Chapter 2

Projective representations of Lie groups

In this chapter, we introduce Lie groups modeled on locally convex vector spaces, or *locally convex Lie groups* for short. This is a generalization of the concept of a finite-dimensional Lie group that captures a wide range of interesting examples (cf. [71] for an overview), including gauge groups, our main object of study. We then summarize the central results from [52], which allow us to go back and forth between smooth projective unitary representations of a locally convex Lie group G and smooth unitary representations of a central Lie group extension $G^\sharp$ of G. On the identity component G_0, these are characterized by representations of the corresponding Lie algebra $\mathfrak{g}^\sharp$.

2.1 Locally convex Lie groups

Let E and F be locally convex spaces, $U \subseteq E$ open and $f: U \to F$ a map. Then, the *derivative of f at x in the direction h* is defined as

$$\partial_h f(x) := \lim_{t \to 0} \frac{1}{t}(f(x + th) - f(x))$$

whenever it exists. We set $Df(x)(h) := \partial_h f(x)$. The function f is called *differentiable at x* if $Df(x)(h)$ exists for all $h \in E$. It is called *continuously differentiable* if it is differentiable at all points of U and

$$Df: U \times E \to F, \quad (x, h) \mapsto Df(x)(h)$$

is a continuous map. Note that this implies that the maps $Df(x)$ are linear (cf. [30, Lemma 1.2.11]). The map f is called a *C^k-map*, $k \in \mathbb{N} \cup \{\infty\}$, if it is continuous, the iterated directional derivatives

$$D^j f(x)(h_1, \ldots, h_j) := (\partial_{h_j} \cdots \partial_{h_1} f)(x)$$

exist for all integers $1 \leq j \leq k$, $x \in U$ and $h_1, \ldots, h_j \in E$, and all maps

$$D^j f: U \times E^j \to F$$

are continuous. As usual, C^∞-maps are called *smooth*.

Once the concept of a smooth function between open subsets of locally convex spaces is established, it is clear how to define a locally convex smooth manifold (cf. [71], [30]).

Definition 2.1. A *locally convex Lie group* G is a group equipped with a smooth manifold structure modeled on a locally convex space for which the group multiplication and the inversion are smooth maps. Morphisms of locally convex Lie groups are smooth group homomorphisms.

We write $\mathbf{1} \in G$ for the identity element. The Lie algebra $\mathfrak{g}$ of G is identified with the tangent space $T_{\mathbf{1}}(G)$, and the Lie bracket is obtained by identification with the Lie algebra of left invariant vector fields. It is a *locally convex Lie algebra* in the following sense.

Definition 2.2. A *locally convex Lie algebra* is a locally convex vector space $\mathfrak{g}$ with a continuous Lie bracket $[\cdot, \cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. Morphisms of locally convex Lie algebras are continuous Lie algebra homomorphisms.

Definition 2.3. A smooth map $\exp\colon \mathfrak{g} \to G$ is called an *exponential function* if each curve $\gamma_x(t) := \exp(tx)$ is a one-parameter group with $\gamma_x'(0) = x$. A Lie group G is said to be *locally exponential* if it has an exponential function for which there is an open 0-neighborhood U in $\mathfrak{g}$ mapped diffeomorphically by exp onto an open subset of G.

2.2 Smooth representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and exponential function $\exp\colon \mathfrak{g} \to G$. In the context of Lie theory, it is natural to study *smooth* (projective) unitary representations on a complex Hilbert space $\mathcal{H}$. We take the scalar product on $\mathcal{H}$ to be linear in the *second* argument, and denote the group of unitary operators by $\mathrm{U}(\mathcal{H})$.

2.2.1 Unitary representations

A *unitary representation* $(\rho, \mathcal{H})$ of G is a Hilbert space $\mathcal{H}$ with a group homomorphism $\rho\colon G \to \mathrm{U}(\mathcal{H})$. A *unitary equivalence* between $(\rho, \mathcal{H})$ and $(\rho', \mathcal{H}')$ is a unitary transformation $U\colon \mathcal{H} \to \mathcal{H}'$ such that

$$U \circ \rho(g) = \rho(g)' \circ U \quad \text{for all } g \in G.$$

Definition 2.4 (Continuous unitary representations). A unitary representation $(\rho, \mathcal{H})$ is called *continuous* if the orbit map $G \to \mathcal{H}\colon g \mapsto \rho(g)\psi$ is continuous for all $\psi \in \mathcal{H}$ (see [78] for more details).

Definition 2.5 (Smooth unitary representations). We call $\psi \in \mathcal{H}$ a *smooth vector* if the orbit map $g \mapsto \rho(g)\psi$ is smooth, and write $\mathcal{H}^\infty \subseteq \mathcal{H}$ for the subspace of smooth vectors. We say that ρ is *smooth* if $\mathcal{H}^\infty$ is dense in $\mathcal{H}$ (see [52] for more details).

Every smooth representation is continuous. A representation ρ is called *bounded* if $\rho\colon G \to \mathrm{U}(\mathcal{H})$ is continuous with respect to the norm topology on $\mathrm{U}(\mathcal{H})$. Boundedness implies continuity, but many interesting continuous representations, including the (smooth!) positive energy representations that are the main focus of this memoir, are unbounded. For some recent results on the automatic smoothness of unbounded unitary representations satisfying certain spectral conditions such as semiboundedness (Definition 6.31), we refer to [114].

Definition 2.6. (Derived representation) For a smooth unitary representation $(\rho, \mathcal{H})$, the *derived representation* $\mathtt{d}\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of the Lie algebra $\mathfrak{g}$ is defined by

$$\mathtt{d}\rho(\xi)\psi := \frac{d}{dt}\Big|_{t=0} \rho(\exp t\xi)\psi.$$

Remark 2.7 (Selfadjoint generators). The closure of any operator $\mathtt{d}\rho(\xi)$ coincides with the infinitesimal generator of the unitary one-parameter group $\rho(\exp t\xi)$. In particular, the operators $i \cdot \mathtt{d}\rho(\xi)$ are essentially selfadjoint by Stone's theorem (cf. [95, Section VIII.4]).

2.2.2 Projective unitary representations

Let $\mathcal{H}$ be a Hilbert space. The projective Hilbert space is denoted by $\mathbb{P}(\mathcal{H})$, and its elements are denoted $[\psi] = \mathbb{C}\psi$ for non-zero $\psi \in \mathcal{H}$. We denote the projective unitary group by

$$\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\mathbb{T}\mathbf{1}$$

and write $\overline{U}$ for the image of $U \in \mathrm{U}(\mathcal{H})$ in $\mathrm{PU}(\mathcal{H})$.

A *projective unitary representation* $(\bar{\rho}, \mathcal{H})$ of a locally convex Lie group G is a complex Hilbert space $\mathcal{H}$ with a group homomorphism $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$. A *unitary equivalence* between $(\bar{\rho}, \mathcal{H})$ and $(\bar{\rho}', \mathcal{H}')$ is a unitary transformation $U\colon \mathcal{H} \to \mathcal{H}'$ such that $\overline{U} \circ \bar{\rho}(g) = \bar{\rho}(g)' \circ \overline{U}$ for all $g \in G$.

A projective unitary representation yields an action of G on $\mathbb{P}(\mathcal{H})$. Since $\mathbb{P}(\mathcal{H})$ is a Hilbert manifold, we can use this to define continuous and smooth projective representations.

Definition 2.8 (Continuous projective unitary representations). A projective unitary representation $(\bar{\rho}, \mathcal{H})$ is called *continuous* if the orbit map $G \to \mathbb{P}(\mathcal{H})\colon g \mapsto \bar{\rho}(g)[\psi]$ is continuous for all $[\psi] \in \mathbb{P}(\mathcal{H})$.

Definition 2.9 (Smooth projective unitary representations). A ray $[\psi] \in \mathbb{P}(\mathcal{H})$ is called *smooth* if its orbit map $g \mapsto \bar{\rho}(g)[\psi]$ is smooth, and we denote the set of smooth rays by $\mathbb{P}(\mathcal{H})^\infty$. A projective unitary representation $(\bar{\rho}, \mathcal{H})$ is called *smooth* if $\mathbb{P}(\mathcal{H})^\infty$ is dense in $\mathcal{H}$ (cf. [52, 78]).

2.3 Central extensions

In this memoir, we are primarily interested in smooth projective unitary representations $\bar{\rho}\colon G \to \mathrm{PU}(\mathcal{H})$ of a locally convex Lie group G. We call $\bar{\rho}$ *linear* if it comes from a smooth unitary representation $\rho\colon G \to \mathrm{U}(\mathcal{H})$.

Although not every smooth projective unitary representation of G is linear, it can always be viewed as a smooth linear representation of a *central extension* of G by the circle group

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}.$$

Definition 2.10 (Central group extensions). A *central extension* of G by $\mathbb{T}$ is an exact sequence

$$1 \to \mathbb{T} \to G^{\sharp} \to G \to 1$$

of locally convex Lie groups (the arrows are smooth group homomorphisms) such that the image of $\mathbb{T}$ is central in $G^{\sharp}$ and $G^{\sharp} \to G$ is a locally trivial principal $\mathbb{T}$-bundle. An *isomorphism* $\Phi\colon G^{\sharp} \to G^{\sharp\prime}$ of central $\mathbb{T}$-extensions is an isomorphism of locally convex Lie groups that induces the identity maps on G and $\mathbb{T}$.

For a smooth projective unitary representation $(\bar{\rho}, \mathcal{H})$ of G, the group

$$G^{\sharp} := \{(g, U) \in G \times \mathrm{U}(\mathcal{H}) : \bar{\rho}(g) = \bar{U}\} \tag{2.1}$$

is a central Lie group extension of G by $\mathbb{T}$ [52, Theorem 4.3]. Its smooth unitary representation

$$\rho\colon G^{\sharp} \to \mathrm{U}(\mathcal{H}), \quad (g, U) \mapsto U$$

reduces to $z \mapsto z\mathbf{1}$ on $\mathbb{T}$ and induces $\bar{\rho}$ on G. Since the restriction of ρ to the identity component $G^{\sharp}_0$ is determined by the derived Lie algebra representation [52, Proposition 3.4], it is worthwhile to take a closer look at central extensions of locally convex Lie algebras.

Definition 2.11 (Central Lie algebra extensions). A *central extension* of a locally convex Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ is an exact sequence

$$0 \to \mathbb{R} \to \mathfrak{g}^{\sharp} \to \mathfrak{g} \to 0$$

of locally convex Lie algebras (the arrows are continuous Lie algebra homomorphisms) such that the image of $\mathbb{R}$ is central in $\mathfrak{g}^{\sharp}$. An *isomorphism* $\varphi\colon \mathfrak{g}^{\sharp} \to \mathfrak{g}^{\sharp\prime}$ of central extensions is an isomorphism of locally convex Lie algebras that induces the identity maps on $\mathfrak{g}$ and $\mathbb{R}$.

The group extensions of Definition 2.10 give rise to Lie algebra extensions in the sense of Definition 2.11. In order to classify the latter, we introduce continuous Lie algebra cohomology.

Definition 2.12. The *continuous Lie algebra cohomology* space $H^n(\mathfrak{g},\mathbb{R})$ of a locally convex Lie algebra $\mathfrak{g}$ is the cohomology of the complex $C^\bullet(\mathfrak{g},\mathbb{R})$, where $C^n(\mathfrak{g},\mathbb{R})$ consists of the continuous alternating linear maps $\mathfrak{g}^n \to \mathbb{R}$ with differential

$$\delta\colon C^n(\mathfrak{g},\mathbb{R}) \to C^{n+1}(\mathfrak{g},\mathbb{R})$$

defined by

$$\delta\omega(\xi_0,\dots,\xi_n) := \sum_{0\le i<j\le n} (-1)^{i+j}\omega\big([\xi_i,\xi_j],\xi_1,\dots,\widehat{\xi}_i,\dots,\widehat{\xi}_j,\dots,\xi_n\big).$$

The second Lie algebra cohomology $H^2(\mathfrak{g},\mathbb{R})$ classifies central extensions up to isomorphism. The 2-cocycle $\omega\colon \mathfrak{g}^2 \to \mathbb{R}$ gives rise to the Lie algebra

$$\mathfrak{g}^\sharp_\omega := \mathbb{R} \oplus_\omega \mathfrak{g}$$

with the Lie bracket

$$[(z,\xi),(z',\xi')] := \big(\omega(\xi,\xi'),[\xi,\xi']\big).$$

Equipped with the obvious maps $\mathbb{R} \to \mathfrak{g}^\sharp_\omega \to \mathfrak{g}$, this defines a central extension of $\mathfrak{g}$. Every central extension is isomorphic to one of this form, and two central extensions are isomorphic if and only if the corresponding cohomology classes $[\omega] \in H^2(\mathfrak{g},\mathbb{R})$ coincide [52, Proposition 6.3].

The following theorem collects some of the main results of our previous paper [52, Corollary 4.5, Theorem 7.3]. It allows us to go back and forth between smooth projective unitary representations of G, smooth unitary representations of a central extension $G^\sharp$ of G, and the corresponding representations of its Lie algebra $\mathfrak{g}^\sharp$.

Theorem 2.13 (Projective G-representations and linear $\mathfrak{g}^\sharp$-representations).

(a) *Every smooth projective unitary representation $(\bar\rho,\mathcal{H})$ of G gives rise to a central extension $\mathbb{T} \to G^\sharp \to G$ of locally convex Lie groups, and a smooth unitary representation $(\rho,\mathcal{H})$ of $G^\sharp$. In turn, this gives rise to the central extension $\mathbb{R} \to \mathfrak{g}^\sharp \to \mathfrak{g}$ of locally convex Lie algebras and the derived representation $d\rho\colon \mathfrak{g}^\sharp \to \mathrm{End}(\mathcal{H}^\infty)$ of $\mathfrak{g}^\sharp$ by essentially skew-adjoint operators.*

(b) *If G is connected, then $(\bar\rho,\mathcal{H})$ and $(\bar\rho',\mathcal{H}')$ are unitarily equivalent if and only if the derived Lie algebra representations $(d\rho,\mathcal{H}^\infty)$ and $(d\rho',\mathcal{H}'^\infty)$ are unitarily equivalent. This means that there exists an isomorphism $\varphi\colon \mathfrak{g}^\sharp \to \mathfrak{g}^{\sharp\prime}$ of central extensions and a unitary isomorphism $U\colon \mathcal{H} \to \mathcal{H}'$ such that*

$$U\mathcal{H}^\infty \subseteq \mathcal{H}^{\infty\prime}$$

and

$$d\rho'(\varphi(\xi)) \circ U = U \circ d\rho(\xi) \quad \text{for all } \xi \in \mathfrak{g}^\sharp.$$

2.4 Integration of projective representations

In this section we discuss the integrability of (projective) unitary representations of Banach–Lie algebras, based on the existence of analytic vectors. Here our main result is the integrability theorem for projective representations of Banach–Lie groups (Theorem 2.18) that we derive with the methods from [76].

Definition 2.14. Let $(\rho, \mathcal{D})$ be a representation of the topological Lie algebra $\mathfrak{g}$ on the pre-Hilbert space $\mathcal{D}$. We say that

(i) ρ is a *$*$-representation* if all operators $\rho(x)$, $x \in \mathfrak{g}$, are skew-symmetric,

(ii) ρ is *strongly continuous* if all the maps $\mathfrak{g} \to \mathcal{D}, x \mapsto \rho(x)\xi$ are continuous,

(iii) $\xi \in \mathcal{D}$ is an *analytic vector* if there exists a 0-neighborhood $U \subseteq \mathfrak{g}$ such that $\sum_{n=0}^{\infty} \frac{\|\rho(x)^n\xi\|}{n!} < \infty$ for every $x \in U$. The analytic vectors form a linear subspace $\mathcal{D}^\omega \subseteq \mathcal{D}$.

Remark 2.15. If $\mathfrak{g}$ is a Banach–Lie algebra, then [76, Proposition 4.10] implies that $\xi \in \mathcal{D}$ is an analytic vector if and only if it is an analytic vector for all operators $\rho(x)$, $x \in \mathfrak{g}$ in the sense that there exists an $s > 0$ such that

$$\sum_{n=0}^{\infty} \frac{s^n \|\rho(x)^n\xi\|}{n!} < \infty.$$

We shall need the following lemma that is not spelled out explicitly in [76].

Lemma 2.16. *Let $(\rho, \mathcal{D})$ be a strongly continuous $*$-representation of the Banach–Lie algebra $\mathfrak{g}$. Then, $\mathcal{D}^\omega$ is a $\rho(\mathfrak{g})$-invariant subspace.*

Proof. Following [76, Definition 3.2], we call a linear functional $\beta\colon U(\mathfrak{g}) \to \mathbb{C}$ on the enveloping algebra of $\mathfrak{g}$ an *analytic functional* if all n-linear maps

$$\mathfrak{g}^n \to \mathbb{C}, \quad (x_1, \ldots, x_n) \mapsto \beta(x_1 \cdots x_n)$$

are continuous and the series $\sum_{n=0}^{\infty} \frac{\beta(x^n)}{n!}$ converges for every x in a 0-neighborhood of $\mathfrak{g}$. According to [76, Proposition 6.3], a vector $\xi \in \mathcal{D}$ is analytic if and only if the functional $\beta_\xi(D) := \langle \xi, \rho(D)\xi \rangle$ is analytic, where $\rho\colon U(\mathfrak{g}) \to \mathrm{End}(\mathcal{D})$ denotes the extension of ρ to the enveloping algebra. For $\xi \in \mathcal{D}^\omega$ and $x \in \mathfrak{g}$, the functional

$$\beta_{\rho(x)\xi}(D) := \langle \rho(x)\xi, \rho(D)\rho(x)\xi \rangle = \beta_\xi((-x)Dx)$$

is also analytic by [76, Theorem 3.6], so that $\rho(x)\xi \in \mathcal{D}^\omega$ by [76, Proposition 6.3]. This shows that $\rho(\mathfrak{g})\mathcal{D}^\omega \subseteq \mathcal{D}^\omega$. ∎

To formulate the integrability theorem for projective representations, we first give a precise definition of a projective $*$-representation of a topological Lie algebra $\mathfrak{g}$.

Definition 2.17. Suppose that $\omega\colon \mathfrak{g}^2 \to \mathbb{R}$ is a continuous 2-cocycle and that

$$\mathfrak{g}^\sharp_\omega = \mathbb{R} \oplus_\omega \mathfrak{g}$$

is the corresponding central extension. Then, any $*$-representation $(\rho^\sharp, \mathcal{D})$ with

$$\rho^\sharp(1,0) = i\mathbf{1}$$

leads to a linear map

$$\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{D}), \quad \rho(x) := \rho^\sharp(0,x)$$

satisfying

$$[\rho(x), \rho(y)] = \rho([x,y]) + \omega(x,y)i\mathbf{1}.$$

We then call $(\rho, \mathcal{D})$ a *projective $*$-representation with cocycle ω.*

Theorem 2.18 (Integrability theorem for projective representations). *Let G be a 1-connected Banach–Lie group with Lie algebra $\mathfrak{g}$, and let $(\rho, \mathcal{D})$ be a projective strongly continuous $*$-representation of $\mathfrak{g}$ on the dense subspace $\mathcal{D}$ of the Hilbert space $\mathcal{H}$. If $\mathcal{D}$ contains a dense subspace of analytic vectors, then there exists a smooth projective unitary representation $\overline{\pi}\colon G \to \mathrm{PU}(\mathcal{H})$ on $\mathcal{H}$ with the property that $\overline{\pi}(\exp x) = q(e^{\overline{\rho(x)}})$ for $x \in \mathfrak{g}$, where $q\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ denotes the quotient map.*

Proof. We proceed as in the proof of [76, Theorem 6.8]. Using Lemma 2.16, we see that we may assume, without loss of generality, that $\mathcal{D} = \mathcal{D}^\omega$, so that $\mathcal{D}$ consists of analytic vectors. According to Nelson's theorem [86], the operators $\rho(x)$, $x \in \mathfrak{g}$, are essentially skew-adjoint, so that their closures generate unitary one-parameter groups. The same holds for the operators $\hat{\rho}(t,x)$, $(t,x) \in \mathfrak{g}^\sharp$. This leads to a map

$$\tilde{\pi}\colon \mathfrak{g}^\sharp \to \mathrm{U}(\mathcal{H}), \quad x \mapsto e^{\overline{\hat{\rho}(t,x)}} = e^{it} e^{\overline{\rho(x)}}.$$

From the proof of [76, Theorem 6.8], we immediately derive that

$$\tilde{\pi}((t,x) * (s,y)) = \tilde{\pi}(t,x)\tilde{\pi}(s,y)$$

holds for $(t,x), (s,y)$ in some open 0-neighborhood $U^\sharp \subseteq \mathfrak{g}^\sharp$. This implies that

$$q(e^{\overline{\rho(x*y)}}) = q(e^{\overline{\rho(x)}})q(e^{\overline{\rho(y)}})$$

for x, y in some open 0-neighborhood $U \subseteq \mathfrak{g}$. Now [12, Chapter 3, Section 6, Lemma 1.1] implies the existence of a unique homomorphism $\overline{\pi}\colon G \to \mathrm{PU}(\mathcal{H})$ such that $\overline{\pi}(\exp x) = q(e^{\overline{\rho(x)}})$ holds for all elements x in some 0-neighborhood of $\mathfrak{g}$.

That $\overline{\pi}$ is a smooth projective representation (Definition 2.5) follows from the analyticity of the orbit maps $G \to \mathbb{P}(\mathcal{H})$, $g \mapsto \overline{\pi}(g)[v]$ for $v \in \mathcal{D}^\omega$, which in turn follows from

$$\overline{\pi}(\exp x)[v] = [e^{\overline{\rho(x)}}v].$$

∎

2.5 Double extensions

Suppose that G is a locally convex Lie group and $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ a homomorphism defining a smooth $\mathbb{R}$-action on G. Then, the semidirect product $G \rtimes_\alpha \mathbb{R}$ is a Lie group with Lie algebra $\mathfrak{g} \rtimes_D \mathbb{R}$, where $D \in \mathrm{der}(\mathfrak{g})$ is the infinitesimal generator of the $\mathbb{R}$-action on $\mathfrak{g}$ induced by α.

If $\bar{\rho}\colon G \rtimes_\alpha \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ is a smooth projective unitary representation, then Theorem 2.13 yields a central extension

$$\mathbb{T} \to \widehat{G} := (G \rtimes_\alpha \mathbb{R})^\sharp \to G \rtimes_\alpha \mathbb{R}$$

with a smooth unitary representation ρ of $\widehat{G}$ on $\mathcal{H}$ that induces $\bar{\rho}$. From Theorem 2.13, we see that the restriction of $\bar{\rho}$ to $(G \rtimes_\alpha \mathbb{R})_0$ is determined up to unitary equivalence by the derived representation $\mathtt{d}\rho$ of the central extension $\widehat{\mathfrak{g}} = (\mathfrak{g} \rtimes_D \mathbb{R})^\sharp$. We identify $\mathfrak{g}$ with the linear subspace $\{0\} \times \mathfrak{g} \times \{0\}$ of $\widehat{\mathfrak{g}}$. We write

$$C := (1, 0, 0) \quad \text{and} \quad D := (0, 0, 1)$$

for the central element and derivation in $\mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R})$ respectively, so that

$$\widehat{\mathfrak{g}} = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D). \tag{2.2}$$

We trust that using the same symbol for the derivation $D \in \mathrm{der}(\mathfrak{g})$ and the Lie algebra element $D \in \widehat{\mathfrak{g}}$ that implements it will not lead to confusion. Note that in the representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}$, the central element C acts by $i\mathbf{1}$. Writing $(z, x, t) = zC + x + tD$, the bracket in $\widehat{\mathfrak{g}}$ takes the form

$$\begin{aligned}&[zC + x + tD, z'C + x' + t'D] \\ &\qquad = \big(\omega(x, x') + t\omega(D, x') - t'\omega(D, x)\big)C + ([x, x'] + tDx' - t'Dx).\end{aligned}$$