Appendix A

Examples

In this appendix we give some examples of C^*-algebras with finite complexity.

A.1 Cuntz algebras

The material in this section is based closely on work of Winter and Zacharias [70, Section 7][1]. Our aim is to establish the following result.

Proposition A.1. *For any n with $2 \leq n < \infty$, the Cuntz algebra $\mathcal{O}_n$ has complexity rank one.*

We should remark that the proof of Proposition A.1 uses classification results for Cuntz algebras, and so depends on prior knowledge of the UCT; it therefore cannot be said that Proposition A.1 gives a new proof of the UCT for Cuntz algebras (and even if it did, it would be quite a complicated one!). Indeed, the main point of establishing Proposition A.1 for us is to use it as an ingredient in Theorem 1.7 from the introduction, not to establish the UCT.

We should also remark that Proposition A.1 was subsequently generalized in [37, Theorem 1.5]; nonetheless, we hope that the different argument given here still has some interest.

We now embark on the proof of Proposition A.1. We will follow the notation from [70, Section 7]. Fix $n \in \mathbb{N}$ with $n \geq 2$. Let H be an n-dimensional Hilbert space, with fixed orthonormal basis $\{e_1, \ldots, e_n\}$. Define

$$\Gamma(n) := \bigoplus_{l=0}^{\infty} H^{\otimes l}, \tag{A.1}$$

where $H^{\otimes l}$ is the lth tensor power of H (and $H^{\otimes 0}$ is by definition a copy of $\mathbb{C}$). Let W_n be the set of all finite words based on the alphabet $\{1, \ldots, n\}$. In symbols

$$W_n := \bigsqcup_{k=0}^{\infty} \{1, \ldots, n\}^k$$

(with $\{1, \ldots, n\}^0$ by definition consisting only of the empty word). For each $\mu = (i_1, \ldots, i_k) \in W_n$, define $e_\mu := e_{i_1} \otimes \cdots \otimes e_{i_k}$, and define $e_\varnothing$ to be any unit-length

[1]More specifically, it is based on the slightly different approach to the material in [70, Section 7] suggested in [70, Remark 7.3].

element of $H^{\otimes 0} = \mathbb{C}$. Then, the set $\{e_\mu \mid \mu \in W_n\}$ is an orthonormal basis of $\Gamma(n)$. For $\mu \in W_n$, write $|\mu|$ for the length of μ, i.e., $|\mu| = k$ means that

$$\mu = (i_1, \dots, i_k)$$

for some $i_1, \dots, i_k \in \{1, \dots, n\}$. Then, the canonical copy of $H^{\otimes k}$ inside $\Gamma(n)$ from line (A.1) has orthonormal basis $\{e_\mu \mid |\mu| = k\}$.

For each $i \in \{1, \dots, n\}$ let T_i be the bounded operator on $\Gamma(n)$ that acts on basis elements via the formula

$$T_i : e_\mu \mapsto e_i \otimes e_\mu.$$

The *Cuntz–Toeplitz algebra* $\mathcal{T}_n$ is defined to be the C^*-subalgebra of $\mathcal{B}(\Gamma(n))$ generated by $T_1, \dots, T_n$. We note that each T_i is an isometry, and that $1 - \sum_{i=1}^n T_i T_i^*$ is the projection onto the span of $e_\varnothing$. It follows directly from this that $\mathcal{T}_n$ contains all matrix units with respect to the basis $\{e_\mu\}$ of $\Gamma(n)$, and therefore contains the compact operators $\mathcal{K}$ on $\Gamma(n)$. Moreover, in the quotient $\mathcal{T}_n/\mathcal{K}$, the images s_i of the generators T_i satisfy the Cuntz relations $s_i^* s_i = 1$ and $\sum_{i=1}^n s_i s_i^* = 1$, and therefore the quotient is a copy of the Cuntz algebra $\mathcal{O}_n$.

Now, for $x \in \mathbb{R}_+$, define $\lceil x \rceil := \min\{n \in \mathbb{N} \mid n \geq x\}$, and define[2]

$$\Gamma_{0,k} := \bigoplus_{l=k}^{2k-1} H^{\otimes l} \quad \text{and} \quad \Gamma_{1,k} := \bigoplus_{l=k+\lceil k/2 \rceil}^{2k+\lceil k/2 \rceil} H^{\otimes l}. \tag{A.2}$$

For $i \in \{0, 1\}$, define $B_{i,k}^{(0)} := \mathcal{B}(\Gamma_{i,k})$. For each $l, m \in \mathbb{N}$, we identify $H^{\otimes l} \otimes H^{\otimes m}$ with $H^{\otimes(l+m)}$ via the bijection of orthonormal bases

$$\left(e_{i_1} \otimes \cdots \otimes e_{i_l}\right) \otimes \left(e_{j_1} \otimes \cdots \otimes e_{j_m}\right) \leftrightarrow e_{i_1} \otimes \cdots \otimes e_{i_l} \otimes e_{j_1} \otimes \cdots \otimes e_{j_m}.$$

Fix for the moment $k \in \mathbb{N}$ (it will stay fixed until Lemma A.2 below). Then, for each $j \in \mathbb{N}$ we get a canonical identification

$$\Gamma_{0,k} \otimes H^{\otimes jk} = \bigoplus_{l=k}^{2k-1} H^{\otimes l} \otimes H^{\otimes jk} = \bigoplus_{l=jk}^{(j+1)k-1} H^{\otimes l}.$$

Combining this with line (A.1) we get a canonical identification

$$\Gamma(n) = \underbrace{\left(\bigoplus_{l=0}^{k-1} H^{\otimes l}\right)}_{=:H_0} \oplus \left(\bigoplus_{j=0}^{\infty} \Gamma_{0,k} \otimes H^{\otimes jk}\right).$$

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[2]In [70, Section 7], $\Gamma_{0,k}$ is written $\Gamma_{k,2k}$ and $\Gamma_{1,k}$ is written $\Gamma_{k+\lceil k/2 \rceil, 2k+\lceil k/2 \rceil}$.

Let id be the identity representation of $B_{0,k}^{(0)}$ on $\Gamma_{0,k}$ and write $B_{0,k}$ for the image of $B_{0,k}^{(0)}$ in the representation on $\Gamma(n)$ that is given by

$$0_{H_0} \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{id} \otimes 1_{H^{\otimes jk}} \right)$$

with respect to the above decomposition above. Similarly, we get a decomposition

$$\Gamma(n) = \underbrace{\left(\bigoplus_{l=0}^{k+\lceil k/2 \rceil -1} H^{\otimes l} \right)}_{=:H_1} \oplus \left(\bigoplus_{j=0}^{\infty} \Gamma_{1,k} \otimes H^{\otimes jk} \right)$$

and define $B_{1,k}$ to be the image of $B_{1,k}^{(0)}$ under the representation

$$0_{H_1} \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{id} \otimes 1_{H^{\otimes jk}} \right).$$

Now, let $f : [0,1] \to [0,1]$ be the piecewise linear function that takes the value 0 on $[0, 1/6]$ and $[5/6, 0]$, the value 1 on $[2/6, 4/6]$, and interpolates linearly between 0 and 1 on $[1/6, 2/6]$ and $[4/6, 5/6]$. Let $h_{0,k}^{(0)} \in B_{0,k}^{(0)}$ be the operator on $\Gamma_{0,k}$ that acts on the summand $H^{\otimes l}$ from line (A.2) by multiplication by the scalar $f((l-k)/(k-1))$. Similarly, let $h_{1,k}^{(0)} \in B_{1,k}^{(1)}$ be the operator on $\Gamma_{1,k}$ that acts on the summand $H^{\otimes l}$ from line (A.2) by multiplication by the scalar $1 - f((l - k - \lceil k/2 \rceil)/(k-1))$. Let $h_{0,k}$ and $h_{1,k}$ be the images of $h_{0,k}^{(0)}$ and $h_{1,k}^{(0)}$ in $B_{0,k}$ and $B_{1,k}$ respectively. Note that the operator on $h_{0,k} + h_{1,k}$ on $\Gamma(n)$ acts on the summand on $H^{\otimes l}$ from line (A.1) by multiplication by 1 as long as $l \geq k + \lceil k/2 \rceil$. In particular,

$$h_{0,k} + h_{1,k} \text{ equals the identity on } \Gamma(n) \text{ up to a finite rank perturbation.} \tag{A.3}$$

We will need two technical lemmas about these operators.

Lemma A.2. *For any T in the Cuntz–Toeplitz algebra $\mathcal{T}_n$ and $i \in \{0, 1\}$, we have that $\|[h_{i,k}, T]\| \to 0$ as $k \to \infty$.*

Proof. We will focus on $h_{0,k}$; the case of $h_{1,k}$ is essentially the same. It suffices to consider the case where T is one of the canonical generators T_i of the Cuntz–Toeplitz algebra. Let e_μ be a basis element with $|\mu| = jk + l$ for some $j, l \in \mathbb{N}$ with $l \in \{0, \ldots, k-1\}$. Then, we compute that $[h_{0,k}, T_i]e_\mu = 0$ if $j = 0$, and that otherwise

$$[h_{0,k}, T_i]e_\mu = \big(f((l+1)/(k-1)) - f(l/(k-1))\big)e_i \otimes e_\mu.$$

As the elements $\{e_i \otimes e_\mu \mid \mu \in W_n\}$ are an orthonormal set, this implies that

$$\|[h_{0,k}, T_i]\| \leq \max_{l \in \{0,\dots,k-1\}} |f((l+1)/(k-1)) - f(l/(k-1))|.$$

The choice of function f implies that the right-hand side above is approximately $6/k$, so we are done. ■

Lemma A.3. *For any T in the Cuntz–Toeplitz algebra $\mathcal{T}_n$ we have that*

(i) *for $i \in \{0,1\}$, $d(h_{i,k}T, B_{i,k}) \to 0$ as $k \to \infty$;*

(ii) *$d(h_{0,k}h_{1,k}T, B_{0,k} \cap B_{1,k}) \to 0$ as $k \to \infty$.*

Proof. We will focus on the case of $h_{0,k}$; the other cases are similar. It suffices to consider T a finite product $S_1 \cdots S_m$, where each S_j is either one of the generators T_i or its adjoint. Using Lemma A.2, we see that $[h_{0,k}^{1/l}, S_j] \to 0$ as $k \to \infty$ for any j, and any $l \in \mathbb{N}$ with $l \geq 1$. Hence, the difference

$$h_{0,k}S_1 \cdots S_m - \big(h_{0,k}^{1/(2m)} S_1 h_{0,k}^{1/(2m)}\big)\big(h_{0,k}^{1/(2m)} S_2 h_{0,k}^{1/(2m)}\big) \cdots \big(h_{0,k}^{1/(2m)} S_m h_{0,k}^{1/(2m)}\big)$$

tends to zero as $k \to \infty$. It thus suffices to prove that the distance between each of the terms $h_{0,k}^{1/(2m)} S_j h_{0,k}^{1/(2m)}$ and $B_{0,k}$ tends to zero as $k \to \infty$. Define p_k to be the strong operator topology limit of $h_{0,k}^{1/l}$ as $l \to \infty$; in other words, p_k is the support projection of $h_{0,k}$. Then, we have that $h_{0,k}^{1/(2m)} S_j h_{0,k}^{1/(2m)} = h_{0,k}^{1/(2m)} p_k S_j p_k h_{0,k}^{1/(2m)}$. As $h_{0,k}^{1/(2m)}$ is in $B_{0,k}$, it suffices to prove that the distance between $p_k T_i p_k$ and $B_{0,k}$ tends to zero as $k \to \infty$. However, $p_k T_i p_k$ is actually in $B_{0,k}$, so we are done. ■

Now, as in the discussion on [70, p. 488], define

$$\Gamma_k(n) := \bigoplus_{l=0}^{k-1} H^{\otimes l}.$$

For a word $\mu \in W_n$ in $\{1,\dots,n\}$, we may uniquely write $\mu = \mu_0\mu_1$, where the lengths $|\mu_0|$ and $|\mu_1|$ satisfy $|\mu_0| \in \{0,\dots,k-1\}$, and $|\mu_1| \in k\mathbb{N}$. Then, the bijective correspondence of orthonormal bases

$$e_\mu \leftrightarrow e_{\mu_0} \otimes e_{\mu_1}$$

gives rise to a decomposition

$$\Gamma(n) = \Gamma_k(n) \otimes \Gamma(n^k).$$

Identify the C^*-algebra $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ with its image in the representation on $\Gamma(n)$ arising from the above decomposition. The following is essentially part of [70, Lemma 7.1].

Lemma A.4. *With notation as above, $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ contains the finite-dimensional C^*-algebras we have called $B_{0,k}$ and $B_{1,k}$, and in particular also contains $h_{0,k}$ and $h_{1,k}$.*

Proof. In the notation of [70, Lemma 7.1], $B_{0,k} = \Lambda_k(\mathcal{B}(\Gamma_{k,2k}))$, and

$$B_{1,k} = \Lambda_k(\mathcal{B}(\Gamma_{k+\lceil k/2\rceil, 2k+\lceil k/2\rceil})).$$

Part (i) of [70, Lemma 7.1] says exactly that the image of Λ_k is contained in

$$\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k},$$

however, so we are done. ∎

It is explained on [70, p. 488] that $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}$ contains $\mathcal{T}_n$, so we get a canonical inclusion.

$$\mathcal{T}_n \to \mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k}. \tag{A.4}$$

The dimension of $\Gamma_k(n)$ is $d_k := 1 + n + n^2 + \cdots + n^{k-1}$, so we may make the identification $\mathcal{B}(\Gamma_k(n)) \otimes \mathcal{T}_{n^k} = M_{d_k}(T_{n^k})$. With respect to this identification, the inclusion in line (A.4) takes the compact operators on $\Gamma(n)$ to $M_{d_k}(\mathcal{K}(\Gamma(n^k)))$. Taking the quotient by the compacts on both sides of line (A.4) thus gives rise to an inclusion

$$\iota : \mathcal{O}_n \to M_{d_k}(\mathcal{O}_{n^k}). \tag{A.5}$$

In this language, we get the following immediate corollary of Lemmas A.2 and A.3. To state it, let $q : \mathcal{B}(\Gamma(n)) \to \mathcal{Q}(\Gamma(n))$ be the quotient map from the bounded operators on $\Gamma(n)$ to the Calkin algebra.

Corollary A.5. *For any $a \in \mathcal{O}_n$, we have that the following all tend to zero as $k \to \infty$: $\|[q(h_{0,k}), \iota(a)]\|$, $\|[q(h_{1,k}), \iota(a)]\|$, $d(q(h_{0,k})\iota(a), q(B_{0,k}))$, $d(q(h_{1,k})\iota(a), q(B_{1,k}))$, and $d(q(h_{0,k}h_{1,k})\iota(a), q(B_{0,k} \cap B_{1,k}))$.* ∎

We are finally ready for the proof of Proposition A.1.

Proof of Proposition A.1. Let $\varepsilon > 0$, and let X be a finite subset of the unit ball of $\mathcal{O}_n$. Corollary A.5 implies that for any large k we have that for all $a \in X$ and $i \in \{0, 1\}$, the quantities $\|[q(h_{i,k}), \iota(a)]\|$, $d(q(h_{i,k})\iota(a), q(B_{i,k}))$, and $d(q(h_{0,k}h_{1,k})\iota(a), q(B_{0,k} \cap B_{1,k}))$ are smaller than $\varepsilon/2$. We may assume moreover that $k \equiv 1$ modulo $n-1$. Fix this k for the remainder of the proof.

As discussed on [70, p. 488], we have a canonical unital inclusion $\mathcal{O}_{n^k} \to \mathcal{O}_n$ by treating suitable products of the generators of $\mathcal{O}_n$ as generators of $\mathcal{O}_{n^k}$. Moreover, d_k is equal to k modulo $n-1$. It follows that the K-theory of $M_{d_k}(\mathcal{O}_n)$ is given by $\mathbb{Z}/(n-1)\mathbb{Z}$ in dimension zero and zero in dimension one, with the class [1] of the unit in K_0 represented by the residue of k in $\mathbb{Z}/(n-1)\mathbb{Z}$. Hence, the K-theory invariants of $M_{d_k}(\mathcal{O}_n)$ and $\mathcal{O}_n$ agree, as we are assuming that $k \equiv 1$ modulo $n-1$. In particular, the Kirchberg–Phillips classification theorem (see for example [53, Corollary 8.4.8]) gives a unital isomorphism $M_{d_k}(\mathcal{O}_n) \cong \mathcal{O}_n$. Combining this with the inclusion $\mathcal{O}_{n^k} \to \mathcal{O}_n$ mentioned above gives a unital inclusion

$$\beta : M_{d_k}(\mathcal{O}_{n^k}) \to \mathcal{O}_n. \tag{A.6}$$

Now, the composition $\beta \circ \iota : \mathcal{O}_n \to \mathcal{O}_n$ of β as in line (A.6) and ι as in line (A.5) is a unital inclusion, whence necessarily induces an isomorphism on K-theory. As $\mathcal{O}_n$ satisfies the UCT, $\beta \circ \iota$ is therefore a KK-equivalence (see for example [55, Proposition 7.3]). Hence, the uniqueness part of the Kirchberg–Phillips classification theorem (see for example [53, Theorem 8.3.3, (iii)]) implies that $\beta \circ \iota : \mathcal{O}_n \to \mathcal{O}_n$ is approximately unitarily equivalent to the identity. Thus, there is a sequence (u_m) of unitaries in $\mathcal{O}_n$ such that

$$\|a - u_m \beta\iota(a) u_m^*\| \to 0$$

for all $a \in \mathcal{O}_n$. Choose m large enough so that $\|a - u_m \beta\iota(a) u_m^*\| < \varepsilon/2$ for all $a \in X$.

Set $h := u_m \beta(q(h_{0,k})) u_m^*$, $C_0 := u_m \beta(q(B_{0,k})) u_m^*$, $D_0 := u_m \beta(q(B_{1,k})) u_m^*$, and $E_0 := u_m \beta(q(B_{1,k} \cap B_{0,k})) u_m^*$. Set C to be the C^*-subalgebra of $\mathcal{O}_n$ spanned by C_0 and the unit, and similarly for D and E. Our choices, plus the fact that

$$q(h_{0,k} + h_{1,k}) = 1$$

(see line (A.3)), imply that this data satisfies the definition of decomposability (Definition 1.1), so we are done. ■

A.2 Groupoids with finite dynamical complexity

In this section, we give another interesting class of C^*-algebras with finite complexity, that is, C^*-algebras of groupoids with finite dynamical complexity. To avoid repeating the same assumptions, let us stipulate that throughout this appendix the word “groupoid” means “locally compact, Hausdorff, étale groupoid”; we will often also assume that G has compact base space, but not always. For background on this class of groupoids and their C^*-algebras, we recommend [10, Section 5.6], [51, Section 2.3], or [59].

Note that if G is a groupoid in this sense, then any open subgroupoid H of G (i.e., H is an open subset of G that is algebraically a groupoid with the inherited operations) is also a groupoid in this sense. Again, to avoid too much repetition, let us say that the word “subgroupoid” means “open subgroupoid”.

The following definitions are essentially contained in the authors’ joint work with Guentner [31, Definition A.4].

Definition A.6. Let G be a groupoid, let H be a subgroupoid of G, and let $\mathcal{C}$ be a set of subgroupoids of G. We say that H is *decomposable* over $\mathcal{C}$ if for any compact subset K of H there exists an open cover $\{U_0, U_1\}$ of $r(K) \cup s(K)$ such that for each $i \in \{0, 1\}$ the subgroupoid of H generated by

$$\{h \in K \mid s(h) \in U_i\}$$

is contained in an element of $\mathcal{C}$.

Definition A.7. For an ordinal number α,

(i) if $\alpha = 0$, let $\mathcal{C}_0$ be the class of groupoids G such that for any compact subset K of G there is a subgroupoid H of G such that $K \subseteq H$, and such that the closure of H is compact;

(ii) if $\alpha > 0$, let $\mathcal{C}_\alpha$ be the class of groupoids that decompose over the collection of their subgroupoids in the class $\bigcup_{\beta<\alpha} \mathcal{C}_\beta$.

We say that a groupoid G has *finite dynamical complexity* if G is contained in $\mathcal{C}_\alpha$ for some ordinal α. If G has finite dynamical complexity, the *complexity rank* of G is the smallest α such that G is in $\mathcal{C}_\alpha$.

The main result of this section is as follows. For the statement, recall that a groupoid is *ample* if it has totally disconnected base space, and *principal* if the units are the elements $g \in G$ that satisfy $s(g) = r(g)$. Recall also that a C^*-algebra is subhomogeneous if it is isomorphic to a C^*-subalgebra of $M_N(C(X))$ for some $N \in \mathbb{N}$ and compact Hausdorff space X. Recall finally the notion of complexity rank relative to a class of C^*-algebras from Definition 1.3.

Proposition A.8. *Let G be a groupoid with compact base space.*

(i) *The complexity rank of $C^*_r(G)$ relative to the class of subhomogeneous C^*-algebras is bounded above by the complexity rank of G.*

(ii) *If G is ample and principal, then the complexity rank of $C^*_r(G)$ (relative to the class of finite-dimensional C^*-algebras) is bounded above by the complexity rank of G.*

In particular, if G is second countable and has finite dynamical complexity, then $C^*_r(G)$ satisfies the UCT.

Before getting into the proof of this, let us discuss some remarks and examples.

Example A.9. Let $G(X)$ be the coarse groupoid associated to a bounded geometry metric space X; see [61, Section 3] or [52, Chapter 10] for background. For such spaces X, Guentner, Tessera and Yu [29] introduced a notion called *finite decomposition complexity*; it comes with a natural complexity rank, defined to be the smallest ordinal α such that X is in the class $\mathfrak{D}_\alpha$ of [30, Definition 2.2.1]. Then, [31, Theorem A.7] shows that $G(X)$ has finite dynamical complexity if and only if X has finite decomposition complexity[3]; moreover, inspection of the proof shows that the two complexity ranks agree. It follows from this and [30, Theorem 4.1] that for any $n \in \mathbb{N}$ there are spaces X such that $G(X)$ is not in $\mathcal{C}_n$, but is in $\mathcal{C}_N$ for some finite $N > n$. Moreover, it follows from [30, discussion below Definition 2.2.1] or the main result of [15] that there are spaces X such that $G(X)$ is in $\mathcal{C}_\alpha$ for some infinite α, but not for any finite α.

Example A.9 shows that the range of possible values of the complexity rank for groupoids is quite rich. As we do not know the corresponding fact for C^*-algebras, the following question is natural.

Question A.10. *Are there any circumstances when the complexity rank of $C_r^*(G)$ is bounded above by that of G?*

It seems very unlikely that there is a positive answer in general, but it is conceivable that there could be a positive answer for coarse groupoids.

Example A.11. Transformation groupoids provide natural examples with finite complexity rank. Using the main result of [2], the complexity rank of the transformation groupoid associated to any free action of a virtually cyclic group on a finite-dimensional space is one. We guess that the techniques used in the proof of [18, Theorem 1.3] should show that for many discrete groups Γ, any free action on the Cantor set X gives rise to a groupoid $X \rtimes \Gamma$ with finite dynamical complexity; however, we did try to look into the details, and would be interested in any progress here. These ideas lead to the following conjecture.

Conjecture A.12. *If Γ has finite decomposition complexity then $X \rtimes \Gamma$ has finite dynamical complexity for* any *free action of Γ on the Cantor set.*

Remark A.13. Proposition A.8 does not give new information on the UCT; this is because all groupoids with finite dynamical complexity are amenable by [31, Theorem A.9], whence their groupoid C^*-algebras satisfy the UCT by Tu's theorem [64, Proposition 10.7]. However, it seems interesting to have an approach to the UCT for a large class of groupoids that does not factor through the Dirac-dual-Dirac machinery employed by Tu.

We now turn to the proof of Proposition A.8. For a subgroupoid H of a groupoid G, write

$$H' := H \cup G^{(0)},$$

which is also a subgroupoid of G.

[3] This result was one of the key motivations for the definition of finite dynamical complexity, and also motivates the terminology.

Lemma A.14. *Let G be a groupoid with compact base space, and let H be a subgroupoid in $\mathcal{C}_\alpha$. Then, $H \cup G^{(0)}$ is a subgroupoid of G that is also in $\mathcal{C}_\alpha$.*

Proof. We proceed by transfinite induction on α. For the base case $\alpha = 0$, let H be a subgroupoid of G in $\mathcal{C}_0$, and let K' be a compact subset of H'. As the base space in an étale groupoid is open, $K := K' \setminus G^{(0)}$ is also a compact set, and is contained in H. As H is in $\mathcal{C}_0$, there exists a subgroupoid L of H that contains K, and that has compact closure. Hence, L' is a subgroupoid of H' that contains K' and has compact closure. Thus, H' is in $\mathcal{C}_0$ too. The inductive step follows the same idea. ■

The lemma below is very similar to [67, Lemma B.3].

Lemma A.15. *Let G be a groupoid with compact base space. Let H be a subgroupoid of G that decomposes over some class $\mathcal{C}$ of subgroupoids of G. Then, H' decomposes over the collection of subgroupoids L', where L is a subgroupoid of H that is in $\mathcal{C}$.*

Proof. Let X be a finite subset of the unit ball of $C_r^*(H')$, and $\varepsilon > 0$. As $C_c(H) + C(G^{(0)})$ is dense in $C_r^*(H')$, perturbing X slightly, we may assume that X is contained in a subset of $C_r^*(H')$ of the form $C_c(K) + C(G^{(0)})$, where K is an open and relatively compact subset of H. The proof of [67, Lemma B.3] gives us open subgroupoids H_1 and H_2 of H and a positive contraction h in $C_c(H_1^{(0)}) \subseteq C_r^*(H_1)$ such that H_1, H_2 and $H_1 \cap H_2$ are in the class $\mathcal{C}$, and such that for all $x \in X$, $hx \in C_r^*(H_1)$, $(1-h)x \in C_r^*(H_2)$, and $(1-h)hx \in C_r^*(H_1 \cap H_2)$. Then, the data h, $C := C_r^*(H_1')$, $D = C_r^*(H_2')$, and $E = C_r^*(H_1' \cap H_2')$ give the desired decomposability statement. ■

Proof of Proposition A.8. For part (i), fix a groupoid G. We show by transfinite induction on α that if H is an open subgroupoid of G in the class $\mathcal{C}_\alpha$, and if

$$H' = H \cup G^{(0)},$$

then $C_r^*(H')$ is in the class $\mathcal{D}_\alpha$ of Definition 1.3, where we define $\mathcal{D}_\alpha$ relative to the class of subhomogeneous C^*-algebras. Applying this to $H = G$ then gives the desired conclusion for $C_r^*(G)$.

For the base case, we need to show that if H is an open subgroupoid of G in the class $\mathcal{C}_0$ and if $H' = H \cup G^{(0)}$, then $C_r^*(H')$ is locally subhomogeneous. Let a finite subset X of $C_r^*(H')$ and $\varepsilon > 0$ be given. As $C_c(H')$ is dense in $C_r^*(H')$, up to a perturbation, we may assume X is contained in $C_c(K)$ for some open and relatively compact subset K of H'. Lemma A.14 implies that H' is in $\mathcal{C}_0$, whence there is an open subgroupoid L of H' with compact closure that contains K, and therefore so that X is contained in $C_r^*(L)$. On the other hand, $C_r^*(L)$ is subhomogeneous by the proof [32, Lemma 8.14], so we are done with the base case.

Assume now that $\alpha > 0$ (and is either a successor ordinal or limit ordinal), and let H be a subgroupoid of G in the class $\mathcal{C}_\alpha$. According to Lemma A.15, we have that H' decomposes over

$$\left\{C_r^*(L') \mid L \text{ an open subgroupoid of } H' \text{ in } \bigcup_{\beta<\alpha} \mathcal{C}_\beta\right\}$$

which completes the proof of part (i) by inductive hypothesis.

We now look at part (ii), so let G be principal and ample. We will show that if G is in $\mathcal{C}_0$, then $C_r^*(G)$ is locally finite dimensional; thanks to our work in part (i), this will suffice for the proof.

Let then G be an element of $\mathcal{C}_0$. We claim that for any compact subset K of G there is a compact open subgroupoid of H of G that contains K. The claim shows that $C_r^*(G)$ is locally finite-dimensional. Indeed, up to a perturbation we can assume any finite subset of $C_r^*(G)$ is contained in $C_c(K)$ for some open and relatively compact subset K of G, and so in $C_r^*(H)$ for some compact, open subgroupoid of G. It is well-known that a compact, Hausdorff, étale, principal groupoid with totally disconnected base space has a locally finite-dimensional C^*-algebra; for example, this follows directly from the structure theorem for "CEERs" in [25, Lemma 3.4].

To establish the claim, let a compact subset K of G be given. According to the definition of $\mathcal{C}_0$ there exists an open subgroupoid L of G with compact closure such that K is contained in L. Note first that as L has compact closure, there is some $m \in \mathbb{N}$ such that L is covered by m open bisections from G. Hence, in particular, for any $x \in L^{(0)}$, we have that the range fibre L^x has at most m elements. Working entirely inside L, it suffices to prove that if K is a compact subset of a principal, ample groupoid L such that $\sup_{x\in L^{(0)}} |L^x| = m < \infty$, then there is a compact, open subgroupoid H of L that contains K.

Now, as L is ample (and étale), each point $l \in K$ is contained in a compact, open subset of L. As finitely many of these compact, open subsets cover K, there is a compact, open subset K' of L such that $K \subseteq K'$. Let H be the subgroupoid of L generated by K'. A subgroupoid generated by an open subset is always open (see for example [32, Lemma 5.2]), so it suffices to prove that H is compact. Let $(h_i)_{i\in I}$ be an arbitrary net consisting of elements from H. Each h_i can be written as a finite product $h_i = k_i^{(1)} \cdots k_i^{(n_i)}$, with $k_i^{(j)}$ in $K'' := K' \cup (K')^{-1} \cup s(K') \cup r(K')$. As each range fibre from L has at most m elements, we may assume that $n_i \leq m$ for all m; in fact we may assume it is exactly m, as otherwise we can just "pad" it with identity elements. Write then $h_i = k_i^{(1)} \cdots k_i^{(m)}$. As K'' is compact, we may pass to a subnet of I, and thus assume that each net $(k_i^{(j)})_{i\in I}$ has a convergent subnet, converging to some $k^{(j)}$ in K''. It follows on passing to this subnet that (h_i) converges to $k^{(1)} \cdots k^{(m)}$. As we have shown that every net in H has a convergent subnet, H is compact, completing the proof. ∎