$ and $[h,u](1-h)$ in line (6.3) above have norms at most $\kappa\varepsilon$ and $\varepsilon$ respectively. Hence, $cd-1$ is within $(\kappa+1)\varepsilon$ of $h^2+h(1-h)u^{-1}+h(1-h)u-2h+h^2$, which equals $h(1-h)(u+u^{-1}-2)$. ∎

**Lemma 6.3.** *Let $B$ be a separable $C^*$-algebra. Let $\kappa \geq 1$, $\varepsilon > 0$, and let $X$ be a subset of the unit ball of $\mathcal{L}_B$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$, and let $u$ be an element of the set $\mathcal{U}^1_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ from Definition 4.11. Let $c = c(u,h)$ and $d = d(u,h)$ be as in line (6.1) above, and let $v = v(u,h)$ be as in line (6.2).*

*Then, $\|v\| \leq (\kappa+2)^3$, $\|v^{-1}\| \leq (\kappa+2)^3$, and the pair*

$$\left(v\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}v^{-1}, \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right)$$

*is an element of $\mathcal{P}^1_{2n,3^6\kappa^6,2^{16}\kappa^5\varepsilon}(X \cup \{h\}, B)$ from Definition 4.9.*

*Proof.* From the definition of $v$ in line (6.2) above,

$$v = \begin{pmatrix}c(dc-2) & 1-cd\\ dc-1 & -d\end{pmatrix} \tag{6.4}$$

and

$$v^{-1} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -c\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ d & 1\end{pmatrix}\begin{pmatrix}1 & -c\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-d & dc-1\\ 1-cd & c(dc-2)\end{pmatrix}.$$

Hence,

$$v\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}v^{-1} = \begin{pmatrix}cd(2-cd) & c(dc-2)(dc-1)\\ (1-dc)d & (dc-1)^2\end{pmatrix}$$

and so

$$v\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}v^{-1} - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}-(cd-1)^2 & (cd-1)c(dc-2)\\ (1-dc)d & (dc-1)^2\end{pmatrix}. \tag{6.5}$$

This formula, part (i) of Lemma 6.2, and the fact that $M_n(\mathcal{K}_B)$ is an ideal in $M_n(\mathcal{L}_B)$ imply that

$$v\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}v^{-1} - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} \in M_{2n}(\mathcal{K}_B),$$

whence $v\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)v^{-1}$ is in $M_{2n}(\mathcal{K}_B^+)$, and $v\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)v^{-1}$ and $\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ have the same image under the image of the canonical quotient map

$$\sigma : M_{2n}(\mathcal{K}_B^+) \to M_{2n}(\mathbb{C}).$$

Note moreover that $\|v\| \leq (\kappa+2)^3$ and $\|v^{-1}\| \leq (\kappa+2)^3$ from the formula for $v$ (whence also $v^{-1}$) as a product of four matrices in line (6.2). As $\kappa \geq 1$, this implies that

$$\left\| v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} \right\| \leq (\kappa+2)^6 \leq 3^6\kappa^6.$$

To complete the proof that the pair

$$\left( v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$$

defines an element of $\mathcal{P}^1_{2n,3^6\kappa^6,2^{16}\kappa^5\varepsilon}(X,B)$ it remains to check the relevant commutator estimates, i.e., condition (ii) from Definition 3.1 with $x$ in $X \cup \{h\}$ and $\varepsilon$ replaced by $2^{16}\kappa^5\varepsilon$. As $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ (and indeed, any scalar matrix) commutes with elements of $X \cup \{h\}$ exactly, it suffices to show that

$$\left\| \left[ x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| \leq 2^{16}\kappa^5\varepsilon \tag{6.6}$$

for all $x \in X \cup \{h\}$. We focus on the case when $x$ is in $X$; the case when $x = h$ follows from similar (and much simpler) estimates that we leave to the reader.

Working towards the estimate in line (6.6), we compute that the element in line (6.5) equals

$$\begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix}. \tag{6.7}$$

The second matrix above satisfies

$$\begin{aligned} \left\| \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right\| &\leq \|1-cd\| + \|c\|\|dc-2\| + \|d\| + \|dc-1\| \\ &\leq ((\kappa+1)^2+1) + (\kappa+1)((\kappa+1)^2+2) \\ &\quad + (\kappa+1) + ((\kappa+1)^2+1). \end{aligned}$$

As $\kappa + 1 \geq 1$, we therefore see that

$$\left\| \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix} \right\| \leq 8(\kappa+1)^4. \tag{6.8}$$

On the other hand, using part (ii) of Lemma 6.2, the first matrix in line (6.7) above is closer than $\varepsilon(\kappa+1)$ to $h(1-h)(u+u^{-1}-2)$ (we identify this as usual with the diagonal matrix with both entries equal to $h(1-h)(u+u^{-1}-2)$). Hence, the difference in line (6.5) is closer than $8(\kappa+1)^5\varepsilon$ to

$$h(1-h)(u+u^{-1}-2) \begin{pmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{pmatrix}.$$

Hence, for $x \in X$,

$$\left\| \left[ x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| < 16(\kappa + 1)^5 \varepsilon + \left\| \left[ x, h(1-h)(u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \right\|. \tag{6.9}$$

As $\|[x, h]\| < \varepsilon$, we have $\|[x, h(1-h)]\| < 2\varepsilon$; combining this with line (6.8) gives

$$\left\| \left[ x, h(1-h)(u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \right\| < 2\varepsilon \cdot 8(\kappa + 1)^5 + \left\| h(1-h) \left[ x, (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \right\|.$$

Combining this with line (6.9) gives

$$\left\| \left[ x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| < 32(\kappa + 1)^5 \varepsilon + \left\| h(1-h) \left[ x, (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \right\|. \tag{6.10}$$

Every entry of the matrix $(u + u^{-1} - 2)\left(\begin{smallmatrix} 1-cd & c(dc-2) \\ -d & dc-1 \end{smallmatrix}\right)$ can be written as a sum of at most 30 terms, each of which is a product of at most 5 elements from the set $\{u, u^{-1}, h, 1\}$, each of which has norm at most $\kappa$. As $\|[h(1-h)x, y]\| < \varepsilon$ for all $y \in \{u, u^{-1}, h, 1\}$, Lemma 4.16 gives

$$\left\| \left[ h(1-h)x, (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \right\| < 4 \cdot 30 \cdot 5 \cdot \kappa^4 \varepsilon. \tag{6.11}$$

On the other hand, $\|[h(1-h), y]\| < 2\varepsilon$ for all $y \in \{u, u^{-1}, h, 1\}$, whence

$$\left\| \left[ h(1-h), (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] x \right\| < 4 \cdot 30 \cdot 5 \cdot \kappa^4 \varepsilon. \tag{6.12}$$

Finally, note that

$$\begin{aligned} h(1-h) &\left[ x, (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \\ &= \left[ h(1-h)x, (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] \\ &\quad + \left[ h(1-h), (u + u^{-1} - 2) \begin{pmatrix} 1 - cd & c(dc-2) \\ -d & dc - 1 \end{pmatrix} \right] x, \end{aligned}$$

so combining lines (6.10), (6.11), and (6.12) implies

$$\left\| \left[ x, v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \right\| < 1232(\kappa + 1)^5 \varepsilon.$$

Recalling that $\kappa \geq 1$, this is enough for the estimate in line (6.6). ■

We are now ready for the proof of Proposition 6.1.

*Proof of Proposition* 6.1. Assume that $w \in \mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$, and let

$$u \in \mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

be in the same path component as $w$ in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$; $u$ is guaranteed to exist by Proposition 4.13 (i). Define $v := v(u,h)$ as in line (6.2), so Lemma 6.3 gives an element

$$\left( v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \in \mathcal{P}_{2n,3^6\kappa^{12},2^{16}\kappa^{11}\varepsilon}(X \cup \{h\}, B).$$

Moreover, if $u_0 := u$, and $u_1$ is another choice of element in

$$\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$$

that is connected to $w$ in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ then Proposition 4.13 (ii) implies that there is a homotopy $(u_t)_{t\in[0,1]}$ that connects $u_0$ and $u_1$ through

$$\mathcal{U}^1_{n,\kappa^4,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B).$$

Let $v_t := v(u_t, h)$ be as in line (6.2). Then, Lemma 6.3 implies that the path

$$t \mapsto \left( v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad t \in [0,1]$$

has image in $\mathcal{P}^1_{2n,3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$. In particular, the class

$$\partial[w] \in KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$$

does not depend on the choice of $u$, so at this point we have a well-defined set map

$$\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{3^6\kappa^{24},2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B).$$

We next claim that this map sends block sums on the left to sums on the right.

For this, assume that $w_1$ and $w_2$ are elements of $\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. Let $u_1$ and $u_2$ be elements of $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are connected to $w_1$ and

$w_2$ respectively in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$. For $i \in \{1,2\}$ let $v_i = v(u_i, h)$ be as in line (6.2), and let $v := v(u_1 \oplus u_2, h) \in M_{4n}(\mathcal{L}_B)$. Then, the pairs

$$\left(v_1 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_1^{-1} \oplus v_2 \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v_2^{-1}, \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right)$$

and

$$\left(v \begin{pmatrix} 1_{2n} & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1_{2n} & 0 \\ 0 & 0 \end{pmatrix}\right)$$

in $M_{4n}(\mathcal{K}_B^+) \oplus M_{4n}(\mathcal{K}_B^+)$ differ by conjugation by the same (scalar) permutation matrix in each component, and so define the same class in $KK^0_{36\kappa^{24}, 2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$.

At this point, we have a semigroup homomorphism

$$\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B) \to KK^0_{36\kappa^{24}, 2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B).$$

We claim that it respects the equivalence relation defining $KK^1_{\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. First, we check that $w \oplus 1_k$ goes to the same class as $w$. As we already know we have a semigroup homomorphism, it suffices to show that $1_k$ goes to zero in $KK^0_{36\kappa^{24}, 2^{16}\kappa^{20}\varepsilon}(X \cup \{h\}, B)$. For this, note that if $v := v(1_k, h)$ is as in line (6.2), then $v = 1_{2k}$, whence the image of $1_k$ in $KK^0_{36\kappa^{24}, 2^{16}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$ is the class $[1_k \oplus 0_k, 1_k \oplus 0_k]$, which is zero by definition.

Let us now show that elements of $\mathcal{U}_{n,\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are homotopic through $\mathcal{U}_{n,2\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$ go to the same class. For this, say that $w_0$ and $w_1$ are homotopic through $\mathcal{U}_{n,2\kappa,\varepsilon}(h(1-h)X \cup \{h\}, B)$. Choose $u_0$ and $u_1$ in $\mathcal{U}^1_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ that are connected to $w_0$ and $w_1$ respectively in $\mathcal{U}_{n,\kappa^2,\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$ as in Proposition 4.13 (i). Using Proposition 4.13 (ii), $u_0$ and $u_1$ are connected by a homotopy $(u_t)_{t\in[0,1]}$ in $\mathcal{U}^1_{n,4\kappa^4,2\kappa\varepsilon}(h(1-h)X \cup \{h\}, B)$. Let $v_t := v(u_t, h)$ be as in line (6.2). Then, Lemma 6.3 implies that the path

$$\left(v_t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v_t^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)$$

defines a homotopy between the images of $w_0$ and $w_1$ in $\mathcal{P}^1_{2n,3^{14}\kappa^{24},2^{27}\kappa^{21}\varepsilon}(X \cup \{h\}, B)$. We thus see that $N_0(\kappa) := 2^{27}\kappa^{24}$ has the desired property, and we are done with the existence of $\partial$.

As the formulas for the boundary map $\partial$ do not depend on the constants $\kappa$ and $\varepsilon$ the naturality statement is clear. ■

## 6.2 Exactness

We now turn to the exactness property of the boundary map. In order to state this, we need two lemmas.

**Lemma 6.4.** *Let $B$ be a separable $C^*$-algebra. Let $X$ and $Y$ be subsets of the unit ball of $\mathcal{L}_B$, $\varepsilon > 0$ and $\kappa \geq 1$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$. With notation as in Definition 3.1, let*

$$(p,q) \in \mathcal{P}_{n,\kappa,\varepsilon}(X \cup Y \cup \{h\}, B)$$

*(respectively, with notation as in Definition 4.9, let $(p,q) \in \mathcal{P}^{(1)}_{n,\kappa,\varepsilon}(X \cup Y \cup \{h\}, B)$). Then,*

$$(p,q) \in \mathcal{P}_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$$

*(respectively, $(p,q) \in \mathcal{P}^1_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$).*

*In particular, there are homomorphisms*

$$\eta_h : KK^0_{\kappa,\varepsilon}(X \cup Y \cup \{h\}, B) \to KK^0_{\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$$

*and*

$$\eta_{1-h} : KK^0_{\kappa,\varepsilon}(X \cup Y \cup \{h\}, B) \to KK^0_{\kappa,2\varepsilon}((1-h)X \cup Y \cup \{h\}, B)$$

*induced by the identity map on cycles $(p,q)$.*

*Proof.* We compute that for $x \in X$,

$$\|[p,hx]\| \leq \|h\|\|[p,x]\| + \|[p,h]\|\|x\| < \varepsilon + \varepsilon.$$

These estimates hold similarly for $q$ so $(p,q) \in \mathcal{P}^1_{n,\kappa,2\varepsilon}(hX \cup Y \cup \{h\}, B)$. As the identity map on cycles takes homotopies to homotopies, and block sums to block sums, existence of the homomorphism $\eta_h$ is clear. Existence of $\eta_{1-h}$ follows on noting that the assumptions on $h$ also holds for $1-h$. ∎

We leave the direct checks needed for the proof of the next lemma for the reader.

**Lemma 6.5.** *Let $B$ be a separable $C^*$-algebra. Let $X$ and $Y$ be subsets of the unit ball of $\mathcal{L}_B$, $\varepsilon > 0$ and $\kappa \geq 1$. Assume moreover that there is $\delta > 0$ such that for all $y \in Y$, $x \in_\delta X$. Then, for any $\gamma \geq \kappa\delta + \varepsilon$ and $\lambda \geq \kappa$, the forget control map of Definition 3.4*

$$KK^0_{\kappa,\varepsilon}(X, B) \to KK_{\lambda,\gamma}(Y, B)$$

*is well-defined.* ∎

The next proposition is the exactness property of the Mayer–Vietoris boundary map that we are aiming for. We refer the reader to Section 1.6 for motivation behind the statement. For the statement, recall that for an element $x$ and subset $Y$ of a metric space, and for $\varepsilon > 0$, we write "$x \in_\varepsilon S$" to mean that there is $y \in Y$ with $d(x,y) < \varepsilon$. Moreover, in the statement below, all unlabeled arrows between controlled $KK$-groups are the forget control maps of Definition 3.4 or Definition 3.7.

**Proposition 6.6.** *The increasing functions $N_1, N_2 : [1,\infty) \to [1,\infty)$ defined by*

$$N_1(\lambda) = 2^{9000000\lambda^3} \quad \textit{and} \quad N_2(\mu) = 2^{3}7\mu^{25}.$$

*satisfy the following properties.*

*Let $\kappa \geq 1$, and let $\varepsilon > 0$. Let $\lambda \geq \kappa$, and let $\delta \geq 3\kappa\varepsilon$. Let $N_1 := N_1(\lambda)$, and let $\mu \geq N_1$ and $\gamma \geq N_1\delta$. With notation as in Proposition 6.1, define*

$$N_0 := N_0(\mu),$$

*and let $N_2 := N_2(\mu)$.*

*Let $B$ be a separable $C^*$-algebra, and let $X$ be a self-adjoint subset of the unit ball of $\mathcal{L}_B$. Let $h \in \mathcal{L}_B$ be a positive contraction such that $\|[h,x]\| < \varepsilon$ for all $x \in X$. Let $Y_h$, $Y_{1-h}$, and $Y$ be self-adjoint subsets of the unit ball of $\mathcal{L}_B$ such that $y \in_\varepsilon Y_h$ and $y \in_\varepsilon Y_{1-h}$ for all $y \in Y$. With notation as in Definition 4.9, let $(p,q)$ be an element of $\mathcal{P}^1_{n,\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B)$. With $\eta_h$ and $\eta_{1-h}$ as in Lemma 6.4, and suing Lemma 6.5 to define the right hand vertical maps in each case, assume that the images of $[p,q]$ under the maps*

$$\begin{array}{ccc}
KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B) & & \\
\downarrow & & \\
KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup \{h\}, B) & \xrightarrow{\eta_h} & KK^0_{\kappa,2\varepsilon}(hX \cup Y_h \cup \{h\}, B) \\
& & \downarrow \\
& & KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B)
\end{array} \tag{6.13}$$

*and*

$$\begin{array}{ccc}
KK^0_{\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B) & & \\
\downarrow & & \\
KK^0_{\kappa,\varepsilon}(X \cup Y_{1-h} \cup \{h\}, B) & \xrightarrow{\eta_{1-h}} & KK^0_{\kappa,2\varepsilon}((1-h)X \cup Y_{1-h} \cup \{h\}, B) \\
& & \downarrow \\
& & KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B)
\end{array} \tag{6.14}$$

*are zero.*

*Then, with notation as in Definition 4.11, there exists an element*

$$u \in \mathcal{U}^1_{\infty,N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Y, B)$$

*such that in the diagram below*

$$\begin{array}{ccc} KK^1_{N_1,N_1\delta}(h(1-h)X\cup\{h\}\cup Y,B) & & KK^0_{\kappa,\varepsilon}(X\cup Y_h\cup Y_{1-h}\cup\{h\},B) \\ \downarrow & & \downarrow \\ KK^1_{\mu,\gamma}(h(1-h)X\cup\{h\},B) & \xrightarrow{\partial} & KK^0_{N_0,N_0\gamma}(X\cup\{h\},B) \\ & & \downarrow \\ & & KK^0_{N_2,N_2\gamma}(X\cup\{h\},B) \end{array}$$

*the images of the classes* $[u]\in KK^1_{N_1,N_1\delta}(h(1-h)X\cup\{h\}\cup Y)$ *and*

$$[p,q]\in KK^0_{\kappa,\varepsilon}(X\cup Y_h\cup Y_{1-h}\cup\{h\},B)$$

*in the bottom right group* $KK^0_{N_2,N_2\gamma}(X\cup\{h\},B)$ *are the same.*

Just as for Proposition 6.1, to make the argument more palatable, we split off some computations as two technical lemmas. As in that earlier case, the arguments we give for these lemmas are elementary, but quite lengthy (in fact, much longer than the earlier ones). We record them for the sake of completeness, but again recommend that the reader skips the proofs.

**Lemma 6.7.** *Let $B$ be a separable $C^*$-algebra. Let $\nu\geq 1$ and let $\gamma>0$. Let $X$ and $Y$ be self-adjoint subsets of the unit ball of $\mathcal{L}_B$. Let $h\in\mathcal{L}_B$ be a positive contraction such that $\|[h,x]\|<\gamma$ for all $x\in X$. Let $(p,q)\in\mathcal{P}^1_{n,\nu,\gamma}(X\cup Y\cup\{h\},B)$ (see Definition 4.9 for notation), and let $u_h\in\mathcal{U}^1_{n,\nu,\gamma}(hX\cup\{h\}\cup Y,B)$ and $u_{1-h}\in\mathcal{U}^1_{n,\nu,\gamma}((1-h)X\cup\{h\}\cup Y,B)$ (see Definition 4.11 for notation).*

*Then, the element*

$$u:=u_{1-h}(1-p)+u_hp \tag{6.15}$$

*is in $\mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X\cup\{h\}\cup Y,B)$.*

*Proof.* We split the computations into the points labeled (i), (ii), (iii), (iv), and (v) below.

(i) As $u_h-1\in M_n(\mathcal{K}_B)$ and $u_{1-h}-1\in M_n(\mathcal{K}_B)$, we compute from line (6.15) that $u-1\in M_n(\mathcal{K}_B)$.

(ii) Note that

$$\|1-p\|\leq\nu \tag{6.16}$$

by Corollary 4.2. Hence, $\max\{\|u_h\|,\|u_{1-h}\|,\|p\|,\|1-p\|\}\leq\nu$, and so by line (6.15), $\|u\|\leq 2\nu^2$.

(iii) Let $y\in Y$. Then, by definition, $\|[a,y]\|<\gamma$ for all $a\in\{u_h,u_{1-h},p,1-p\}$. Hence, the definition of $u$ from line (6.15) implies that $\|[y,u]\|$ is bounded above by

$$\|[y,u_{1-h}]\|\|1-p\|+\|u_{1-h}\|\|[y,1-p]\|+\|[y,u_h]\|\|p\|+\|u_h\|\|[y,p]\|<4\nu\gamma.$$

(iv) Using the definition of $u$ from line (6.15) and the assumptions on $u_h$, $u_{1-h}$ and $p$ directly together with line (6.16) implies that

$$\begin{aligned}\|[u,h]\| &\le \|[h,u_{1-h}]\|\|1-p\| + \|u_{1-h}\|\|[h,1-p]\| \\ &\quad + \|[h,u_h]\|\|p\| + \|u_h\|\|[h,p]\| \\ &< 4\nu\gamma.\end{aligned}$$

(v) Let $x \in X$ and note that

$$[h(1-h)x,u_h] = (1-h)[hx,u_h] + [h,u_h](1-h)x.$$

As $\|[hx,u_h]\| < \gamma$, as $\|[h,u_h]\| < \gamma$, as $h$ is a positive contraction, and as $x$ is a contraction, we get

$$\|[h(1-h)x,u_h]\| \le \|[hx,u_h]\|\|1-h\| + \|hx\|\|[1-h,u_h]\| < 2\gamma. \tag{6.17}$$

Completely analogously, we see that

$$\|[h(1-h)x,u_{1-h}]\| < 2\gamma. \tag{6.18}$$

We see also that

$$\begin{aligned}\|[h(1-h)x,p]\| &\le \|[x,p]\|\|h(1-h)\| + \|[1-h,p]\|\|hx\| + \|[h,p]\|\|(1-h)x\| \\ &< 3\gamma.\end{aligned}$$

Combining this with lines (6.16), (6.17), (6.18), we get

$$\begin{aligned}\|[h(1-h)x,u]\| &\le \|[h(1-h)x,u_{1-h}]\|\|1-p\| + \|u_{1-h}\|\|[h(1-h)x,1-p]\| \\ &\quad + \|[h(1-h)x,u_h]\|\|p\| + \|u_h\|\|[h(1-h)x,p]\| \\ &< 2\nu\gamma + 3\nu\gamma + 2\nu\gamma + 3\nu\gamma \\ &= 10\nu\gamma.\end{aligned}$$

Putting the points (i), (ii), (iii), (iv), and (v) above together (and using that $\nu \ge 1$) we conclude that, $u$ is an element of $\mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X \cup \{h\} \cup Y, B)$ as claimed. ∎

**Lemma 6.8.** *With assumptions as in Lemma 6.7, let*

$$u := u_{1-h}(1-p) + u_h p \in \mathcal{U}^1_{n,2\nu^2,10\nu\gamma}(h(1-h)X \cup \{h\} \cup Y, B)$$

*be the element considered there. Let $v := v(u,h)$ be as in line (6.2) above, and define*

$$w := \begin{pmatrix} u_{1-h}(1-p) & -q \\ p & (1-p)u_{1-h}^{-1} \end{pmatrix} \in M_{2n}(\mathcal{L}_B).$$

*Then, $w$ is invertible, and $vw^{-1}$ is in $\mathcal{U}_{2n,(2\nu)^8,2^{37}\nu^{25}\gamma}(X \cup \{h\}, B)$.*

*Proof.* Using the assumptions on $\|p\|$, $\|u_{1-h}\|$, $\|u_{1-h}^{-1}\|$ and line (6.16) to estimate $\|1-p\|$, we have

$$\|w\| \leq \|u_{1-h}(1-p)\| + \|q\| + \|p\| + \|(1-p)u_{1-h}^{-1}\| \leq 4\nu^2.$$

A direct computation shows that $w$ is invertible with inverse

$$w^{-1} = \begin{pmatrix} (1-p)u_{1-h}^{-1} & p \\ -q & u_{1-h}(1-p) \end{pmatrix}. \tag{6.19}$$

This satisfies the same norm estimate as $w$, and so we get the norm estimates

$$\|w\| \leq (2\nu)^2 \quad \text{and} \quad \|w^{-1}\| \leq (2\nu)^2. \tag{6.20}$$

Lemma 6.3 and the fact that $\|u\| \leq 2\nu^2$ implies that $\|v\| \leq (2\nu^2+2)^3$ and $\|v^{-1}\| \leq (2\nu^2+2)^3$. As $\nu \geq 1$, we thus see that

$$\|v\| \leq (2\nu)^6 \quad \text{and} \quad \|v^{-1}\| \leq (2\nu)^6. \tag{6.21}$$

Lines (6.20) and (6.21) then imply

$$\|vw^{-1}\| \leq (2\nu)^8 \quad \text{and} \quad \|wv^{-1}\| \leq (2\nu)^8. \tag{6.22}$$

To complete the proof, we need to show that for all $x \in X \cup \{h\}$, we have $\|[vw^{-1}, x]\| < 2^{37}\nu^{25}\gamma$ and $\|[wv^{-1}, x]\| < 2^{37}\nu^{25}\gamma$. We focus first on the case of $vw^{-1}$, and look first at $[h, vw^{-1}]$.

Let $c := hu + (1-h)$ and $d := hu^{-1} + (1-h)$ be as in line (6.1). It will be technically convenient to define

$$S := \{h, 1-h, p, q, 1-p, 1-q, u_h, u_h^{-1}, u_{1-h}, u_{1-h}^{-1}, u, u^{-1}, c, d\}, \tag{6.23}$$

and to define $S^n$ to be the set of all products of at most $n$ elements from $S$. Note that for every $s \in S$ we have $\|s\| \leq (2\nu)^2$, and $\|[s,h]\| < 10\nu\gamma$. Hence, by Lemma 4.16, for all $n \in \mathbb{N}$ we have

$$s \in S^n \Rightarrow \|[h,s]\| < n(2\nu)^{2(n-1)}10\nu\gamma. \tag{6.24}$$

Using the formula in line (6.4) above,

$$[h, v] = \begin{pmatrix} [cdc, h] - 2[c,h] & [cd, h] \\ [h, dc] & [d, h] \end{pmatrix}$$

and so

$$\|[h,v]\| \leq \|[cdc,h]\| + 2\|[c,h]\| + \|[cd,h]\| + \|[h,dc]\| + \|[d,h]\|.$$

Each summand on the right-hand side above is of the form $\|[h,s]\|$ where $s \in S^3$ for $S$ as in line (6.23). Hence, line (6.24) implies that

$$\|[h,v]\| < 6 \cdot 3 \cdot (2\nu)^4 \cdot 10\nu\gamma \leq 2^{11}\nu^5\gamma. \tag{6.25}$$

We also compute that

$$[h,w^{-1}] = \begin{pmatrix} [h,(1-p)u_{1-h}^{-1}] & [h,p] \\ [q,h] & [h,u_{1-h}(1-p)] \end{pmatrix},$$

whence

$$\|[h,w^{-1}]\| \leq \|[h,(1-p)u_{1-h}^{-1}]\| + \|[h,p]\| + \|[q,h]\| + \|[h,u_{1-h}(1-p)]\|.$$

Each commutator appearing above is of the form $[h,s]$ for some $s \in S^2$ as in line (6.23), whence line (6.24) gives

$$\|[h,w^{-1}]\| < 4 \cdot (2\nu)^2 \cdot 10\nu\gamma \leq 2^7\nu^3\gamma. \tag{6.26}$$

On the other hand,

$$\|[h,vw^{-1}]\| \leq \|[h,v]\|\|w^{-1}\| + \|v\|\|[h,w^{-1}]\|.$$

Combining this with lines (6.20), (6.21), (6.25), and (6.26), as well as that $\nu \geq 1$, we see that

$$\|[h,vw^{-1}]\| < 2^{11}\nu^5\gamma \cdot (2\nu)^2 + (2\nu)^6 \cdot 2^7\nu^3\gamma \leq 2^{14}\nu^9\gamma. \tag{6.27}$$

Now, let us look at $[x,vw^{-1}]$ for $x \in X$. The definition of $v$ from line (6.2) gives

$$\begin{aligned} vw^{-1} &= \begin{pmatrix} c(dc-1) & 1-cd \\ dc-1 & 0 \end{pmatrix} w^{-1} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} w^{-1} \\ &= \begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix} w^{-1} - \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} w^{-1}. \end{aligned}$$

Hence, the formula for $w^{-1}$ from line (6.19) gives

$$\begin{aligned} vw^{-1} = {} & \underbrace{\begin{pmatrix} cd-1 & 0 \\ 0 & dc-1 \end{pmatrix} \begin{pmatrix} c(1-p)u_{1-h}^{-1} & cp-u_{1-h}(1-p) \\ (1-p)u_{1-h}^{-1} & p \end{pmatrix}}_{y_1} \\ & \underbrace{-h\begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}}_{y_2} \\ & \underbrace{-(1-h)\begin{pmatrix} (1-p)u_{1-h}^{-1} & p \\ -q & u_{1-h}(1-p) \end{pmatrix}}_{y_3}. \end{aligned} \tag{6.28}$$

We now estimate $\|[vw^{-1}, x]\|$ for each $x \in X$ by looking at each of the terms $y_1$, $y_2$, and $y_3$ separately.

(i) First, we look at $y_1$ from line (6.28). Let $x \in X$. Lemma 6.2 implies that

$$\left\| \begin{pmatrix} cd - 1 & 0 \\ 0 & dc - 1 \end{pmatrix} - h(1-h)(u + u^{-1} - 2) \right\| < (\nu + 1)\gamma \tag{6.29}$$

(where, as usual, we identify $h(1-h)(u + u^{-1} - 2)$ with the corresponding diagonal matrix). Let

$$z_1 := \begin{pmatrix} c(1-p)u_{1-h}^{-1} & cp - u_{1-h}(1-p) \\ (1-p)u_{1-h}^{-1} & p \end{pmatrix}. \tag{6.30}$$

As in line (6.16), $\|1 - p\| \le \nu$, whence using that $\nu \ge 1$,

$$\begin{aligned} \|z_1\| &\le \|c\|\|1-p\|\|u_{1-h}^{-1}\| + \|c\|\|p\| + \|u_{1-h}\|\|1-p\| \\ &\quad + \|1-p\|\|u_{1-h}^{-1}\| + \|p\| \\ &\le (2\nu^2 + 1)\nu^2 + (2\nu^2+1)\nu + \nu^2 + \nu^2 + \nu \\ &\le 9\nu^4. \end{aligned} \tag{6.31}$$

Combining this with line (6.29), we see that

$$\begin{aligned} &\|y_1 - h(1-h)(u + u^{-1} - 2)z_1\| \\ &\le \left\| \begin{pmatrix} cd - 1 & 0 \\ 0 & dc - 1 \end{pmatrix} - h(1-h)(u + u^{-1} - 2) \right\| \|z_1\| \\ &< 9\nu^4(\nu + 1)\gamma \le (2\nu)^5\gamma. \end{aligned}$$

As $\|x\| \le 1$, this implies that

$$\begin{aligned} \|[x, y_1]\| &\le \|[x, y_1 - h(1-h)(u + u^{-1} - 2)z_1]\| \\ &\quad + \|[x, h(1-h)(u + u^{-1} - 2)z_1]\| \\ &< (2\nu)^5\gamma + \|[x, h(1-h)(u + u^{-1} - 2)z_1]\|. \end{aligned}$$

Hence, we see that

$$\begin{aligned} \|[x, y_1]\| &< (2\nu)^5\gamma + \|[[x, h(1-h)], (u + u^{-1} - 2)z_1]\| \\ &\quad + \|[h(1-h)x, (u + u^{-1} - 2)z_1]\| \\ &\quad + \|[h(1-h), (u + u^{-1} - 2)z_1]x\|. \end{aligned} \tag{6.32}$$

Looking at line (6.30), every entry of the matrix $(u + u^{-1} - 2)z_1$ is a sum of at most 8 elements from the set $S^4$, where $S$ is as in line (6.23). Hence, by line (6.24), we see that

$$\|[h(1-h), (u + u^{-1} - 2)z_1]\| < 4 \cdot 2 \cdot 8 \cdot 4 \cdot (2\nu)^6 \cdot 12\nu^2\gamma \le 2^{18}\nu^8\gamma. \tag{6.33}$$

We have $\|[x, h(1-h)]\| < 2\gamma$, and line (6.31) implies

$$\|(u+u^{-1}-2)z_1\| \leq (4\nu^2+2)\cdot 9\nu^4 \leq 2^6\nu^6,$$

whence

$$\|[[x, h(1-h)], (u+u^{-1}-2)z_1]\| \leq 2^8\nu^6\gamma. \tag{6.34}$$

Combining lines (6.32), (6.33), and (6.34) thus implies that

$$\|[x, y_1]\| \leq 2^{19}\nu^8\gamma + \|[h(1-h)x, (u+u^{-1}-2)z_1]\|. \tag{6.35}$$

Note now that for every element $s \in S$ we have that at least one of the following holds: (a) $\|[s, x]\| < 16\nu^2\gamma$ for all $x \in X$; or (b) $\|[s, (1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$; or (c) $\|[s, (1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$; or (d) $\|[s, h(1-h)x]\| < 16\nu^2\gamma$ for all $x \in X$. In any of these cases, using that $\|[s, h]\| \leq 12\nu^2\gamma$ for any $s \in S$, we get that for any $s \in S$ and $x \in X$, $\|[s, h(1-h)x]\| < 40\nu^2\gamma$. Applying Lemma 4.16, we therefore see that

$$s \in S^n \Rightarrow \|[h(1-h)x, s]\| < n(2\nu)^{2(n-1)}40\nu^2\gamma. \tag{6.36}$$

As we have observed above already, every entry in the matrix $(u+u^{-1}-2)z_1$ is a sum of at most 8 elements from the set $S^4$, where $S$ is as in line (6.23). From line (6.36) we therefore see that

$$\|[h(1-h)x, (u+u^{-1}-2)z_1]\| < 4\cdot 4\cdot (2\nu)^4\cdot 40\nu^2\gamma \leq 2^{14}\nu^6\gamma.$$

Combining this with line (6.35) above therefore implies

$$\|[x, y_1]\| < 2^{20}\nu^8\gamma.$$

(ii) Now, we look at the element $y_2$ from line (6.28) above. If $x \in X$, we see that

$$[x, y_2] = \left[xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right] + \left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right]x. \tag{6.37}$$

We have that

$$\left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right] = \begin{pmatrix} [q, h] & [h, u_h p] \\ [u_h^{-1}q, h] & [p, h] \end{pmatrix}.$$

Each entry in the matrix on the right is the commutator of $h$ with an element of $S^2$, where $S$ is as in line (6.23) above. Hence, by line (6.24), we see that

$$\left\|\left[h, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1}q & 1-p \end{pmatrix}\right]\right\| < 4\cdot 2\cdot (2\nu)^2\cdot 12\nu^2\gamma \leq 2^9\nu^4\gamma.$$

Combining this with line (6.37) gives

$$\|[x, y_2]\| < \left\| \left[ xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| + 2^9 \nu^4 \gamma. \tag{6.38}$$

On the other hand

$$\begin{aligned} \left[ xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] &= \left[ [x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \\ &\quad + \left[ hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right]. \end{aligned} \tag{6.39}$$

As $\|[h, x]\| < \gamma$, we have

$$\left\| \left[ [x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| \leq 2\gamma \left\| \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right\|.$$

As $\|1-p\| \leq \nu$ and $\|1-q\| \leq \nu$ by Corollary 4.2, every entry of the matrix on the right has norm at most $\nu^2$, and so

$$\left\| \left[ [x,h], \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < 2^3 \nu^2 \gamma.$$

Hence, line (6.39) implies that

$$\left\| \left[ xh, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < \left\| \left[ hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| + 2^3 \nu^2 \gamma. \tag{6.40}$$

The commutator appearing on the right above equals

$$\begin{pmatrix} [q, hx] & [hx, u_h]p + u_h[hx, p] \\ [u_h^{-1}, hx]q - u_h^{-1}[hx, q] & [p, hx] \end{pmatrix}.$$

Using that $u_h \in \mathcal{U}^1_{n,\nu,\gamma}(hX, B)$, and applying Lemma 6.4, the norm of each entry above is at most $2\nu\gamma$, whence

$$\left\| \left[ hx, \begin{pmatrix} 1-q & u_h p \\ -u_h^{-1} q & 1-p \end{pmatrix} \right] \right\| < 2^3 \nu \gamma.$$

Combining this with lines (6.38) and (6.40) therefore implies that

$$\|[x, y_2]\| < 2^{10} \nu^4 \gamma.$$

(iii) Finally, we look at $y_3$ from line (6.28). This can be handled very similarly to the case of $y_2$, giving the estimate $\|[x, y_3]\| < 2^{10}\nu^4\gamma$ for all $x \in X$; we leave the details to the reader.

Putting together the concluding estimates of points (i), (ii), and (iii) above, we see that $\|[x, vw^{-1}]\| < 2^{21}\nu^8\gamma$ for all $x \in X$. Combining this with line (6.27), we see that

$$\|[x, vw^{-1}]\| < 2^{21}\nu^9\gamma \tag{6.41}$$

for all $x \in X \cup \{h\}$.

To complete the proof, let us estimate $\|[x, wv^{-1}]\|$ for $x \in X \cup \{h\}$. Using the formula $[x, wv^{-1}] = wv^{-1}[vw^{-1}, x]wv^{-1}$, we see that

$$\|[x, wv^{-1}]\| \leq \|wv^{-1}\|\|[vw^{-1}, x]\|\|wv^{-1}\|.$$

Lines (6.41) and (6.22) therefore imply that

$$\|[x, wv^{-1}]\| \leq 2^{37}\nu^{25}\gamma$$

and we are finally done. ■

Finally, we are ready for the proof of Proposition 6.6.

*Proof of Proposition* 6.6. With notation as in the statement, let

$$(p, q) \in \mathcal{P}^1_{n,\kappa,\varepsilon}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B),$$

and assume that the images of $[p, q]$ in $KK^0_{\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ and $KK^0_{\lambda,\delta}((1-h)X \cup Y \cup \{h\}, B)$ under the maps in lines (6.13) and (6.14) are zero.

Note first that the map in line (6.13) is induced by the identity map on cycles, so Lemma 3.3 applied to the cycle $(p, q)$ in $\mathcal{P}_{n,\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ implies that there exists $k \in \mathbb{N}$ such that $(p \oplus 1_k \oplus 0_k, q \oplus 1_k \oplus 0_k)$ is in the same path component of $\mathcal{P}_{n+2k,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ as an element of the form $(r, r)$. Replacing $(r, r)$ with $(yry^*, yry^*)$ for some appropriate unitary $y \in M_{n+2k}(\mathbb{C})$ and using that the unitary group of $M_{n+2k}(\mathbb{C})$ is connected, we may assume that $(r, r)$ is in $\mathcal{P}^1_{n+2k,2\lambda,\delta}(hX \cup Y_h \cup \{h\}, B)$ (see Definition 4.9 for notation). Moreover, as

$$(p, q) \in \mathcal{P}^1_{n,\lambda,\delta}(X \cup Y_h \cup Y_{1-h} \cup \{h\}, B)$$

there is a unitary $z \in M_{n+2k}(\mathbb{C})$ such that $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ is in $\mathcal{P}^1_{n,\lambda,\delta}(hX \cup Y \cup \{h\}, B)$. As the elements $(r, r)$ and $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ of $\mathcal{P}^1_{n,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$ are connected by a path $\mathcal{P}_{n,2\lambda,\delta}(hX \cup Y \cup \{h\}, B)$, we may use Proposition 4.10 (ii) to connect them by a path in $\mathcal{P}^1_{n,2\lambda,4\delta}(hX \cup Y \cup \{h\}, B)$. Precisely analogously (increasing $k$ if necessary), we may assume that $(z(p \oplus 1_k \oplus 0_k)z^*, z(q \oplus 1_k \oplus 0_k)z^*)$ is in the same path component of

$$\mathcal{P}^1_{n,2\lambda,4\delta}((1-h)X \cup Y_{1-h} \cup \{h\}, B)$$

as an element of the form $(s, s)$.

For notational simplicity, write $m = n + 2k$, and let $M := 4 \cdot 2^{(200\lambda)^3}$. Then, (with notation as in Definition 4.11), Proposition 4.17 gives $j \in \mathbb{N}$ and elements

$$u_h \in \mathcal{U}^1_{m+2j,M,M\delta}(hX \cup \{h\} \cup Y, B)$$

and

$$u_{1-h} \in \mathcal{U}^1_{m+2j,M,M\delta}((1-h)X \cup \{h\} \cup Y, B)$$

such that

$$u_h(z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j)u_h^{-1} = z(q \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j \qquad (6.42)$$

and

$$u_{1-h}(z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j)u_{1-h}^{-1} = z(q \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j. \quad (6.43)$$

For notational simplicity, rename $z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j$ and $z(p \oplus 1_k \oplus 0_k)z^* \oplus 1_j \oplus 0_j$ as $p$ and $q$ respectively and rewrite $m + 2j$ as $n$; if the conclusion of the proposition holds for this new pair then it also holds for the original pair thanks to the definition of the controlled $KK^0$ groups (see Definition 3.1), so this makes no real difference. In this new language, lines (6.42) and (6.43) can be rewritten $u_h p u_h^{-1} = q$ and $u_{1-h} p u_{1-h}^{-1} = q$ respectively.

Define now

$$u := u_{1-h}(1-p) + u_h p,$$

which we claim has the properties in the statement. Using Lemma 6.7 with $\nu = M$ and $\gamma = M\delta$, we see that (with notation as in Definition 4.11), $u$ is an element of $\mathcal{U}^1_{n,2M^2,10M^2\delta}(h(1-h)X \cup \{h\} \cup Y, B)$. Recalling that $M = 4 \cdot 2^{(200\lambda)^3}$, we see that

$$N_1(\lambda) = 2^{9000000\lambda^3}$$

has the desired property.

To complete the proof, it remains to show that if $N_2 = N_2(\mu) = 2^{252000000\mu^3}$, then $\partial[u] = [p, q]$ in $KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B)$.

Now, $v := v(u, h)$ is as in line (6.2), we have

$$\partial[u] = \left[ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right].$$

Define now

$$w := \begin{pmatrix} u_{1-h}(1-p) & -q \\ p & (1-p)u_{1-h}^{-1} \end{pmatrix} \in M_{2n}(\mathcal{L}_B).$$

Applying Lemma 6.8 with

$$\nu = M \quad \text{and} \quad \gamma = M\delta,$$

we see that $w$ is in $\mathcal{U}_{2n,(2M)^8,2^{37}M^{25}\delta}(X \cup \{h\}, B)$. For notational simplicity, set $M_1 := 2^{37}M^{25}$. Proposition 4.6 implies that in $KK^0_{M_1^3,3M_1^3\delta}(X \cup \{h\}, B)$

$$\begin{aligned} \partial[u] &= \left[ v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= \left[ (vw^{-1})^{-1} v \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v^{-1}(vw^{-1}), \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \\ &= \left[ w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right]. \end{aligned}$$

Computing, we see that

$$w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} = \begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix},$$

whence

$$\partial[u] = \left[ \begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right] \tag{6.44}$$

in the group $KK^0_{M_1^3,3M_1^3\delta}(X \cup \{h\}, B)$.

Note now that the matrix $\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \in M_{2n}(\mathcal{K}_B^+)$ has norm at most $2\lambda$ (as $\|q\| \leq \kappa \leq \lambda$, and so $\|1-q\| \leq \lambda$ by Corollary 4.2), and that it satisfies

$$\left\| \left[ x, \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \right] \right\| < \varepsilon < \delta$$

for all $x \in X \cup \{h\}$. Hence, $\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \in \mathcal{U}_{2n,2\lambda,\delta}(X \cup \{h\}, B)$. Applying Proposition 4.6 again and using that $\lambda \leq M_1$, the identity

$$\begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1-q & q \\ q & 1-q \end{pmatrix} = \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix}$$

shows that the class on the right-hand side of line (6.44) is the same as the class

$$\left[ \begin{pmatrix} 1-q & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix} \right]$$

in $KK^0_{M_1^6,9M_1^9\delta}(X \cup \{h\}, B)$. Using a rotation homotopy, this is the same as $[p, q]$ by definition of $KK^0_{M_1^6,9M_1^9\delta}(X \cup \{h\}, B)$; recalling that

$$M_1 := 2^{37}M^{25}, \quad M = 4 \cdot 2^{(200\lambda)^3},$$

and that $\mu \geq 2^{9000000\lambda^3}$ we see that $N_2(\mu) = 2^{37}\mu^{25}$ indeed has the right properties. ■