Chapter 7

Main theorems

In this chapter (as throughout), if B is a separable C^*-algebra, then $\mathcal{L}_B$ and $\mathcal{K}_B$ are respectively the adjointable and compact operators on the standard Hilbert B-module $\ell^2 \otimes B$. We identify $\mathcal{L}_B$ with the "diagonal subalgebra" $1_{M_n} \otimes \mathcal{L}_B \subseteq M_n \otimes \mathcal{L}_B = M_n(\mathcal{L}_B)$ for each n.

In this chapter we prove our main result; that is, the class of separable and nuclear C^*-algebras with the UCT is closed under decomposability.

7.1 Two technical "local" controlled vanishing results

In order to make the structure of the proof of Theorem 1.2 as clear as we can, in this section we split off two "local" technical results. These are based on our work in Chapters 5 and 6; given the material in these earlier chapters, at this point the proofs are essentially bookkeeping.

The next result is the first key technical ingredient we need; it is based on the material from Chapter 5. For the statement, recall that if x and S are respectively an element and subset of a metric space, and $\varepsilon > 0$, then "$x \in_\varepsilon S$" means that there is $s \in S$ with $d(x, s) < \varepsilon$.

Proposition 7.1. *There exists a function $M : [1, \infty) \to [1, \infty)$ with the following property. Let $\kappa \geq 1$, and let $M := M(\kappa)$. Let B be a separable C^*-algebra such that*

$$K_*(B) = 0.$$

Let $\varepsilon > 0$, and let X be a finite subset of the unit ball of $\mathcal{L}_{SB}$. Let $F \subseteq \mathcal{L}_{SB}$ be a separable, nuclear, unital C^*-subalgebra of $\mathcal{L}_{SB}$ such that the identity representation $F \to \mathcal{L}_{SB}$ is strongly unitally absorbing (see Definition 2.5), such that for all $x \in X$, $x \in_\varepsilon F$, and such that F satisfies the UCT.

Then, for each $i \in \{0, 1\}$ there exists a finite subset Z of F_1 such that the forget control map

$$KK^i_{\kappa,\varepsilon}(Z, B) \to KK^i_{M,M\varepsilon}(X, B)$$

of Definition 3.4 (for $i = 0$) or Definition 3.7 (for $i = 1$) is zero.

Proof. Let us focus on the case of $i = 0$ first. Let Y be a finite subset of F_1 such that for all $x \in X$ there exists $y \in Y$ with $\|x - y\| < \varepsilon$. Then, for any n, any $\delta > 0$, we see that with notation in Definition 3.1

$$\mathcal{P}_{n,\kappa,\delta}(Y, SB) \subseteq \mathcal{P}_{n,\kappa,\delta+2\kappa\varepsilon}(X, SB).$$

Hence, the forget control map

$$KK^0_{\kappa,\delta}(Y, SB) \to KK^0_{\kappa,\delta+2\kappa\varepsilon}(X, SB) \tag{7.1}$$

is defined. On the other hand, Corollary 5.3 implies that there is a finite subset Z of F_1 such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\kappa,160\varepsilon}(Y, SB)$$

is defined and zero. Taking $\delta = 160\varepsilon$, and composing this with the forget control map in line (7.1) above, we see that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\kappa,(160+2\kappa)\varepsilon}(Y, SB)$$

is well-defined and zero. We are therefore done in the case $i = 0$; any function M satisfying $M(\kappa) \geq 160 + 2\kappa$ will work. The case of $i = 1$ is similar (although requiring a much larger $M(\kappa)$), using Lemma 5.5 in place of Corollary 5.3. ■

The second key technical result we need is as follows; it is based on the material from Chapter 6.

Proposition 7.2. *Let X be a finite subset of the unit ball of $\mathcal{L}_B$, let $\varepsilon > 0$, and let $\kappa \geq 1$. Assume there exists a positive contraction $h \in \mathcal{L}_B$, finite self-adjoint subsets Z_h, Z_{1-h}, and $Z_{h(1-h)}$ of the unit ball of $\mathcal{L}_B$, and $\lambda, \mu \geq 1$ and $\delta, \gamma > 0$ with the following properties:*

(i) $\|[h, x]\| < \varepsilon$ *for all* $x \in X$*;*

(ii) *for each* $z \in Z_{h(1-h)}$, $z \in_\varepsilon Z_h$ *and* $z \in_\varepsilon Z_{1-h}$*;*

(iii) *with* $N_1 := N_1(\lambda)$ *as in Proposition 6.6, the forget control map*

$$KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}) \to KK^1_{\mu,\gamma}(h(1-h)X \cup \{h\}, B)$$

of Definition 3.7 is defined and zero;

(iv) *the forget control map*

$$KK^0_{4\kappa^2,2\varepsilon}(Z_h \cup hX \cup \{h\}, B) \\ \to KK^0_{\lambda,\delta}(hX \cup \{h\} \cup Z_{h(1-h)}, B)$$

of Definition 3.4 is defined and zero;

(v) *the forget control map*

$$KK^0_{4\kappa^2,2\varepsilon}(Z_{1-h} \cup (1-h)X \cup \{h\}, B) \\ \to KK^0_{\lambda,\delta}((1-h)X \cup \{h\} \cup Z_{h(1-h)}, B)$$

of Definition 3.4 is defined and zero.

Then, if $Z := Z_h \cup Z_{1-h} \cup X \cup \{h\}$ *and* $N_2 := N_2(\mu)$ *is as in Proposition* 6.6*, we have that the forget control map*

$$KK^0_{\kappa,\varepsilon}(Z,B) \to KK^0_{N_2,N_2\gamma}(X,B)$$

of Definition 3.4 *is zero.*

Proof. We need to show that an arbitrary class $\alpha \in KK^0_{\kappa,\varepsilon}(X,B)$ vanishes under the forget control map

$$KK^0_{\kappa,\varepsilon}(Z,B) \to KK^0_{N_2,N_2\gamma}(X,B).$$

Using Proposition 4.10 (i), with notation as in Definition 4.9, we may assume that there is a cycle $(p,q) \in \mathcal{P}^1_{n,4\kappa^3,\varepsilon}(Z,B)$ such that $[p,q] \in KK^0_{4\kappa^3,\varepsilon}(Z,B)$ agrees with the image of α under the forget control map

$$KK^0_{\kappa,\varepsilon}(Z,B) \to KK^0_{4\kappa^3,\varepsilon}(Z,B).$$

It thus suffices to show that $[p,q] \in KK^0_{4\kappa^3,\varepsilon}(Z,B)$ vanishes under the forget control map

$$KK^0_{4\kappa^2,\varepsilon}(Z,B) \to KK^0_{N_2,N_2\gamma}(X,B)$$

(we leave the check that this map is defined under our assumptions to the reader). Now, with notation as in Proposition 6.6, the composition

$$\begin{array}{ccc} KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup Z_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup \{h\}, B) & \xrightarrow{\eta_h} & KK^0_{4\kappa^2,2\varepsilon}(hX \cup Z_h \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}(hX \cup Z_{h(1-h)} \cup \{h\}, B) \end{array}$$

(compare line (6.13)) is the zero map; indeed, the right-hand vertical map is zero by assumption (iv). Similarly, using assumption (v), we see that the composition

$$\begin{array}{ccc} KK^0_{4\kappa^2,\varepsilon}(X \cup Z_h \cup Z_{1-h} \cup \{h\}, B) & & \\ \downarrow & & \\ KK^0_{4\kappa^2,\varepsilon}(X \cup Z_{1-h} \cup \{h\}, B) & \xrightarrow{\eta_{1-h}} & KK^0_{4\kappa^2,2\varepsilon}((1-h)X \cup Z_{1-h} \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{\lambda,\delta}((1-h)X \cup Z_{h(1-h)} \cup \{h\}, B) \end{array}$$

(compare line (6.14)) is zero. Hence, Proposition 6.6 gives us an element

$$u \in \mathcal{U}^1_{\infty,N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}, B)$$

such that in the diagram below (with $N_0 = N_0(\mu)$ as in Proposition 6.1)

$$\begin{array}{ccc} KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)}, B) & & KK^0_{4\kappa^2,2\varepsilon}(Z,B) \\ \downarrow & & \downarrow \\ KK^1_{\mu,\gamma}(h(1-h)X \cup \{h\}, B) & \xrightarrow{\partial} & KK^0_{N_0,N_0\gamma}(X \cup \{h\}, B) \\ & & \downarrow \\ & & KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B) \end{array} \qquad (7.2)$$

the images of the classes $[u] \in KK^1_{N_1,N_1\delta}(h(1-h)X \cup \{h\} \cup Z_{h(1-h)})$ and $[p,q] \in KK^0_{\kappa,\varepsilon}(Z,B)$ in the bottom right group $KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B)$ are the same; a fortiori their images are also the same if we further compose with the forget control map

$$KK^0_{N_2,N_2\gamma}(X \cup \{h\}, B) \to KK^0_{N_2,N_2\gamma}(X, B).$$

Assumption (iii) implies, however, that the left-hand vertical map in line (7.2) is zero, however, so we are done. ■

7.2 Proof of the main theorems

We are now ready for our main results. For the statement of the first of these, we recall what it means for a C^*-algebra to decompose over a class of C^*-algebras from Definition 1.1 above. After giving a proof of this, we will use it to establish the theorems from the introduction.

Theorem 7.3. *Let $\kappa \geq 1$ and $\gamma > 0$. Let $M_1 := M(4)$ be as in Proposition 7.1. Let $N_1 := N_1(M_1)$ be as in Proposition 6.6. Let $M_2 := M(N_1)$ be as in Proposition 7.1. Let $N_2 := N_2(M_2)$ be as in Proposition 6.6. Then, any $\nu \geq N_2$ and $\varepsilon \in (0, \gamma(2N_2M_2N_1M_1)^{-1})$ have the following property.*

Let A be a separable, unital C^*-algebra that decomposes over the class of nuclear C^*-algebras that satisfy the UCT. Let B be any separable C^*-algebra such that $K_*(B) = 0$. Then, for any finite subset X of A_1, and $\varepsilon > 0$, there is a finite subset Z of A_1, such that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

of Definition 3.4 is defined and zero.

In particular, A satisfies the UCT.

Proof. The claim that A satisfies the UCT follows as the vanishing property in the statement of Theorem 7.3 implies condition (iii) from Corollary 5.3. It thus suffices to prove the vanishing property. Let ν and ε satisfy the given assumptions.

As A is decomposable with respect to the family of nuclear C^*-subalgebras that satisfy the UCT, there are nuclear, UCT C^*-subalgebras C, D and E of A and a positive contraction $h \in E$ such that for all $x \in X$, $\|[h, x]\| < \varepsilon$, $hx \in_\varepsilon C$, $(1-h)x \in_\varepsilon D$, and $h(1-h)x \in_\varepsilon E$, and such that all $e \in E$ we have that $e \in_\varepsilon C$, and $e \in_\varepsilon D$. Replacing C, D, and E by the C^*-subalgebra of A spanned by the algebra and the unit of A, we may assume that C, D, and E are unital subalgebras of A (note that the unitization of a nuclear C^*-algebra that satisfies the UCT is nuclear and satisfies the UCT; see [10, Exercise 2.3.5] for nuclearity and [55, Proposition 2.3 (a)] for the UCT).

Represent A on $\mathcal{L}_{SB}$ using a representation with the properties in Corollary 2.7 (with B replaced by SB), and identify A (therefore also C, D, and E) with unital C^*-subalgebras of $\mathcal{L}_{SB}$ using this representation. Note that the restrictions of this representation to each of E, C, D, (and the representation of A itself) are strongly unitally absorbing.

Throughout the rest of the proof, all unlabeled arrows are forget control maps as in Definitions 3.4 or 3.7 as appropriate.

Using Proposition 7.1 there exists a finite self-adjoint subset $Z_{h(1-h)}$ of E_1 such that the forget control map

$$KK^1_{N_1,2N_1M_1\varepsilon}(h(1-h)X \cup Z_{h(1-h)} \cup \{h\}, SB) \\ \to KK^1_{M_2,2M_2N_1M_1\varepsilon}(h(1-h)X \cup \{h\}, SB) \tag{7.3}$$

is zero. Similarly, Proposition 7.1 and the facts that for all $z \in Z_{h(1-h)} \subseteq E_1$, $z \in_\varepsilon C$ and $z \in_\varepsilon D$ gives finite self-adjoint subsets Z_h and Z_{1-h} of C_1 and D_1 respectively such that the forget control maps

$$KK^0_{4,2\varepsilon}(hX \cup Z_h \cup \{h\}, SB) \\ \to KK^0_{M_1,2M_1\varepsilon}(hX \cup Z_{h(1-h)} \cup \{h\}, SB) \tag{7.4}$$

and

$$KK^0_{4,2\varepsilon}((1-h)X \cup Z_{1-h} \cup \{h\}, SB) \\ \to KK^0_{M_1,2M_1\varepsilon}((1-h)X \cup Z_{h(1-h)} \cup \{h\}, SB) \tag{7.5}$$

are defined and zero. Expanding Z_h and Z_{1-h} if necessary (using that for all $e \in E$, $e \in_\varepsilon C$, and $e \in_\varepsilon D$), we may assume that,

$$\text{for all } z \in Z,\ z \in_\varepsilon Z_h \text{ and } z \in_\varepsilon Z_{1-h}. \tag{7.6}$$

We are now in a position to apply Proposition 7.2 with the given ε and κ, $\lambda = M_1$, $\delta = 2M_1\varepsilon$, $\mu = M_2$ and γ as given: assumption (i) follows by choice of h; assumption (ii) follows from line (7.6); assumption (iii) follows as the map in line (7.3) is zero; assumption (iv) follows as the map in line (7.4) is zero; and assumption (v) follows as the map in line (7.5) is zero. Therefore, Proposition 7.2 implies that the forget control map

$$KK^0_{\kappa,\varepsilon}(Z, SB) \to KK^0_{\nu,\gamma}(X, SB)$$

is zero and we are done. ■

To establish the main results as stated in the introduction, we need a basic lemma.

Lemma 7.4. *The class of unital, nuclear C^*-algebras is closed under decomposability.*

Proof. Let A be a unital C^*-algebra that decomposes over the class of unital nuclear C^*-algebras. Let a finite subset X of A and $\varepsilon \in (0, 1)$ be given. To show that A is nuclear, it will suffice to construct a finite rank ccp map

$$\phi : A \to A$$

such that $\phi(x) \approx_\varepsilon x$ for all $x \in X$ (compare for example [8, Lemma IV.3.1.6, (iii)]). We may assume that X contains the unit of A.

Let then C, D, E[1], and h be as in the definition of decomposability (Definition 1.1) with respect to the finite set X and the parameter $\delta = \frac{1}{18}(\varepsilon/(1+\varepsilon))^2$, and with C and D nuclear. Note that for any $x \in X$, $\|[h^{1/2}, x]\| \leq \frac{5}{4}\|[h, x]\|^{1/2}$ by the main result of [49], whence

$$\|hx - h^{1/2}xh^{1/2}\| \leq \frac{5}{4}\|[h, x]\|^{1/2} < \frac{5}{4}\delta^{1/2} < 2\delta^{1/2}; \tag{7.7}$$

as $hx \in_\delta C$, and as $\delta < 1$, this implies that $h^{1/2}xh^{1/2} \in_{3\delta^{1/2}} C$. Choose a finite subset Y of C such that for all $x \in X$ there is $y_x \in Y$ with

$$\|y_x - h^{1/2}xh^{1/2}\| < 3\delta^{1/2}. \tag{7.8}$$

Similarly, there is a finite subset Z of D such that for all $x \in X$ there is $z_x \in Z$ with

$$\|z_x - (1-h)^{1/2}x(1-h)^{1/2}\| < 3\delta^{1/2}.$$

Now, as C and D are nuclear there are diagrams

$$C \xrightarrow{\psi_C} F_C \xrightarrow{\phi_C} C \quad \text{and} \quad D \xrightarrow{\psi_D} F_D \xrightarrow{\phi_D} C$$

[1] One does not actually need E at all in the proof.

where all the arrows are ccp maps; F_C and F_D are finite dimensional C^*-algebras; and for all $y \in Y$, and all $z \in Z$,

$$\phi_C(\psi_C(y)) \approx_{\delta^{1/2}} y \quad \text{and} \quad \psi_D(\phi_D(z)) \approx_{\delta^{1/2}} z. \tag{7.9}$$

Using Arveson's extension theorem (see for example [10, Theorem 1.6.1]), extend ψ_C and ψ_D to ccp maps defined on all of A, which we keep the same notation for. Define

$$\phi_0 : A \to A, \quad a \mapsto \phi_C(\psi_C(h^{1/2}xh^{1/2})) + \phi_D(\psi_D((1-h)^{1/2}x(1-h)^{1/2})),$$

and note that ϕ_0 is completely positive. For any $x \in X$, let y_x have the property in line (7.8). As ψ_C is contractive, this and lines (7.9) and (7.7) imply that

$$\phi_C(\psi_C(h^{1/2}xh^{1/2})) \approx_{3\delta^{1/2}} \phi(\psi_C(y_x)) \approx_{\delta^{1/2}} y_x \approx_{3\delta^{1/2}} h^{1/2}xh^{1/2} \approx_{2\delta^{1/2}} hx.$$

Precisely analogously, for any $x \in X$,

$$\phi_D(\psi_D((1-h)^{1/2}x(1-h)^{1/2})) \approx_{9\delta^{1/2}} (1-h)x$$

and so for any $x \in X$, $\phi_0(x) \approx_{18\delta^{1/2}} x$. Applying this to $x = 1$ implies in particular that $\|\phi_0\| = \|\phi_0(1)\| \geq 1 - 18\delta^{1/2}$. Hence, if we define

$$\phi : A \to A, \quad a \mapsto \frac{\phi_0(a)}{\|\phi_0(1)\|}$$

then ϕ is a ccp map such that

$$\|\phi(x) - x\| \leq \frac{18\delta^{1/2}}{1 - 18\delta^{1/2}}$$

for all $x \in X$. Using the choice of δ, this completes the proof. ∎

The next corollary is Theorem 1.2 from the introduction; it is an immediate consequence of Lemma 7.4 and Theorem 7.3.

Corollary 7.5. *If a separable, unital C^*-algebra decomposes over the class of nuclear, unital C^*-algebras that satisfies the UCT, then it is nuclear and satisfies the UCT.* ∎

The next result is Theorem 1.4 from the introduction. For the definition of finite complexity and the classes $\mathcal{D}_\alpha$ used below, see Definition 1.3.

Corollary 7.6. *Let $\mathcal{C}$ be a class of separable, unital, nuclear C^*-algebras that satisfy the UCT. Then, the class of separable unital C^*-algebras that have finite complexity relative to $\mathcal{C}$ consists of nuclear C^*-algebras that satisfy the UCT.*

In particular, every separable C^*-algebra of finite complexity is nuclear and satisfies the UCT.

Proof. With notation as in Definition 1.3, let $\mathcal{D}_0 = \mathcal{C}$, and for each ordinal α, let $\mathcal{D}_{\alpha,sep}$ consist of the separable C^*-algebras in the class $\mathcal{D}_\alpha$ from Definition 1.3. We proceed by transfinite induction to show that each $\mathcal{D}_{\alpha,sep}$ consists of nuclear, UCT C^*-algebras. If $\alpha = 0$, this is just the well-known fact that AF C^*-algebras satisfy the UCT. If $\alpha > 0$ (and either a successor or limit ordinal) then any C^*-algebra in $\mathcal{D}_{\alpha,sep}$ decomposes over C^*-algebras in $\bigcup_{\beta<\alpha} \mathcal{D}_{\beta,sep}$, and so is nuclear and UCT by Corollary 7.5 and the inductive hypothesis. ∎