Chapter 3

Positive energy representations

In this chapter we introduce positive energy representations and some tools to handle them. In Section 3.1 we give the precise definition on both the linear and the projective level, and in Section 3.2 we define equivariant positive energy representations. In Section 3.3, we use the Borchers–Arveson theorem to reduce the classification of positive energy representations to the so-called *minimal* ones. Finally, in Section 3.4 we describe the key tool of this memoir in a first general form: the Cauchy–Schwarz estimates for projective positive energy representations. Here we will discuss them for general groups, but they will be refined in the context of gauge algebras in Section 5.3 below.

3.1 Positive energy representations

Let G be a locally convex Lie group with Lie algebra $\mathfrak{g}$ and let $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism defining a smooth $\mathbb{R}$-action on G. Then, it also induces a smooth action $\alpha^{\mathfrak{g}}$ on $\mathfrak{g}$ and we write $D \in \mathrm{der}(\mathfrak{g})$ for its infinitesimal generator

$$Dx := \frac{d}{dt}\Big|_{t=0} \alpha_t^{\mathfrak{g}}(x) \quad \text{for } x \in \mathfrak{g}.$$

In this section, we investigate smooth projective unitary representations of G that extend to projective *positive energy* representations of $G \rtimes_\alpha \mathbb{R}$.

Definition 3.1 (Projective positive energy representations). A smooth, projective, unitary representation $\bar{\rho}\colon G \rtimes_\alpha \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ is called a *positive energy representation* if one (hence any) strongly continuous homomorphic lift $U\colon \mathbb{R} \to \mathrm{U}(\mathcal{H})$ of $\overline{U}\colon \mathbb{R} \to \mathrm{PU}(\mathcal{H}), t \mapsto \bar{\rho}(\mathbf{1}, t)$ has a generator

$$H := i\frac{d}{dt}\Big|_{t=0} U_t$$

whose spectrum is bounded below. We then call H a *Hamiltonian* and note that $U_t = e^{-itH}$ holds in the sense of functional calculus.

Remark 3.2. By adding a constant, we can always choose a Hamiltonian H that satisfies $\mathrm{Spec}(H) \subseteq [0, \infty)$.

We have seen in Section 2.5 that every smooth projective unitary representation $\bar{\rho}$ of $G \rtimes_\alpha \mathbb{R}$ gives rise to a smooth linear representation $(\rho, \mathcal{H})$ of a locally convex

Lie group $\widehat{G} = (G \rtimes_\alpha \mathbb{R})^\sharp$, a central $\mathbb{T}$-extension of $G \rtimes_\alpha \mathbb{R}$ with Lie algebra

$$\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R}) = \mathbb{R}C \oplus_\omega (\mathfrak{g} \rtimes \mathbb{R}D)$$

as in (2.2).

Definition 3.3 (Linear positive energy representations). Let $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$ be a smooth unitary representation of $\widehat{G}$. Then, ρ gives rise to a derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}$ on the space $\mathcal{H}^\infty$ of smooth vectors. We call

$$H := i\,\mathtt{d}\rho(D)$$

the *Hamiltonian* and we say that ρ is a *positive energy representation* if

$$\mathtt{d}\rho(C) = i\mathbf{1} \quad \text{and if } \operatorname{Spec}(H) \subseteq [0, \infty).$$

Remark 3.4. (a) If $\mathtt{d}\rho(C) = i\mathbf{1}$ and $\operatorname{Spec}(H) \subseteq [E_0, \infty)$ is bounded below, we can always replace D by $D + E_0 C$ to obtain a positive Hamiltonian. Note that this does not change the cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$, only the isomorphism between $\widehat{\mathfrak{g}}$ and $(\mathfrak{g} \rtimes_D \mathbb{R})^\sharp$.

(b) For a cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$, the relation

$$\omega(D, [\xi, \eta]) = \omega(D\xi, \eta) + \omega(\xi, D\eta)$$

shows that the linear functional $i_D\omega$ measures the non-invariance of the restriction of ω to $\mathfrak{g} \times \mathfrak{g}$ under the derivation D. It also shows that if the Lie algebra $\mathfrak{g}$ is perfect, then the linear functional $i_D\omega\colon \mathfrak{g} \to \mathbb{R}$ is completely determined by the restriction of ω to $\mathfrak{g} \times \mathfrak{g}$.

3.2 Equivariant positive energy representations

We will also need an equivariant version of positive energy representations. Let P be a Lie group with Lie algebra $\mathfrak{p}$ and let $\alpha\colon P \to \operatorname{Aut}(G)$ be a homomorphism defining a smooth P-action on G.

Definition 3.5 (Equivariant projective positive energy representations). A smooth, projective, unitary representation $\bar{\rho}\colon G \rtimes_\alpha P \to \mathrm{PU}(\mathcal{H})$ is called a *positive energy representation with respect to* $p \in \mathfrak{p}$ if the projective representation

$$\bar{\rho}_p\colon G \rtimes_{\alpha \circ \exp_p} \mathbb{R} \to \mathrm{PU}(\mathcal{H})$$

defined by

$$\bar{\rho}_p(g, t) := \bar{\rho}(g, \exp(pt))$$

is of positive energy in the sense of Definition 3.1. The *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$ is the set of all elements $p \in \mathfrak{p}$ for which $\bar{\rho}$ is a positive energy representation.

Note that $\mathcal{C}$ is an Ad_P-invariant cone. In particular, the representation $\bar{\rho}$ is of positive energy with respect to $p \in \mathfrak{p}$ if and only if it is of positive energy for all elements in the cone generated by the adjoint orbit $\mathrm{Ad}_P(p) \subseteq \mathfrak{p}$ of p.

The homomorphism $\alpha \colon P \to \mathrm{Aut}(G)$ can be twisted by an inner automorphism Ad_{g_0}, $g_0 \in G$, yielding

$$\alpha' = \mathrm{Ad}_{g_0} \alpha \mathrm{Ad}_{g_0}^{-1}.$$

Essentially, these inner twists do not affect the class of equivariant projective positive energy representations.

Proposition 3.6. *Let $(\bar{\rho}, \mathcal{H})$ be an equivariant projective positive energy representation of $G \rtimes_\alpha P$, and let*

$$\bar{U}_0 := \bar{\rho}(g_0).$$

Then

$$\bar{\rho}'(g, p) := \bar{U}_0 \bar{\rho}(\mathrm{Ad}_{g_0}^{-1}(g), p) \bar{U}_0^{-1}$$

is an equivariant projective positive energy representation of $G \rtimes_{\alpha'} P$ with the same restriction to G, and with the same positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Proof. To see that $\bar{\rho}'$ is a projective representation of $G \rtimes_{\alpha'} P$, one checks that the following is a commutative diagram of group homomorphisms:

$$\begin{array}{ccc} G \rtimes_\alpha P & \xrightarrow{\bar{\rho}} & \mathrm{PU}(\mathcal{H}) \\ {\scriptstyle (\mathrm{Ad}_{g_0}, \mathrm{Id}_P)} \downarrow & & \downarrow {\scriptstyle \mathrm{Ad}_{U_0}} \\ G \rtimes_{\alpha'} P & \xrightarrow{\bar{\rho}'} & \mathrm{PU}(\mathcal{H}). \end{array}$$

For the positive energy condition, note that any lift $t \mapsto V_t$ of $t \mapsto \bar{\rho}(\exp(tp))$ yields a lift $t \mapsto U_0 V_t U_0^{-1}$ of $t \mapsto \bar{\rho}'(\exp(tp))$ whose generator has the same spectrum. ∎

3.3 Minimal representations

The following refinement of the Borchers–Arveson theorem [13] will be used in the proof of Corollary 3.9 below.

Theorem 3.7. *Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra. Further, let $(U_t)_{t \in \mathbb{R}}$ be a strongly continuous unitary one-parameter group on $\mathcal{H}$ for which $\mathcal{M}$ is invariant under conjugation with the operators U_t, so that we obtain a one-parameter group $\alpha \colon \mathbb{R} \to \mathrm{Aut}(\mathcal{M})$ by*

$$\alpha_t(M) := \mathrm{Ad}(U_t)M := U_t M U_t^* \quad \textit{for } M \in \mathcal{M}.$$

If $U_t = e^{-itH}$ *with* $H \geq 0$*, then the following assertions hold:*

(i) *there exists a strongly continuous unitary one-parameter group* $(V_t)_{t\in\mathbb{R}}$ *in* $\mathcal{M}$ *with*

$$\mathrm{Ad}(V_t) = \alpha_t \quad \text{and} \quad V_t = e^{-itH_0}$$

with $H_0 \geq 0$*. It is uniquely determined by the requirement that it is minimal in the sense that, for any other one-parameter group* $(V_t')_{t\in\mathbb{R}}$ *with these properties, the central one-parameter group* $V_t'V_{-t} = e^{-itZ}$ *in* $\mathcal{M}$ *satisfies* $Z \geq 0$*,*

(ii) *if* $V_T = \mathbf{1}$ *for some* $T > 0$ *and* $\mathcal{F} \subseteq \mathcal{H}$ *is an* $\mathcal{M}$*-invariant subspace, then the subspace*

$$\mathcal{F}_0 := \{\xi \in \mathcal{F} : H_0\xi = 0\}$$

is $\mathcal{M}$*-generating in* $\mathcal{F}$*,*

(iii) *if* $\alpha_T = \mathrm{id}_{\mathcal{M}}$ *for some* $T > 0$*, then* $V_T = \mathbf{1}$*.*

Proof. (i) This is the Borchers–Arveson theorem (see [11, Theorem II.4.6]; also [13, Theorem 3.2.46] and [8] for a detailed discussion).

(ii) If $V_T = \mathbf{1}$, then $\mathrm{Spec}(H_0) \subseteq \frac{2\pi}{T}\mathbb{Z}$. In particular, H_0 is diagonalizable. If $\mathcal{F}_0$ is not $\mathcal{M}$-generating in $\mathcal{F}$, then

$$\mathcal{E} := (\mathcal{M}\mathcal{F}_0)^\perp \cap \mathcal{F}$$

is a non-zero $\mathcal{M}$-invariant subspace of $\mathcal{F}$ with

$$\inf \mathrm{Spec}(H_0|_{\mathcal{E}}) \geq \frac{2\pi}{T}.$$

As

$$\mathcal{H}_0 := \ker H_0 \subseteq \mathcal{E}^\perp,$$

we also have $\mathcal{M}\mathcal{H}_0 \subseteq \mathcal{E}^\perp$. Since $\mathcal{M}\mathcal{H}_0$ is invariant under $\mathcal{M}$ and $\mathcal{M}'$, the orthogonal projection Z onto

$$\mathcal{H}_1 := (\mathcal{M}\mathcal{H}_0)^\perp$$

is central in $\mathcal{M}$. On this subspace we have $\inf \mathrm{Spec}(H_0|_{\mathcal{H}_1}) \geq \frac{2\pi}{T}$, so that

$$H := H_0 - Z\frac{2\pi}{T} \geq 0,$$

contradicting minimality.

(iii) If $\alpha_T = \mathrm{id}_{\mathcal{M}}$, then V_T is contained in the center $\mathcal{Z}(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$ of $\mathcal{M}$. As $\mathcal{Z}(\mathcal{M})$ is a direct sum of L^∞-algebras, there exists a non-negative $Z \geq 0$ in $\mathcal{Z}(\mathcal{M})$ with $\mathrm{Spec}(Z) \subseteq [0, \frac{2\pi}{T}]$ and $V_T = e^{iTZ}$. Now

$$V_t' := e^{-it(H_0+Z)} = V_t e^{-itZ}$$

also has a non-negative generator $H_1 := H_0 + Z$ and satisfies $V'_T = \mathbf{1}$. In particular,

$$\mathrm{Spec}(H_1) \subseteq \frac{2\pi}{T}\mathbb{Z}.$$

We claim that the minimality of V implies that, for every $\varepsilon > 0$, the central support of the spectral projection $P := P^{H_0}[0, \varepsilon]$ of H_0 in $\mathcal{M}$ equals $\mathbf{1}$. To see this, note that the central support Q of P is the orthogonal projection onto the closed subspace generated by $\mathcal{M}P\mathcal{H}$. If this subspace is proper, then the restriction H_1 of H_0 to $\mathcal{H}_1 := (\mathbf{1} - Q)\mathcal{H}$ satisfies $H_1 \geq \varepsilon\mathbf{1}$, so that

$$H' := H_0 - \varepsilon(\mathbf{1} - Q) \geq 0.$$

The minimality of H_0 now yields $\mathbf{1} = Q$.

We now show that $\mathrm{Spec}(Z) \subseteq \{0, \frac{2\pi}{T}\}$, which implies that

$$V_T = V'_T e^{-iTZ} = V'_T = \mathbf{1}.$$

Assume that this is not the case. Then, there exists a non-zero spectral value $0 < a < \frac{2\pi}{T}$ of Z. Let $\varepsilon > 0$ be such that $0 < a - 2\varepsilon < a + 2\varepsilon < \frac{2\pi}{T}$ and consider the spectral projection $Q := P^Z([a - \varepsilon, a + \varepsilon])$ for Z, which is contained in $\mathcal{Z}(\mathcal{M})$. Since the central support of $P^{H_0}[0, \varepsilon]$ is $\mathbf{1}$, we have $QP^{H_0}([0, \varepsilon]) \neq 0$, so that

$$\mathrm{Spec}(QH_0) \cap [0, \varepsilon] \neq \emptyset.$$

Since $\mathrm{Spec}(QZ) \subseteq [a - \varepsilon, a + \varepsilon]$, this leads to

$$\mathrm{Spec}((H_0 + Z)Q) \cap [a - \varepsilon, a + 2\varepsilon] \neq \emptyset.$$

This contradicts

$$\mathrm{Spec}((H_0 + Z)Q) = \mathrm{Spec}(H_1 Q) \subseteq \mathrm{Spec}(H_1) \subseteq \frac{2\pi}{T}\mathbb{Z}. \qquad \blacksquare$$

Using the Borchers–Arveson theorem, every smooth positive energy representation $(\rho, \mathcal{H})$ can be brought in the following standard form.

Definition 3.8 (Minimal representations). A positive energy representation $(\rho, \mathcal{H})$ of $\widehat{G}$ is called *minimal* if the 1-parameter group $U_t = \rho(\exp(tD))$ is minimal with respect to the von Neumann algebra $\rho(\widehat{G})''$.

Corollary 3.9. *Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$ and let $G^\sharp \subseteq \widehat{G}$ be the inverse image of the subgroup G of $G \rtimes_\alpha \mathbb{R}$, so that $\widehat{G} \cong G^\sharp \rtimes \mathbb{R}$. Then, there exists a unitary 1-parameter group $(W_t)_{t\in\mathbb{R}}$ in the commutant $\rho(\widehat{G})'$ such that the representation $\rho_0(g, t) := \rho(g, t)W_t^{-1}$ has the following properties:*

(i) $\rho_0(\widehat{G})'' = \rho(G^\sharp)''$,

(ii) *if ρ is irreducible, then so is $\rho|_{G^\sharp}$,*

(iii) *if $\alpha_T = \mathrm{id}_G$, then $\rho_0(\mathbf{1}, T) = \mathbf{1}$ and, for every closed $\rho(\widehat{G})$-invariant subspace $\mathcal{F} \subseteq \mathcal{H}$, the subspace $\mathcal{F}_0 := \{\xi \in \mathcal{F} : H_0\xi = 0\}$ is $\widehat{G}$-generating in $\mathcal{F}$,*

(iv) *ρ_0 is a smooth positive energy representation.*

Proof. (i) Theorem 3.7 implies that $U_t := \rho(\exp tD)$ can be written as $U_t = V_t W_t$, where $(V_t)_{t\in\mathbb{R}}$ is a continuous unitary one-parameter group in the von Neumann algebra $\mathcal{M} := \rho(G^\sharp)''$ and $W_t \in \rho(G^\sharp)'$.

(ii) If ρ is irreducible, then Schur's Lemma implies that $W_t \in \mathbb{T}\mathbf{1}$, hence that the restriction $\rho|_{G^\sharp}$ remains irreducible.

(iii) follows from Theorem 3.7 (iii) and (ii).

(iv) As $V_t = \rho_0(\mathbf{1}, t)$ has a positive generator, ρ_0 also is a positive energy representation. It remains to see that ρ_0 is smooth. Since $(W_t)_{t\in\mathbb{R}}$ lies in the commutant $\rho(\widehat{G})'$, all its spectral subspaces are invariant under $\widehat{G}$. Therefore, ρ is a direct sum of subrepresentations for which W is norm continuous. We may therefore assume, without loss of generality, that W is norm continuous. Then, we can consider W as a smooth representation of $\widehat{G}$ and therefore $\rho_0(g,t) = \rho(g,t)W_{-t}$ is a smooth representation of $\widehat{G}$. ■

In view of the factorization $\rho(g,t) = \rho_0(g,t)W_t$, we can adopt the point of view that we know all positive energy representations if we know the minimal ones. On the level of the irreducible representations, the only difference is a phase factor corresponding to the minimal energy level. In general, the ambiguity consists in unitary one-parameter groups of the commutant, and these can be classified in terms of spectral measures.

3.4 Cauchy–Schwarz estimates (general case)

We show that the requirement that a representation be of positive energy severely restricts the class of cocycles that may occur.

Let ρ be a positive energy representation of $\widehat{G}$. For a smooth unit vector $\psi \in \mathcal{H}^\infty$ the expectation values

$$\langle H\rangle_\psi := \langle \psi, H\psi\rangle \quad \text{and} \quad \langle i\,\mathrm{d}\rho(\xi)\rangle_\psi := \langle \psi, i\,\mathrm{d}\rho(\xi)\psi\rangle$$

of H and $\xi \in \mathfrak{g}$ are defined. The following is a non-commutative adaptation of [85, Theorem 2.8].

Lemma 3.10 (Cauchy–Schwarz estimate). *Let ρ be a positive energy representation of $\widehat{G}$, and let $\xi \in \mathfrak{g}$ be such that $[\xi, D\xi] = 0$. Then, for every unit vector $\psi \in \mathcal{H}^\infty$,* *we have*

$$\big(\langle i\,\mathrm{d}\rho(D\xi)\rangle_\psi + \omega(\xi, D)\big)^2 \leq 2\omega(\xi, D\xi)\langle H\rangle_\psi,$$

and further $\omega(\xi, D\xi) \geq 0$.

Proof. Since $H = i\,\mathrm{d}\rho(D)$ has non-negative spectrum, the expectation value of the energy in the state defined by $\exp(t\,\mathrm{d}\rho(\xi))\psi$ is non-negative for all $t \in \mathbb{R}$;

$$0 \leq \langle H\rangle_{\exp(t\,\mathrm{d}\rho(\xi))\psi} = \langle e^{-t\,\mathrm{ad}_{\mathrm{d}\rho(\xi)}} H\rangle_\psi. \tag{3.1}$$

Since $[\xi, D\xi] = 0$, the exponential series terminates at order 2,

$$\begin{aligned}\exp(-t\,\mathrm{ad}_{\mathrm{d}\rho(\xi)})(H) &= i\,\mathrm{d}\rho(e^{-t\,\mathrm{ad}_\xi} D)\\ &= i\,\mathrm{d}\rho\Big(D + tD\xi - t\omega(\xi, D)C - \frac{t^2}{2}\omega(\xi, D\xi)C\Big)\\ &= H + t\big(i\,\mathrm{d}\rho(D\xi) + \omega(\xi, D)\big) + \frac{t^2}{2}\omega(\xi, D\xi),\end{aligned} \tag{3.2}$$

so that substitution in (3.1) yields the inequality

$$0 \leq \langle H\rangle_\psi + t\big(\langle i\,\mathrm{d}\rho(D\xi)\rangle_\psi + \omega(\xi, D)\big) + \frac{t^2}{2}\omega(\xi, D\xi) \quad \text{for } t \in \mathbb{R}.$$

The proposition now follows from the simple observation that $at^2 + bt + c \geq 0$ for all $t \in \mathbb{R}$ is equivalent to $0 \leq a, c$ and $b^2 \leq 4ac$. ∎

The Cauchy–Schwarz estimate will play an important role in the rest of the memoir. We will use it mainly in situations where $\omega(D, \mathfrak{g}) = \{0\}$, so that the bilinear form $(\xi, \eta) \mapsto \omega(\xi, D\eta)$ is symmetric. This is the case for gauge algebras (cf. Remark 5.8), but also more generally for locally convex Lie algebras with an admissible derivation in the sense of [52, Definition 9.1, Proposition 9.10].

In Chapter 5 we use Lemma 3.10 to show that $(\xi, \eta) \mapsto \omega(\xi, D\eta)$ is a positive semidefinite form on the gauge algebra $\mathfrak{g}$, and that every cocycle coming from a positive energy representation can be represented by a *measure* (Theorem 5.7). In Chapter 6, we make extensive use of the bound on the expectation value $\langle i\,\mathrm{d}\rho(D\xi)\rangle_\psi$ in terms of the average energy $\langle H\rangle_\psi$ afforded by Lemma 3.10. In fact, we shall need such bounds also for Lie algebra elements which are not in the image of D. The following refinement of the Cauchy–Schwarz estimate was designed for this purpose.

We start out with a proposition on Lie algebras which are *Mackey complete*, in the sense that every smooth curve $\zeta\colon [0, 1] \to \mathfrak{g}$ has a weak integral $\int_0^1 \zeta(t)dt$ in $\mathfrak{g}$. For a Mackey complete Lie algebra $\mathfrak{g}$, the operator

$$\int_0^1 e^{s\,\mathrm{ad}_y} ds$$

on $\mathfrak{g}$ is denoted $\frac{e^{\mathrm{ad}_y} - \mathbf{1}}{\mathrm{ad}_y}$.

Proposition 3.11. *Let $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega \mathfrak{g}$ be a central extension of a Mackey complete Lie algebra $\mathfrak{g}$ of the Lie group G with exponential function* exp. *Then, the adjoint action* $\mathrm{Ad}^{\widehat{\mathfrak{g}}}$ *of G on $\widehat{\mathfrak{g}}$ satisfies*

$$\mathrm{Ad}^{\widehat{\mathfrak{g}}}_{\exp y}(z, x) = \left(z + \omega\left(y, \frac{e^{\mathrm{ad}_y} - \mathbf{1}}{\mathrm{ad}_y}(x)\right), e^{\mathrm{ad}_y} x\right).$$

Proof. This is verified by solving the ODE

$$\gamma'(t) = [(0, y), \gamma(t)] \quad \text{with } \gamma(0) = (z, x).$$

Writing $\gamma(t) = (\alpha(t), e^{t\,\mathrm{ad}_y}(x))$, it leads to $\alpha'(t) = \omega(y, e^{t\,\mathrm{ad}_y} x)$. ∎

Lemma 3.12 (Refined Cauchy–Schwarz estimate). *Let $\mathfrak{g}$ be a Mackey complete Lie algebra, and let ρ be a positive energy representation of $\widehat{G}$. Let $\xi, \eta \in \mathfrak{g}$ be such that $[\xi, D\xi] = 0$ and $[\eta, D\eta] = 0$. Then, for all $s \in \mathbb{R}$, we have*

$$\left(\left\langle i\, d\rho(e^{-s\,\mathrm{ad}_\eta} D\xi)\right\rangle_\psi + \omega(\xi, D) + \omega\left(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\right)\right)^2$$
$$\leq 2\omega(\xi, D\xi)\left(\langle H\rangle_\psi + s(\langle i\, d\rho(D\eta)\rangle_\psi + \omega(\eta, D)) + \frac{s^2}{2}\omega(\eta, D\eta)\right).$$

In particular, if $\omega(\xi, D) = 0$, $\omega(\eta, D) = 0$ and $\omega(\mathrm{ad}^n_{d\rho(\eta)}(D\xi), \eta) = 0$ for all $n \geq 0$, then

$$\left\langle i\, d\rho\left(e^{-s\,\mathrm{ad}_\eta} D\xi\right)\right\rangle^2_\psi \leq 2\omega(\xi, D\xi)\left(\langle H\rangle_\psi + s\langle i\, d\rho(D\eta)\rangle_\psi + \frac{s^2}{2}\omega(\eta, D\eta)\right).$$

Proof. We write $W_{s,t} := \exp(t\,\mathrm{d}\rho(\xi))\exp(s\,\mathrm{d}\rho(\eta))$, and exploit the fact that the operator $H_{s,t} := W^*_{s,t} H W_{s,t}$ has non-negative spectrum. Repeated use of (3.2) on

$$H_{s,t} = \exp(-s\,\mathrm{ad}_{\mathrm{d}\rho(\eta)})\big(\exp(-t\,\mathrm{ad}_{\mathrm{d}\rho(\xi)})H\big)$$

yields

$$H_{s,t} = A_0(s) + A_1(s)t + A_2 t^2$$

with

$$\begin{aligned} A_0(s) &= H + s(i\,\mathrm{d}\rho(D\eta) + \omega(\eta, D)\mathbf{1}) + \frac{s^2}{2}\omega(\eta, D\eta)\mathbf{1}, \\ A_1(s) &= \omega(\xi, D)\mathbf{1} + \exp(-s\,\mathrm{ad}_{\mathrm{d}\rho(\eta)})(i\,\mathrm{d}\rho(D\xi)), \\ A_2 &= \frac{1}{2}\omega(\xi, D\xi)\mathbf{1}. \end{aligned}$$

With the preceding proposition, we obtain for $\exp(-s\,\mathrm{ad}_{\mathrm{d}\rho(\eta)})(i\,\mathrm{d}\rho(D\xi))$ the expression

$$i\,\mathrm{d}\rho(e^{-s\,\mathrm{ad}_\eta} D\xi) + \omega\left(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\right)\mathbf{1},$$

and thus

$$A_1(s) = \omega(\xi, D)\mathbf{1} + i\,\mathrm{d}\rho\big(e^{-s\,\mathrm{ad}_\eta}(D\xi)\big) + \omega\bigg(\frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(D\xi), \eta\bigg)\mathbf{1}.$$

Consider the expectation value $\langle H_{s,t}\rangle_\psi \geq 0$. Setting

$$\alpha_0(s) := \langle A_0(s)\rangle_\psi, \quad \alpha_1(s) := \langle A_1(s)\rangle_\psi \quad \text{and} \quad \alpha_2 := \langle A_2\rangle_\psi,$$

we observe that

$$\langle H_{s,t}\rangle_\psi = \alpha_0(s) + \alpha_1(s)t + \alpha_2 t^2$$

is a non-negative polynomial in t of degree at most 2. From this, we obtain the inequality $\alpha_1(s)^2 \leq 4\alpha_2\alpha_0(s)$. This is the first inequality mentioned above, the second one is a direct consequence. ■