Chapter 4

Covariant extensions of gauge algebras

The results in the preceding chapter concerned the general level of Lie groups of the form $G \rtimes_\alpha \mathbb{R}$. Now we turn to the specifics of gauge groups. After introducing gauge groups and their Lie algebras in Section 4.1, we describe in Section 4.2 a procedure that provides a reduction from semisimple to simple structure Lie algebras, at the expense of replacing M by a finite covering manifold $\widehat{M}$. In Section 4.3, we recall the classification [51] of 2-cocycles for the extended gauge algebra $\mathfrak{g} \rtimes_D \mathbb{R}$.

4.1 Gauge groups and gauge algebras

Let $\mathcal{K} \to M$ be a smooth bundle of Lie groups, and let $\mathfrak{K} \to M$ be the corresponding Lie algebra bundle with fibers

$$\mathfrak{K}_x = \mathrm{Lie}(\mathcal{K}_x).$$

If M is connected, then the fibers $\mathcal{K}_x$ of $\mathcal{K} \to M$ are all isomorphic to a fixed structure group K, and the fibers $\mathfrak{K}_x$ of $\mathfrak{K}$ are isomorphic to its Lie algebra

$$\mathfrak{k} = \mathrm{Lie}(K).$$

Definition 4.1 (Gauge group). The *gauge group* is the group $\Gamma(M, \mathcal{K})$ of smooth sections of $\mathcal{K} \to M$, and the *compactly supported gauge group* is the group $\Gamma_c(M, \mathcal{K})$ of smooth compactly supported sections.

Definition 4.2 (Gauge algebra). We define the *gauge algebra* as the Fréchet–Lie algebra $\Gamma(M, \mathfrak{K})$ of smooth sections of $\mathfrak{K} \to M$, equipped with the pointwise Lie bracket. The *compactly supported gauge algebra* $\Gamma_c(M, \mathfrak{K})$ is the LF-Lie algebra of smooth compactly supported sections.

The compactly supported gauge group $\Gamma_c(M, \mathcal{K})$ is a locally convex Lie group, whose Lie algebra is the compactly supported gauge algebra $\Gamma_c(M, \mathfrak{K})$. It is locally exponential, with $\exp\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(M, \mathcal{K})$ given by pointwise exponentiation [51, Proposition 2.3].

Definition 4.3. In the following we write $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ for the simply connected covering group of the identity component $\Gamma_c(M, \mathcal{K})_0$ and

$$q_\Gamma\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(M, \mathcal{K})_0$$

for the covering map. Then, $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ has the same Lie algebra $\Gamma_c(M, \mathfrak{K})$ as the gauge group $\Gamma_c(M, \mathcal{K})$, and its exponential function Exp satisfies $q_\Gamma \circ \mathrm{Exp} = \exp$.

4.1.1 Gauge groups from principal fiber bundles

The motivating example of a gauge group is of course the group $\mathrm{Gau}(\Xi)$ of vertical automorphisms of a principal K-bundle $\pi\colon \Xi \to M$.

Definition 4.4. A *vertical automorphism* of a principal fiber bundle $\pi\colon \Xi \to M$ is a K-equivariant diffeomorphism $\alpha\colon \Xi \to \Xi$ such that $\pi \circ \alpha = \pi$. The group $\mathrm{Gau}(\Xi)$ of vertical automorphisms is called the *gauge group* of Ξ.

In order to interpret $\mathrm{Gau}(\Xi)$ as a gauge group in the sense of Definition 4.1, define the bundle of groups $\mathrm{Ad}(\Xi) \to M$ with typical fiber K by

$$\mathrm{Ad}(\Xi) := \Xi \times K/\sim,$$

where the relation $\sim$ is given by $(pk, h) \sim (p, khk^{-1})$ for $p \in \Xi$ and $k, h \in K$. We obtain an isomorphism

$$\mathrm{Gau}(\Xi) \simeq \Gamma(M, \mathrm{Ad}(\Xi))$$

by mapping the section $\sigma \in \Gamma(M, \mathrm{Ad}(\Xi))$ to the corresponding vertical automorphism $\alpha_\sigma \in \mathrm{Gau}(\Xi)$, defined by

$$\alpha_\sigma(p) = p \cdot k$$

if $\sigma(\pi(p))$ is the class of (p, k) in $\mathrm{Ad}(\Xi) = \Xi \times K/\sim$.

The bundle of Lie algebras associated to Ξ is the *adjoint bundle* $\mathrm{ad}(\Xi) \to M$, defined as the quotient

$$\mathrm{ad}(\Xi) := \Xi \times_{\mathrm{Ad}} \mathfrak{k}$$

of $\Xi \times \mathfrak{k}$ modulo the relation $(pk, X) \sim (p, \mathrm{Ad}_k(X))$ for $p \in \Xi$, $X \in \mathfrak{k}$ and $k \in K$. Here $\mathrm{Ad}_k \in \mathrm{Aut}(\mathfrak{k})$ is the Lie algebra automorphism induced by the group automorphism $h \mapsto khk^{-1}$.

The *compactly supported gauge group* $\mathrm{Gau}_c(\Xi) \subseteq \mathrm{Gau}(\Xi)$ is the group of vertical bundle automorphisms of Ξ that are trivial outside the preimage of some compact subset of M. Since it is isomorphic to $\Gamma_c(M, \mathrm{Ad}(\Xi))$, it is a locally convex Lie group with Lie algebra $\mathfrak{gau}_c(\Xi) = \Gamma_c(M, \mathrm{ad}(\Xi))$.

Remark 4.5. In applications to gauge theory on noncompact manifolds M, the relevant group $\mathscr{G}$ of gauge transformations may be smaller than $\mathrm{Gau}(\Xi)$ due to boundary conditions at infinity. One expects $\mathscr{G}$ to contain at least $\mathrm{Gau}_c(\Xi)$, or perhaps even some larger Lie group of gauge transformations specified by a decay condition at infinity (cf. [31, 110]). In Part II of this series of papers, we will focus on the case where $M = \mathbb{R}^d$ is Minkowski space, and $\mathscr{G} \subset \Gamma(\mathbb{R}^d, \mathrm{Ad}(\Xi))$ is the group of gauge transformations that extend continuously to the conformal completion of Minkowski space. If the extension of Ξ to the conformal completion is trivial, then $\mathscr{G}$ contains global as well as compactly supported gauge transformations.

4.1.2 Gauge groups and space-time symmetries

An *automorphism* of $\pi\colon \mathcal{K} \to M$ is a pair $(\gamma, \gamma_M) \in \mathrm{Diff}(\mathcal{K}) \times \mathrm{Diff}(M)$ with $\pi \circ \gamma = \gamma_M \circ \pi$, such that for each fiber $\mathcal{K}_x$, the map $\gamma|_{\mathcal{K}_x}\colon \mathcal{K}_x \to \mathcal{K}_{\gamma_M(x)}$ is a group homomorphism. Since γ_M is determined by γ, we will omit it from the notation. We denote the group of automorphisms of $\mathcal{K}$ by $\mathrm{Aut}(\mathcal{K})$.

Definition 4.6 (Geometric $\mathbb{R}$-actions). In the context of gauge groups, we will be interested in $\mathbb{R}$-actions $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathcal{K}))$ which are of *geometric* type, i.e., derived from a 1-parameter group $\gamma\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ by

$$\alpha_t(\sigma) := \gamma_{-t} \circ \sigma \circ \gamma_{M,t}.$$

The $\mathbb{R}$-action on $\Gamma(M, \mathcal{K})$ preserves the subgroup $\Gamma_c(M, \mathcal{K})_0$ on which it defines a smooth action. Moreover, it lifts to a smooth action on the simply connected covering group $\widetilde{\Gamma}_c(M, \mathcal{K})_0$ (cf. [62, Theorem VI.3]).

Remark 4.7. If $\mathcal{K}$ is of the form $\mathrm{Ad}(\Xi)$ for a principal fiber bundle $\Xi \to M$, then a 1-parameter group of automorphisms of Ξ induces a 1-parameter group of automorphisms of $\mathcal{K}$.

The 1-parameter group $\alpha\colon \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathcal{K}))$ of group automorphisms differentiates to a 1-parameter group $\alpha^{\mathfrak{g}}\colon \mathbb{R} \to \mathrm{Aut}(\Gamma(M, \mathfrak{K}))$ of Lie algebra automorphisms given by

$$\alpha^{\mathfrak{g}}_t(\xi) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \gamma_{-t} \circ e^{\varepsilon\xi} \circ \gamma_{M,t}.$$

The corresponding derivation $D := \frac{d}{dt}\big|_{t=0}\alpha^{\mathfrak{g}}_t$ of $\Gamma(M, \mathfrak{K})$ can be described in terms of the infinitesimal generator of γ,

$$\mathbf{v} := \frac{d}{dt}\bigg|_{t=0} \gamma_{-t} \in \mathcal{V}(\mathcal{K}).$$

We identify the element $\xi \in \Gamma(M, \mathfrak{K})$ with the vertical, fiberwise left invariant vector field $\Xi_\xi \in \mathcal{V}(\mathcal{K})$ defined by $\Xi_\xi(k_x) = \frac{d}{d\varepsilon}\big|_{\varepsilon=0} k_x e^{\varepsilon\xi(x)}$. Using the equality $[\mathbf{v}, \Xi_\xi] = \Xi_{D(\xi)}$, we write

$$D(\xi) = L_{\mathbf{v}}\xi.$$

For $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, the Lie algebra $\mathfrak{g} \rtimes_D \mathbb{R}$ then has the bracket

$$[\xi \oplus t, \xi' \oplus t'] = \big([\xi, \xi'] + (tL_{\mathbf{v}}\xi' - t'L_{\mathbf{v}}\xi)\big) \oplus 0. \tag{4.1}$$

Remark 4.8. Alternatively, we can consider $\gamma\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ as a smooth 1-parameter group of bisections of the gauge groupoid $\mathcal{G}(\mathcal{K}) \rightrightarrows M$, the Lie groupoid whose objects are points $x, y \in M$, and whose morphisms are Lie group isomorphisms $\mathcal{K}_x \to \mathcal{K}_y$. It gives rise to a smooth 1-parameter family $\dot{\gamma}$ of bisections of the Lie groupoid $\mathcal{G}(\mathfrak{K}) \rightrightarrows M$, whose morphisms from x to y are Lie algebra isomorphisms $\mathfrak{K}_x \to \mathfrak{K}_y$. Its generator $\mathbf{v} = -\frac{d}{dt}|_{t=0}\dot{\gamma}$ is thus a section of its Lie algebroid $\mathfrak{a}(\mathfrak{K}) \to M$, called the *Atiyah algebroid*. A section $\xi \in \Gamma(M, \mathfrak{K})$ can be considered as an element of $\Gamma(M, \mathfrak{der}(\mathfrak{K})) \subseteq \Gamma(M, \mathfrak{a}(\mathfrak{K}))$, and we interpret $L_{\mathbf{v}}\xi$ as the commutator $[\mathbf{v}, \xi]$ in $\Gamma(M, \mathfrak{a}(\mathfrak{K}))$. We will need this picture in Section 4.2, where the bundle of Lie groups is not available.

4.2 Reduction to simple structure algebras

In this memoir, we consider gauge algebras with a *semisimple* structure algebra $\mathfrak{k}$. The following theorem shows that, without further loss of generality, we may restrict attention to the case where $\mathfrak{k}$ is *simple*.

Theorem 4.9 (Reduction from semisimple to simple structure algebras). *If $\mathfrak{K} \to M$ is a smooth locally trivial bundle of Lie algebras with semisimple fibers, then there exists a finite cover $\widehat{M} \to M$ and a smooth locally trivial bundle of Lie algebras $\widehat{\mathfrak{K}} \to \widehat{M}$ with simple fibers such that there exist isomorphisms $\Gamma(M, \mathfrak{K}) \simeq \Gamma(\widehat{M}, \widehat{\mathfrak{K}})$ and $\Gamma_c(M, \mathfrak{K}) \simeq \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ of locally convex Lie algebras.*

This is proven in [51, Theorem 3.1]. In brief, one uses local trivializations of $\mathfrak{K} \to M$ to give a manifold structure to

$$\widehat{M} := \bigcup_{x \in M} \mathrm{Spec}(\mathfrak{K}_x),$$

where $\mathrm{Spec}(\mathfrak{K}_x)$ is the finite set of maximal ideals $I_x \subset \mathfrak{K}_x$. The bundle of Lie algebras is then defined by

$$\widehat{\mathfrak{K}} := \bigcup_{I_x \in \widehat{M}} \mathfrak{K}_x / I_x,$$

and one shows that the natural projection $\pi : \widehat{\mathfrak{K}} \to \widehat{M}$ is a locally trivial vector bundle. Note that the finite cover $\widehat{M} \to M$ is not necessarily connected, and that the isomorphism classes of the fibers of $\widehat{\mathfrak{K}} \to \widehat{M}$ are not necessarily the same over different connected components of $\widehat{M}$.

Remark 4.10. Since a smooth 1-parameter family of automorphisms of $\mathfrak{K} \to M$ acts naturally on the maximal ideals, we obtain a smooth action on the Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$. We denote the corresponding section of the Atiyah algebroid $\mathfrak{a}(\widehat{\mathfrak{K}}) \to \widehat{M}$ by $\widehat{\mathbf{v}} \in \Gamma(\widehat{M}, \mathfrak{a}(\widehat{\mathfrak{K}}))$, and we denote the corresponding vector field on $\widehat{M}$ by

$$\mathbf{v}_{\widehat{M}} := \pi_* \widehat{\mathbf{v}}.$$

Since $\widehat{\mathfrak{K}}$ has simple fibers, the Atiyah algebroid $\mathfrak{a}(\widehat{\mathfrak{K}})$ fits in the exact sequence

$$\widehat{\mathfrak{K}} \to \mathfrak{a}(\widehat{\mathfrak{K}}) \to T\widehat{M},$$

where the first map is given by the pointwise adjoint action, and the second by the anchor. Note that the action on $\widehat{M}$ is locally free or periodic if and only if the action on M is. In that case, the period on $\widehat{M}$ is a multiple of the period on M.

In many situations, the connected components of $\widehat{M}$ are diffeomorphic to M. However, non-trivial covers $\widehat{M} \to M$ do occur naturally, for example in connection to non-orientable 4-manifolds.

Example 4.11. If the fibers of $\mathfrak{K} \to M$ are simple, then $\widehat{M} = M$.

Example 4.12. If $\mathfrak{K} = M \times \mathfrak{k}$ is trivial, then $\widehat{M} = M \times \mathrm{Spec}(\mathfrak{k})$ and all connected components of $\widehat{M}$ are diffeomorphic to M.

Example 4.13. Suppose that M is connected, and that the typical fiber $\mathfrak{k}$ of $\mathfrak{K} \to M$ is a semisimple Lie algebra with r simple ideals that are mutually non-isomorphic. Then,

$$\widehat{M} = \bigsqcup_{i=1}^{r} M$$

is a disjoint union of copies of M.

Example 4.14 (Frame bundles of 4-manifolds). Let M be a 4-dimensional Riemannian manifold. Let $\Xi := \mathrm{OF}(M)$ be the principal $\mathrm{O}(4,\mathbb{R})$-bundle of orthogonal frames, and let $\mathfrak{K} = \mathrm{ad}(\Xi)$. Then, $K = \mathrm{O}(4,\mathbb{R})$ and $\mathfrak{k} = \mathfrak{so}(4,\mathbb{R})$ is isomorphic to

$$\mathfrak{su}_L(2,\mathbb{C}) \oplus \mathfrak{su}_R(2,\mathbb{C}).$$

The group $\pi_0(K)$ is of order 2, the non-trivial element acting by conjugation with $T = \mathrm{diag}(-1,1,1,1)$. Since this permutes the two simple ideals, the manifold $\widehat{M}$ is the orientable double cover of M. This is the disjoint union $\widehat{M} = M_L \sqcup M_R$ of two copies of M if M is orientable, and a connected twofold cover $\widehat{M} \to M$ if it is not.

4.3 Central extensions of gauge algebras

Let $\mathfrak{g}$ be the compactly supported gauge algebra $\Gamma_c(M,\mathfrak{K})$, where $\mathfrak{K} \to M$ is a Lie algebra bundle with simple fibers. In this section, we classify all possible central extensions of $\mathfrak{g} \rtimes_D \mathbb{R}$. This amounts to calculating the continuous second Lie algebra cohomology $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ with trivial coefficients. In Chapter 5, we will characterize those cocycles coming from a positive energy representation.

4.3.1 Universal invariant symmetric bilinear forms

Let $\mathfrak{k}$ be a finite-dimensional, simple real Lie algebra. Then, its automorphism group $\mathrm{Aut}(\mathfrak{k})$ is a closed subgroup of $\mathrm{GL}(\mathfrak{k})$, hence a Lie group with Lie algebra $\mathrm{der}(\mathfrak{k}) \simeq \mathfrak{k}$.

Since $\mathfrak{k}$ acts trivially on the space

$$V(\mathfrak{k}) := S^2(\mathfrak{k}) \big/ (\mathfrak{k} \cdot S^2(\mathfrak{k}))$$

of $\mathfrak{k}$-coinvariants of the twofold symmetric tensor power $S^2(\mathfrak{k})$, the $\mathrm{Aut}(\mathfrak{k})$-representation on $V(\mathfrak{k})$ factors through $\pi_0(\mathrm{Aut}(\mathfrak{k}))$. The *universal $\mathfrak{k}$-invariant symmetric bilinear form* is defined by

$$\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k}), \quad \kappa(x, y) := [x \otimes_s y] = \frac{1}{2}[x \otimes y + y \otimes x].$$

We associate to $\lambda \in V(\mathfrak{k})^*$ the $\mathbb{R}$-valued, $\mathrm{der}(\mathfrak{k})$-invariant, symmetric, bilinear form

$$\kappa_\lambda := \lambda \circ \kappa.$$

This correspondence is a bijection between $V(\mathfrak{k})^*$ and the space of $\mathrm{der}(\mathfrak{k})$-invariant symmetric bilinear forms on $\mathfrak{k}$.

Since $\mathfrak{k}$ is simple, we have $V(\mathfrak{k}) \simeq \mathbb{C}$ if $\mathfrak{k}$ admits a complex structure, and $V(\mathfrak{k}) \simeq \mathbb{R}$ if it does not (cf. [84, Appendix B]). In the latter case, $\mathfrak{k}$ is called *absolutely simple*. The universal invariant symmetric bilinear form can be identified with the Killing form of the real Lie algebra $\mathfrak{k}$ if $V(\mathfrak{k}) \simeq \mathbb{R}$ and with the Killing form of the underlying complex Lie algebra if $V(\mathfrak{k}) \simeq \mathbb{C}$. In particular, in the important special case that $\mathfrak{k}$ is a compact simple Lie algebra, a universal invariant bilinear form $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is the negative definite Killing form given by $\mathrm{tr}(\mathrm{ad}\, x\, \mathrm{ad}\, y)$. However, in the following, we shall always use the normalized invariant positive definite symmetric bilinear form κ that satisfies

$$\kappa(i\alpha^\vee, i\alpha^\vee) = 2 \tag{4.2}$$

for the coroots $\alpha^\vee$ corresponding to long roots in the root decomposition of $\mathfrak{k}_{\mathbb{C}}$ (cf. [68, 94] and Appendix A).

4.3.2 The flat bundle $\mathbb{V} = V(\mathfrak{K})$

If $\mathfrak{K} \to M$ is a bundle of Lie algebras with simple fibers, then we denote by $\mathbb{V} \to M$ the vector bundle with fibers $\mathbb{V}_x = V(\mathfrak{K}_x)$. It carries a canonical flat connection d, defined by

$$\mathrm{d}\kappa(\xi, \eta) := \kappa(d_\nabla \xi, \eta) + \kappa(\xi, d_\nabla \eta) \quad \text{for } \xi, \eta \in \Gamma(M, \mathfrak{K}),$$

where ∇ is a *Lie connection* on $\mathfrak{K}$, meaning that

$$d_\nabla[\xi, \eta] = [d_\nabla \xi, \eta] + [\xi, d_\nabla \eta] \quad \text{for all } \xi, \eta \in \Gamma(M, \mathfrak{K}).$$

Since the fibers are assumed to be simple, any two Lie connections differ by a $\mathfrak{K}$-valued 1-form, so that the preceding definition is independent of the choice of ∇ (cf. [53]).

Let $\mathfrak{k}_i$ be the fiber of $\mathfrak{K}$ over a connected component M_i of M. If $\mathfrak{k}_i$ is absolutely simple (hence, in particular, when $\mathfrak{k}$ is compact), we have $V(\mathfrak{k}_i) \simeq \mathbb{R}$, and $\pi_0(\mathrm{Aut}(\mathfrak{k}))$ acts trivially on $V(\mathfrak{k}_i)$. In this case, $\mathbb{V} \to M_i$ is the trivial line bundle $M_i \times \mathbb{R} \to M_i$.

If $\mathfrak{k}_i$ possesses a complex structure, then $V(\mathfrak{k}_i) \simeq \mathbb{C}$, and $\alpha \in \mathrm{Aut}(\mathfrak{k}_i)$ flips the complex structure on $\mathbb{C}$ if and only if it flips the complex structure on $\mathfrak{k}_i$. In this case, $\mathbb{V} \to M_i$ is a vector bundle of real rank 2.

Remark 4.15. In the context of positive energy representations, we will see in Theorem 6.2 below that $\mathfrak{k}$ must be compact, so that $\mathbb{V} \to M$ is the trivial real line bundle. Although we need to consider the *a priori* possibility of non-trivial bundles, then, it will become clear in the course of our analysis that they will not give rise to positive energy representations.

4.3.3 Classification of central extensions

We define 2-cocycles $\omega_{\lambda,\nabla}$ on $\mathfrak{g} \rtimes_D \mathbb{R}$ whose classes span the cohomology group $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$. They depend on a $\mathbb{V}$-*valued* 1-*current* $\lambda \in \Omega^1_c(M, \mathbb{V})'$, and on a *Lie connection* ∇ on $\mathfrak{K}$. A 1-current $\lambda \in \Omega^1_c(M, \mathbb{V})'$ is said to be

(L1) *closed* if $\lambda(\mathrm{d}C^\infty_c(M, \mathbb{V})) = 0$,

(L2) $\mathbf{v}_M$-*invariant* if $\lambda(L_{\mathbf{v}_M}\Omega^1_c(M, \mathbb{V})) = \{0\}$.

Given a closed, $\mathbf{v}_M$-invariant current $\lambda \in \Omega^1_c(M, \mathbb{V})'$, we define the 2-cocycle $\omega_{\lambda,\nabla}$ on $\mathfrak{g} \rtimes_D \mathbb{R}$ by skew-symmetry and the equations

$$\omega_{\lambda,\nabla}(\xi, \eta) = \lambda(\kappa(\xi, d_\nabla \eta)), \tag{4.3}$$

$$\omega_{\lambda,\nabla}(D, \xi) = \lambda(\kappa(L_{\mathbf{v}}\nabla, \xi)), \tag{4.4}$$

where we write ξ for $(\xi, 0) \in \mathfrak{g} \rtimes_D \mathbb{R}$ and D for $(0, 1) \in \mathfrak{g} \rtimes_D \mathbb{R}$ as in (2.2). We define the $\mathrm{der}(\mathfrak{K})$-valued 1-form $L_{\mathbf{v}}\nabla \in \Omega^1(M, \mathrm{der}(\mathfrak{K}))$ by

$$(L_{\mathbf{v}}\nabla)_w(\xi) = L_{\mathbf{v}}(d_\nabla \xi)_w - \nabla_w L_{\mathbf{v}}\xi = L_{\mathbf{v}}(\nabla_w \xi) - \nabla_w L_{\mathbf{v}}\xi - \nabla_{[\mathbf{v}_M, w]}\xi \tag{4.5}$$

for all $w \in \mathcal{V}(M)$, $\xi \in \Gamma(M, \mathfrak{K})$. Since the fibers of $\mathfrak{K} \to M$ are simple, all derivations are inner, so we can identify $L_{\mathbf{v}}\nabla$ with an element of $\Omega^1(M, \mathfrak{K})$. Using the formulae

$$\mathrm{d}\kappa(\xi, \eta) = \kappa(d_\nabla \xi, \eta) + \kappa(\xi, d_\nabla \eta), \tag{4.6}$$

$$L_{\mathbf{v}_M}\kappa(\xi, \eta) = \kappa(L_{\mathbf{v}}\xi, \eta) + \kappa(\xi, L_{\mathbf{v}}\eta), \tag{4.7}$$

$$L_{\mathbf{v}}(d_\nabla \xi) - d_\nabla L_{\mathbf{v}}\xi = [L_{\mathbf{v}}\nabla, \xi], \tag{4.8}$$

it is not difficult to check that $\omega_{\lambda,\nabla}$ is a cocycle. Skew-symmetry follows from (4.6) and (L1). The vanishing of $\delta\omega_{\lambda,\nabla}$ on $\mathfrak{g}$ follows from (4.6), the derivation property of ∇ and invariance of κ. Finally, $i_D\delta\omega_{\lambda,\nabla} = 0$ follows from skew-symmetry, (4.8), (4.7), (L2) and the invariance of κ.

Note that the class $[\omega_{\lambda,\nabla}]$ in $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ depends only on λ, not on ∇. Indeed, two connection 1-forms ∇ and ∇' differ by $A \in \Omega^1(M, \mathrm{der}(\mathfrak{K}))$. Using $\mathrm{der}(\mathfrak{K}) \simeq \mathfrak{K}$, we find

$$\omega_{\lambda,\nabla'} - \omega_{\lambda,\nabla} = \delta\chi_A \quad \text{with } \chi_A(\xi \oplus t) := \lambda(\kappa(A, \xi)).$$

According to the following theorem, every continuous Lie algebra 2-cocycle on $\mathfrak{g} \rtimes_D \mathbb{R}$ is cohomologous to one of the type $\omega_{\lambda,\nabla}$ as defined in (4.3) and (4.4).

Theorem 4.16 (Central extensions of extended gauge algebras). *Let $\mathcal{K} \to M$ be a bundle of Lie groups with simple fibers, equipped with a 1-parameter group of automorphisms with generator $\mathbf{v} \in \mathcal{V}(\mathcal{K})$. Let $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ be the compactly supported gauge algebra, and let $\mathfrak{g} \rtimes_D \mathbb{R}$ be the Lie algebra* (4.1). *Then, the map $\lambda \mapsto [\omega_{\lambda,\nabla}]$ induces an isomorphism*

$$\left(\Omega^1_c(M, \mathbb{V}) / (d\Omega^0_c(M, \mathbb{V}) + L_{\mathbf{v}_M} \Omega^1_c(M, \mathbb{V}))\right)' \xrightarrow{\sim} H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$$

between the space of closed, $\mathbf{v}_M$-invariant $\mathbb{V}$-valued currents and $H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$.

This is proven in [51, Theorem 5.3]. The proof relies heavily on the description of $H^2(\mathfrak{g}, \mathbb{R})$ provided in [53, Proposition 1.1].

Remark 4.17 (Temporal gauge). If the Lie connection ∇ on $\mathfrak{K}$ can be chosen so as to make $\mathbf{v} \in \mathcal{V}(\mathcal{K})$ horizontal, $\nabla_{\mathbf{v}_M}\xi = L_{\mathbf{v}}\xi$ for all $\xi \in \Gamma(M, \mathfrak{K})$, then equation (4.5) shows that $L_{\mathbf{v}}\nabla = i_{\mathbf{v}_M} R$, where R is the curvature of ∇. For such connections, (4.4) is equivalent to

$$\omega_{\lambda,\nabla}(D, \xi) = \lambda(\kappa(i_{\mathbf{v}_M} R, \xi)).$$