Chapter 1

Introduction

This is the first in a series of papers where we analyze the projective positive energy representations of gauge groups.

Our main motivation is the Wigner–Mackey classification [60, 113] of projective unitary representations of the Poincaré group. Every irreducible such representation is labeled by an $SO^{\uparrow}(1, d-1)$-orbit in momentum space $\mathbb{R}^d$, together with an irreducible unitary representation of the corresponding little group. It is called a *positive energy* representation if for every 1-parameter group of timelike translations, the corresponding Hamilton operator is bounded from below. This excludes the tachyonic orbits, leaving the positive mass hyperboloids $p_\mu p^\mu = m^2$, the positive light cone $p_\mu p^\mu = 0$, $p_0 > 0$, and the origin $p = 0$. The corresponding little groups yield an intrinsic description of spin (for the positive mass hyperboloids) and helicity (for the positive light cone).

In this series of papers we aim to extend this picture with an infinite-dimensional group $\mathcal{G}$ of gauge transformations, placing internal symmetries and space-time symmetries on the same footing.

1.1 Outline of Part I and II of this series

For a gauge theory with structure group K, the fields over the space-time manifold M are associated to a principal K-bundle $\Xi \to M$. We consider the equivariant setting, where the group P of space-time symmetries acts by automorphisms on $\Xi \to M$, and the Lie algebra $\mathfrak{p}$ of P contains a distinguished cone $\mathcal{C} \subseteq \mathfrak{p}$ of "timelike generators". For Minkowski space, this is of course the Poincaré group P with the cone $\mathcal{C}$ of timelike translations.

The relevant group $\mathcal{G}$ of gauge transformations depends on the context. It always contains the group

$$\mathcal{G}_c := \Gamma_c(M, \mathrm{Ad}(\Xi))$$

of compactly supported vertical automorphisms of $\Xi \to M$, and it is this group that we will focus on in Part I of this series. In Part II we consider also global gauge transformations. The group $\mathcal{G}$ is then larger than $\mathcal{G}_c$, but it may be smaller than the group $\Gamma(M, \mathrm{Ad}(\Xi))$ of all vertical automorphisms due to boundary conditions at infinity.

A projective unitary representation of $\mathcal{G} \rtimes P$ assigns to every timelike generator $p_0 \in \mathcal{C}$ a one-parameter group of projective unitary transformations, and hence a selfadjoint Hamiltonian $H(p_0)$ that is well defined up to a constant.

Our main objective is to study the projective unitary representations of $\mathcal{G} \rtimes P$ that are of positive energy, in the sense that the Hamiltonians $H(p_0)$ are bounded from below.

Perhaps surprisingly, this places rather stringent restrictions on the representation theory of $\mathcal{G}$, leading to a complete classification in favorable cases.

1.1.1 Outline of Part I

In the first part of this series, we focus on the group $\mathcal{G}_c = \Gamma_c(M, \mathrm{Ad}(\Xi))$ of *compactly supported* gauge transformations. Our main result concerns the case where M has no fixed points for the cone $\mathcal{C}$ of timelike generators, and K is a 1-connected, semisimple Lie group.

Localization theorem. *For every projective positive energy representation $(\bar{\rho}, \mathcal{H})$ of the identity component $\Gamma_c(M, \mathrm{Ad}(\Xi))_0$, there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ and a positive energy representation $\bar{\rho}_S$ of $\Gamma_c(S, \mathrm{Ad}(\Xi))$ such that the following diagram commutes,*

$$\begin{array}{ccc} \Gamma_c(M, \mathrm{Ad}(\Xi))_0 & \xrightarrow{\ \bar{\rho}\ } & \mathrm{PU}(\mathcal{H}) \\ {\scriptstyle r_S}\big\downarrow & \nearrow{\scriptstyle \bar{\rho}_S} & \\ \Gamma_c(S, \mathrm{Ad}(\Xi)), & & \end{array}$$

where the vertical arrow denotes restriction to S.

This effectively reduces the classification of projective positive energy representations to the 1-dimensional case. If M is compact, then

$$S = \bigcup_{j=1}^{k} S_j$$

is a finite union of circles. If K is noncompact and simple, then we show that all positive energy representations are trivial. If K is compact, then the group

$$\Gamma(S, \mathrm{Ad}(\Xi)) \cong \prod_{j=1}^{k} \Gamma(S_j, \mathrm{Ad}(\Xi))$$

is a finite product of twisted loop groups, yielding a complete classification in terms of tensor products of highest weight representations for the corresponding affine Kac–Moody algebras [32, 54, 94, 104].

To some extent these results generalize to the case of noncompact manifolds M, where S can then have infinitely many connected components. We are able to classify the projective positive energy representations under the additional assumption that they admit a cyclic ground state vector which is unique up to scalar. These *vacuum representations* are classified in terms of infinite tensor products of vacuum representations of affine Kac–Moody algebras. In particular, every such representation is of type I. Without the vacuum condition, the classification is considerably more involved. We study in detail the case where all connected components of S are circles. Under a geometric "spectral gap" condition, we reduce the classification of projective positive energy representations to the representation theory of UHF C^*-algebras, yielding a rich source of representations of type II and III.

1.1.2 Outline of Part II

In the second part of this series, we consider the case where $\mathcal{G}$ contains *global* as well as compactly supported gauge transformations. To study the projective positive energy representations, we use the exact sequence

$$1 \to \mathcal{G}_c \to \mathcal{G} \to \mathcal{G}/\mathcal{G}_c \to 1.$$

By the results from Part I on the positive energy representations of $\mathcal{G}_c$, the problem essentially reduces to the group $\mathcal{G}/\mathcal{G}_c$ of gauge transformations "at infinity".

Needless to say, the resulting representation theory is very sensitive to the boundary conditions at infinity. We focus on the situation where M is an *asymptotically simple* space-time in the sense of Penrose [24, 37, 91, 92], and $\mathcal{G}$ consists of gauge transformations that extend smoothly to the conformal boundary. For the motivating example of the Poincaré group acting on d-dimensional Minkowski space, we obtain the following detailed account of the projective positive energy representation theory.

Minkowski space in dimension $d > 2$. In this setting we show that the projective positive energy representations of $\mathcal{G}$ depend only on the 1-*jets* of the gauge transformation at spacelike infinity ι_0 and at past and future timelike infinity $\iota_\pm$. This reduces the problem to the classification of projective positive energy representations of the (finite-dimensional!) semidirect product

$$\big(\mathrm{SO}^{\uparrow}(1, d-1) \times K^3\big) \ltimes \big(\mathbb{R}^d \oplus (\mathfrak{k}^3 \otimes \mathbb{R}^{d*})\big),$$

where $\mathrm{SO}(1, d-1)$ acts on $\mathbb{R}^d$ in the usual fashion, and K acts on its Lie algebra $\mathfrak{k}$ by the adjoint representation. The three copies of K encode the values of the gauge transformation at ι_0 and $\iota_\pm$, whereas the three copies of the additive group $\mathfrak{k} \otimes \mathbb{R}^{d*}$ encode the derivatives.

In the special case where the derivatives act trivially, we recover a projective positive energy representations of the Poincaré group $\mathbb{R}^d \rtimes \mathrm{SO}^{\uparrow}(1, d-1)$, together with 3 projective unitary representations of the structure group K. More generally, by

Mackey's theorem of imprimitivity, the irreducible projective positive energy representations are labeled by an orbit of $\mathrm{SO}^{\uparrow}(1, d-1) \times K^3$ in $\mathbb{R}^d \oplus (\mathfrak{k}^3 \otimes \mathbb{R}^{d*})$ whose energy is bounded from below, together with a projective unitary representation of the corresponding little group. In general these little groups will not contain the three copies of K, giving rise to phenomena that are reminiscent of spontaneous symmetry breaking.

Minkowski space in dimension $d = 2$. In contrast to the higher dimensional case, the projective positive energy representations in $d = 2$ do *not* in general factor through a finite-dimensional Lie group. For simplicity, we consider the group $\mathcal{G}$ of gauge transformations that extend smoothly to the *conformal compactification* of 2-dimensional Minkowski space. Here the three points ι_0 and $\iota_{\pm}$ at space- and timelike infinity are collapsed to a single point I, and past and future null infinity $\mathcal{I}^-$ and $\mathcal{I}^+$ are identified along lightlike geodesics (cf. [92]). The boundary of this space is a union of two circles $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$ (corresponding to left and right moving modes) that intersect transversally in a single point I. We prove that the positive energy representations of $\mathcal{G}$ depend only on the *values* of the gauge transformations at null infinity

$$\mathcal{I} = (\mathbb{S}^1_L \cup \mathbb{S}^1_R) \setminus \{I\},$$

and on the *2-jets* at the single point I.

At the Lie algebra level, the problem therefore reduces to classifying the projective positive energy representations of the abelian extension

$$0 \to |\mathfrak{k}| \to \mathfrak{g} \to \mathfrak{g}_{\mathrm{eq}} \to 0.$$

Here, the equalizer Lie algebra

$$\mathfrak{g}_{\mathrm{eq}} = \big\{(\xi, \eta) \in \Gamma(\mathbb{S}^1_L, \mathrm{ad}(\Xi)) \times \Gamma(\mathbb{S}^1_R, \mathrm{ad}(\Xi)); \xi(I) = \eta(I)\big\}$$

represents the values of the infinitesimal gauge transformations on the conformal boundary $\mathbb{S}^1_L \cup \mathbb{S}^1_R$, the abelian Lie algebra $|\mathfrak{k}|$ with underlying vector space $\mathfrak{k}$ represents the mixed second derivatives at I, and $\mathfrak{g}_{\mathrm{eq}}$ acts on $|\mathfrak{k}|$ by evaluating at I and composing with the adjoint representation.

Even in the untwisted case, where Ξ is the trivial K-bundle, the classification of the projective positive energy representations is by no means trivial. This is because the positive energy condition is *not* with respect to rigid rotations of $\mathbb{S}^1_{L/R}$, but with respect to the translations of the real projective line

$$\mathbb{S}^1_{L/R} = \mathbb{R} \cup \{I\}$$

fixing the point I at infinity.

Under the restrictive additional condition that the projective unitary representations are of positive energy with respect to rotations as well as translations, we obtain a classification in terms of highest weight representations of the two untwisted affine Kac–Moody algebras corresponding to $\mathbb{S}^1_L$ and $\mathbb{S}^1_R$, together with a projective positive energy representation of the finite-dimensional Lie group of 2-jets of gauge transformations at I.

Although the Kac–Moody representations are familiar from the construction of loop group nets in conformal field theory, the positive energy representations involving 2-jets (which are Poincaré invariant but not conformally invariant) appear to be a novel feature.

1.2 Structure of the present memoir

For a closed quantum system that is described by a Hilbert space $\mathcal{H}$, any two states that differ by a global phase are physically indistinguishable. The state space of the system is therefore described by the projective Hilbert space $\mathbb{P}(\mathcal{H})$. By Wigner's theorem, a connected Lie group G acts on the projective Hilbert space $\mathbb{P}(\mathcal{H})$ by projective unitary transformations, resulting in a projective unitary representation $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$.

1.2.1 Positive energy representations

Since we are interested in the group of compactly supported gauge transformations, we need to work with *infinite-dimensional* Lie groups modeled on locally convex spaces, or *locally convex Lie groups* for short. In Chapter 2 we recall and extend some recent results from [52, 76] that allow us to go back and forth between smooth projective unitary representations of a locally convex Lie group G, smooth unitary representations of a central Lie group extension $G^\sharp$, and the derived representations of its Lie algebra $\mathfrak{g}^\sharp$.

In Chapter 3 we introduce projective *positive energy representations* in the context of a Lie group P that acts smoothly on G by automorphisms. For a distinguished *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$, we require that the spectrum of the corresponding selfadjoint operators in the derived representation is bounded from below. Since a representation is of positive energy for the cone $\mathcal{C}$ if and only if it is of positive energy for the 1-parameter subgroups generated by $\mathcal{C} \subseteq \mathfrak{p}$, we can always reduce to the case $P = \mathbb{R}$, where the non-negative spectrum condition pertains to a single Hamilton operator H. Using the Borchers–Arveson theorem, we further reduce the classification to the *minimal* representations, where $H \geq 0$ is the smallest possible Hamilton operator with non-negative spectrum.

In Chapter 4 we then turn to our subject proper, namely the locally convex Lie group $\mathcal{G}_c$ of compactly supported gauge transformations. We consider the setting where M is a manifold, P is a Lie group acting smoothly on M, and $\mathcal{K} \to M$ is a bundle of 1-connected semisimple Lie groups that is equipped with a lift of this action. The group $\mathcal{G}_c = \Gamma_c(M, \mathcal{K})$ of compactly supported sections then carries a smooth action of P by automorphisms, and we consider the smooth projective unitary representations of the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes P$.

The motivating example is of course the case where $\mathcal{K} = \mathrm{Ad}(\Xi)$ is the adjoint bundle of a principal fiber bundle $\Xi \to M$, and $\Gamma_c(M, \mathrm{Ad}(\Xi))$ is the group of vertical automorphisms of Ξ that are trivial outside a compact subset of M. The reason for the minor generalization to bundles of Lie groups is purely technical; the reduction to simple structure groups in Section 4.2 is somewhat easier in that setting.

1.2.2 The localization theorem

The main result in the present memoir is the following localization result (a minor generalization of the one in Section 1.1.1), which essentially reduces the classification of projective positive energy representations to the 1-dimensional case.

Localization theorem (Theorem 7.19). *Let $(\bar{\rho}, \mathcal{H})$ be a projective positive energy representation of $\Gamma_c(M, \mathcal{K}) \rtimes P$. If the cone $\mathcal{C}$ has no fixed points in M, then there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ such that on the connected component $\Gamma_c(M, \mathcal{K})_0$, the projective representation $\bar{\rho}$ factors through the restriction homomorphism $r_S \colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$.*

We sketch the proof in the special case that the structure group K of $\mathcal{K}$ is a compact simple Lie group. The result for (not necessarily compact) semisimple Lie groups is reduced to the simple case in Section 4.2, and to the compact simple case in Section 6.1. We require K to be 1-connected, but this is by no means essential; results beyond 1-connected groups are discussed in Sections 7.1 and 8.3.

Further, we will assume without loss of generality that P is the additive group $\mathbb{R}$ of real numbers. The corresponding flow is then given by a non-vanishing vector field $\mathbf{v}_M$ on M, which lifts to a vector field $\mathbf{v}$ on $\mathcal{K}$. We denote the corresponding derivation of the gauge algebra by $D\xi := L_{\mathbf{v}}\xi$. The reduction from P to $\mathbb{R}$ is carried out in Section 7.5 by considering the 1-parameter subgroups of P that are generated by elements of the positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Step 1. Let $\mathfrak{K} \to M$ be the bundle of Lie algebras derived from $\mathcal{K} \to M$. Then, every smooth projective unitary representation of $\Gamma_c(M, \mathcal{K}) \rtimes \mathbb{R}$ gives rise to an $\mathbb{R}$-invariant 2-cocycle ω on the compactly supported gauge algebra $\Gamma_c(M, \mathfrak{K})$. In Section 4.3 we show that every such cocycle is cohomologous to one of the form

$$\omega(\xi, \eta) = \lambda(\kappa(\xi, d_\nabla \eta)) \quad \text{for } \xi, \eta \in \Gamma_c(M, \mathfrak{K}), \tag{1.1}$$

where ∇ is a Lie connection on $\mathfrak{K} \to M$, κ is a positive definite invariant bilinear form on the Lie algebra $\mathfrak{k}$ of K, and $\lambda \colon \Omega^1_c(M) \to \mathbb{R}$ is a closed current that is invariant under the flow.

Step 2. The positive energy condition for $(\rho, \mathcal{H})$ gives rise to a Cauchy–Schwarz inequality for the derived Lie algebra representation $\mathtt{d}\rho$. In Section 3.4 we show that if $[\xi, D\xi] = 0$ and $\omega(\xi, D) = 0$, then

$$\langle \psi, i\,\mathtt{d}\rho(D\xi)\psi \rangle^2 \leq 2\omega(\xi, D\xi)\langle \psi, H\psi \rangle \tag{1.2}$$

for every smooth unit vector ψ. Moreover, $\omega(\xi, D\xi)$ is non-negative. In Chapter 5 this is used to show that the closed current λ from (1.1) takes the form

$$\lambda(\alpha) = \int_M (i_{\mathbf{v}_M}\alpha)(x)d\mu(x)$$

for a flow-invariant regular Borel measure μ on M. In terms of this measure, the Cauchy–Schwarz inequality (1.2) becomes

$$\langle \psi, i\,\mathtt{d}\rho(L_{\mathbf{v}}\xi)\psi \rangle^2 \leq 2\langle \psi, H\psi \rangle \|L_{\mathbf{v}}\xi\|_{\mu}^2. \tag{1.3}$$

In other words, if ξ is in the image of the derivation

$$D = L_{\mathbf{v}},$$

then the expectation value of the unbounded operator $i\,\mathtt{d}\rho(\xi)$ is controlled in terms of the energy $\langle \psi, H\psi \rangle$, and the L^2-norm of ξ with respect to the measure μ. In fact, a small but important refinement allows one to control the expectation of $i\,\mathtt{d}\rho(\xi)$ in terms of similar data if ξ is not in the image of the derivation.

Step 3. In Chapter 6 we use the Cauchy–Schwarz estimate (1.3) and its refinement to show that

$$\pm\, i\,\mathtt{d}\rho(\xi) \leq \|\xi\|_{\nu}\mathbf{1} + \|\xi\|_{\mu}H \tag{1.4}$$

as unbounded operators. The measure ν is absolutely continuous with respect to μ, with a density that is upper semi-continuous and invariant under the flow. From a technical point of view, this is the heart of the proof. It allows us to extend $\mathtt{d}\rho$ to a positive energy representation of the Banach–Lie algebra $H^2_\partial(M, \mathcal{K})$ of sections that are twice differentiable in the direction of the flow, but only ν-*measurable* in the direction perpendicular to the flow.

Step 4. The final steps of the proof are carried out in Chapter 7. Every point in M admits a flow box $U_0 \times I \simeq U \subseteq M$, where the flow fixes all points in U_0 and acts by translation on the interval $I \subseteq \mathbb{R}$ for small times. Accordingly, the flow-invariant measure on U decomposes as

$$\mu = \mu_0 \otimes dt.$$

Since the sections in $H^2_\partial(U, \mathcal{K}) \subseteq H^2_\partial(M, \mathcal{K})$ need only be measurable in the direction perpendicular to the flow, we can continuously embed $C^\infty_c(I, \mathfrak{k})$ as a Lie subalgebra of $H^2_\partial(U, \mathcal{K})$ by multiplying with an indicator function χ_E for a Borel subset $E \subseteq U_0$ of finite measure. This yields a projective unitary representation of $C^\infty_c(I, \mathfrak{k})$ with central charge $2\pi\mu_0(E)$.

Step 5. Since the dense space of analytic vectors for H is analytic for the extension of $\mathtt{d}\rho$ to $H^2_\partial(M,\mathcal{K})$, the projective unitary representation of $C^\infty_c(I,\mathfrak{k})$ extends to the 1-connected Lie group G that integrates $C^\infty_c(I,\mathfrak{k})$. This gives rise to a smooth central $\mathbb{T}$-extension $G^\sharp \to G$. For every smooth map $\sigma\colon \mathbb{S}^2 \to G$, the pullback $\sigma^*G^\sharp \to \mathbb{S}^2$ is a principal circle bundle, and integrality of the corresponding Chern class implies that $2\pi\mu_0(E) \in \mathbb{N}_0$. Since this holds for every Borel set, we conclude that μ_0 is a locally finite sum of point measures, and hence that $\mu = \mu_0 \otimes dt$ is concentrated on a closed embedded submanifold $S_U \subseteq U$ of dimension 1. Since the argument is local, the measure μ is concentrated on a closed, embedded, 1-dimensional submanifold $S \subseteq M$. Using (1.4), one shows that $\mathtt{d}\rho$ vanishes on the ideal of sections that vanish μ-almost everywhere. This proves the theorem at the Lie algebra level. The result at the group level follows because $\Gamma_c(S,\mathcal{K})$ is 1-connected.

1.2.3 Classification of positive energy representations

For manifolds with a fixed point free $\mathbb{R}$-action, the Localization theorem effectively reduces the projective positive energy representation theory to the 1-dimensional setting.

Compact manifolds. For *compact* manifolds M, we show in Chapter 8 that the localization theorem leads to a full classification. Indeed, since

$$S = \bigcup_{j=1}^{k} S_j$$

is a finite union of periodic orbits S_j, the group $\Gamma(S,\mathcal{K})$ is a finite product of *twisted loop groups* $\Gamma(S_j,\mathcal{K})$. The projective positive energy representations of twisted loop groups are classified in Section 8.1, using the rich structure and representation theory of affine Kac–Moody Lie algebras [54], combined with the method of holomorphic induction for Fréchet–Lie groups developed in [77, 79].

This leads to a full classification of projective positive energy representations of $\Gamma(M,\mathcal{K})$, which is detailed in Section 8.2. Up to unitary equivalence, every irreducible projective positive energy representation is determined by the following data.

- Finitely many periodic $\mathbb{R}$-orbits $S_j \subseteq M$, each equipped with a central charge $c_j \in \mathbb{N}$.
- For every pair (S_j, c_j), an anti-dominant integral weight λ_j of the corresponding affine Kac–Moody algebra with central charge $c_j \in \mathbb{N}$.

Moreover, every projective positive energy representation is a direct sum of irreducible ones.

Noncompact manifolds. For *noncompact* manifolds M, the situation is somewhat more intricate. Here S is a union of countably many $\mathbb{R}$-orbits S_j, each of which is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. These two cases are considered separately in Chapter 9.

In Section 9.1 we consider the case where S consists of countably many lines. Since the bundle $\mathcal{K}$ trivializes over every line, the gauge group $\Gamma_c(S, \mathcal{K})$ is a weak direct product of countably many copies of $C_c^\infty(\mathbb{R}, K)$. In order to arrive at a (partial) classification, we impose the additional condition that the projective positive energy representation admit a cyclic ground state vector $\Omega \in \mathcal{H}$ that is unique up to a scalar. In Theorem 9.11 we show that these *vacuum representations* are classified up to unitary equivalence by a non-zero central charge $c_j \in \mathbb{N}$ for every connected component $S_j \simeq \mathbb{R}$. The proof proceeds by reducing to the (important) special case $M = \mathbb{R}$, where the classification is essentially due to Tanimoto [102].

In Section 9.2 we consider the case where S consists of infinitely many circles. Here we impose the much less restrictive condition that $\mathcal{H}$ is a *ground state representation*. This means that $\mathcal{H}$ is generated under $\Gamma_c(S, \mathcal{K})$ by the space of ground states, but we do not require these ground states to be unique. We show that under an (essentially geometric) *spectral gap* condition, every positive energy representation is automatically a ground state representation. Since $\Gamma_c(S, \mathcal{K})$ admits projective positive energy representations of Type II and III, it is necessary to consider factor representations instead of irreducible ones. If all orbits in M are periodic, we show that the minimal, factorial ground state representations of $\Gamma_c(M, \mathcal{K})$ are classified up to unitary equivalence by 3 pieces of data. The first two are the same as in the case of compact manifolds.

- Countably many periodic orbits $S_j \subseteq M$, equipped with a central charge $c_j \in \mathbb{N}$.
- For every pair (S_j, c_j) an anti-dominant integral weight λ_j of the corresponding affine Kac–Moody algebra with central charge c_j.

The integral weight λ_j gives rise to a unitary lowest weight representation $\mathcal{H}_{\lambda_j}$ of the corresponding affine Kac–Moody algebra. Using the ground state projections P_j, we consider the collection of finite tensor products of the compact operators $K(\mathcal{H}_{\lambda_j})$ as a directed system of C^*-algebras. Its direct limit

$$\mathcal{B} = \bigotimes_j K(\mathcal{H}_{\lambda_j})$$

has a distinguished ground state projection

$$P_\infty = \bigotimes_j P_j .$$

The third datum needed to characterize a minimal factorial ground state representation is the following.

- A factorial representation of $\mathcal{B}$ that is generated by fixed points of the projection P_∞.

Since $P_\infty \mathcal{B} P_\infty$ is a UHF C^*-algebra, this provides a rich supply of representations of type II and III, in marked contrast with the compact case.

1.3 Connection to the existing literature

Abelian structure groups. If the structure Lie algebra $\mathfrak{k}$ is merely assumed to be reductive, then it decomposes as a direct sum $\mathfrak{k} = \mathfrak{z} \oplus \mathfrak{k}'$, where $\mathfrak{z}$ is abelian and the commutator algebra $\mathfrak{k}$ is semisimple. Since this decomposition is invariant under all automorphism, we obtain a corresponding decomposition on the level of Lie algebra bundles $\mathfrak{K} \cong \mathfrak{Z} \oplus \mathfrak{K}'$. Accordingly, the Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ decomposes as a direct sum $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}'$ and this decomposition is orthogonal with respect to any 2-cocycle because $\mathfrak{g}'$ is perfect. Therefore, the classification of the positive energy representations basically reduces to the cases where $\mathfrak{k}$ is semisimple and where $\mathfrak{k}$ is abelian. We refer to Solecki's paper [100] for some interesting results concerning groups of maps with values in the circle group, and to [98] for related results pertaining to defects in conformal field theory. G. Segal's paper [97] contains a number of interesting results on projective positive energy representations of loop groups with values in a torus.

Integrating representations of infinite-dimensional Lie groups. The technique to integrate representations of infinite-dimensional Lie algebras to groups by first verifying suitable estimates has already been used by R. Goodman and N. Wallach in [32] to construct the irreducible unitary positive energy representations of loop groups and diffeomorphism groups. Their technique has later been refined by V. Toledano–Laredo [104] to larger classes of infinite-dimensional Lie algebras. Related results on integrating Lie algebra representations can be found in [52].

Non-commutative distributions. In [3] an irreducible unitary representation of $\mathcal{G}_c = \Gamma_c(M, \mathrm{Ad}(\Xi))$ is called a *non-commutative distribution*. In view of the Borchers–Arveson theorem [13], an irreducible projective positive energy representation of $\mathcal{G}_c \rtimes_\alpha \mathbb{R}$ remains irreducible when restricted to $\mathcal{G}_c$. In this sense we contribute to the program outlined in [3] by classifying those non-commutative distributions for M compact and K compact semisimple for which extensions to positive energy representations exist.

Tensor product representations. For any, not necessary compact, Lie group K, the group $C(X, K)$ has unitary representations obtained as finite tensor products of evaluation representations. However, for some noncompact groups, such as $K = \widetilde{\mathrm{SU}}_{1,n}(\mathbb{C})$, one even has "continuous" tensor product representations which are irreducible and extend to groups of measurable maps (cf. [101] for finite-dimensional target groups, [48], [9], [15], [16], [21], [23, 108], [14, 107] for semisimple target groups, [35] for a general discussion and classification results for locally compact target groups, [6] for classification results for compact and nilpotent target groups, and [88] for an example where the target group $\mathrm{U}(\infty)$ is infinite-dimensional). In the algebraic context, these representations also appear in [47] which contains a classification of various types of unitary representations generalizing highest weight representations. All these representations are most naturally defined on groups of measurable maps, so that they neither require a topology nor a smooth structure on X.

Derivative and energy representations. One of the first references concerning unitary representations of groups of smooth maps such as $C^\infty(\mathbb{R}, \mathrm{SU}(2,\mathbb{C}))$ is [22], where the authors introduce the concept of a *derivative representation* which depends only on the derivatives up to some order N in some point $t_0 \in \mathbb{R}$. These ideas can be combined with continuous tensor product representations to obtain factorizable representations that do not extend to groups of measurable maps [89], [90]. Further, there exist factorial representations of mapping groups defined most naturally on groups of Sobolev H^1-maps, the so-called energy representations (cf. [45, 46], [2], [109], [3], [4], [99], [5], [1]).

Central extensions. The problem of classifying all smooth projective irreducible unitary representations of gauge groups is still wide open. Our treatment in Chapter 5 of the present memoir, as well as our earlier work on bounded representations [51], suggests that a classification of the central extensions of gauge algebras can be a key step towards this goal. The second Lie algebra homology of $\mathfrak{sl}_n(\mathcal{A})$ for a unital ring $\mathcal{A}$ is due to Bloch [10] and Kassel–Loday [56], and the full homology ring of $\mathfrak{gl}(\mathcal{A})$ was characterized in terms of the cyclic homology of $\mathcal{A}$ by Tsygan and Loday–Quillen [57, 58, 106]. Some of these arguments were adapted to $C^\infty(M, \mathfrak{k})$ with semisimple $\mathfrak{k}$ by Pressley–Segal [94, Section 4.2], and to $\mathcal{A} \otimes \mathfrak{k}$ for general Lie algebras $\mathfrak{k}$ by [36, 82, 115] in the setting where $\mathcal{A}$ is commutative. For non-trivial Lie algebra bundles, the universal central extension of the gauge algebra was obtained in [53] from the compactly supported trivial case [61] using a localization trick.

The case where *M* is a torus. In [105] (see also [3, Section 5.4]) Torresani studies projective unitary "highest weight representations" of $C^\infty(\mathbb{T}^d, \mathfrak{k})$, where $\mathfrak{k}$ is compact simple. Besides the finite tensor products of so-called evaluation representations (*elementary representations*) he finds finite tensor products of evaluation representations of $C^\infty(\mathbb{T}^d, \mathfrak{k}) \cong C^\infty(\mathbb{T}^{d-1}, C^\infty(\mathbb{T}, \mathfrak{k}))$, where the representations of the target algebra $C^\infty(\mathbb{T}, \mathfrak{k})$ are projective highest weight representations (*semi-elementary representations*). Our results in Chapter 8 reduce to this picture in the special case of a circle action on a torus.

Norm continuous representations. In a previous paper [50], we considered the related problem of classifying norm continuous unitary representations of the connected groups $\Gamma_c(M, \mathcal{K})_0$. In this case the problem also reduces to the case where $\mathfrak{k}$ is compact semisimple and the representations are linear rather than projective. For every irreducible representation ρ, there exists an embedded 0-dimensional submanifold S, i.e., a locally finite subset, $S \subseteq M$ such that ρ factors through the restriction map $\Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}) \cong \mathfrak{k}^{(S)}$. If M is compact, it follows that ρ is a finite tensor product of irreducible representations obtained by composing an irreducible representation of $\mathfrak{k}$ with the evaluation in a point $s \in S$. In particular, it is finite-dimensional. If M is noncompact, then the bounded representation theory of the LF-Lie algebra $\Gamma_c(M, \mathfrak{K})$ is "wild" in the sense that there exist bounded factor representations of type II and III. The main result in [50] is a complete reduction of the classification of bounded irreducible representations to the classification of irreducible representations of UHF C^*-algebras.

Type III representations from noncompact orbits. For noncompact M, a different source of representations comes from the group $C_c^\infty(\mathbb{R}, K)$ corresponding to a single noncompact connected component of S. Here representations of Type III_1 were constructed in [19, 112]. Other results in this context have recently been obtained in [17], where solitonic representations of conformal nets on the circle are constructed from non-smooth diffeomorphisms. These in turn provide positive energy representations of $C_c^\infty(\mathbb{R}, K) \cong C_c^\infty(\mathbb{T} \setminus \{-1\}, K)$ which do not extend to loop group representations [17, Theorem 3.4, Section 4.2]. In particular, irreducible representations of this type are obtained.