Chapter 5

Cocycles for positive energy representations

Having classified all the possible 2-cocycles on $\Gamma_c(M, \mathfrak{K}) \rtimes \mathbb{R}$, we now address the restrictions that are imposed on these cocycles by the Cauchy–Schwarz estimates from Section 3.4.

In Section 5.1 we derive a local normal form of the cocycle ω in a flow box around a point $m \in M$, where $\mathbf{v}_M \in \mathcal{V}(M)$ does not vanish. In Section 5.2, we use this to derive a global normal form for ω, provided that $\mathbf{v}_M$ is nowhere vanishing. It turns out that ω is characterized by a *measure* μ on the covering space $\widehat{M}$. In Section 5.3, we plug this information back into the Cauchy–Schwarz estimate. This yields the basic estimates needed for the continuity results in Chapter 6.

The setting of this chapter is as follows. As before, $\pi \colon \mathcal{K} \to M$ is a bundle of Lie groups with semisimple fibers, and $\mathfrak{K} \to M$ is the corresponding bundle of Lie algebras. We consider positive energy representations of $\widehat{G}$, where $G = \Gamma_c(M, \mathcal{K})$ is the compactly supported gauge group with Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$. In fact, we will work mainly at the Lie algebra level, so our results continue to hold for the slightly more general case that $G = \widetilde{\Gamma}_c(M, \mathcal{K})_0$ is the simply connected cover of the identity component. Using Section 4.2, we identify $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ with $\mathfrak{g} = \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with simple fibers over a covering space $\widehat{M}$ of M. We assume that the 1-parameter group of automorphisms is of geometric type in the sense of Definition 4.6. The analogs of the generators $\mathbf{v}_M \in \mathcal{V}(M)$ and $\mathbf{v} \in \Gamma(M, \mathfrak{a}(\mathfrak{K}))$ for $\widehat{\mathfrak{K}}$ are denoted by $\pi_* \widehat{\mathbf{v}} \in \mathcal{V}(\widehat{M})$ and $\widehat{\mathbf{v}} \in \Gamma(\widehat{M}, \mathfrak{a}(\widehat{\mathfrak{K}}))$.

5.1 Local gauge algebras

The following simple lemma will be used extensively throughout the rest of the memoir. It gives a normal form for the pair $(\Gamma_c(M, \mathfrak{K}), \mathbf{v})$ in the neighborhood of a point $m \in M$ where the vector field $\mathbf{v}_M$ does not vanish.

Definition 5.1 (Good flowbox). A *good flowbox* is a $\mathbf{v}$-equivariant, local trivialization $(I \times U_0) \times K \to \mathcal{K}$ of $\mathcal{K}$ over an open neighborhood $U \subseteq M$ that is equivariantly diffeomorphic to $I \times U_0$. Here $I \subseteq \mathbb{R}$ is a bounded open interval, and $U_0 \subseteq \mathbb{R}^{n-1}$ is open. Note that for $n = 1$, we may take $U_0 = \{0\}$.

In particular, we have coordinates $t := x_0$ for I and $\vec{x} := (x_1, \ldots, x_{n-1})$ for U_0 such that $\mathbf{v}_M \in \mathcal{V}(U)$ corresponds to $\partial_t \in \mathcal{V}(I \times U_0)$.

Lemma 5.2. *For any point $m \in M$ with $\mathbf{v}_M(m) \neq 0$, there exists a good flowbox $U \simeq I \times U_0$ containing m. Under the trivialization $U \times \mathfrak{k} \to \mathfrak{K}|_U$, the induced* *isomorphism* $C_c^\infty(U,\mathfrak{k}) \simeq \Gamma_c(U,\mathfrak{K})$ *yields an inclusion*

$$I_U\colon C_c^\infty(U,\mathfrak{k}) \rtimes_{\partial_t} \mathbb{R} \hookrightarrow \Gamma_c(M,\mathfrak{K}) \rtimes_D \mathbb{R}.$$

Proof. Since $\mathbf{v}_M(m) \neq 0$, we can find a neighborhood $U \subseteq M$ of m and local coordinates $t, x_1, \ldots, x_{n-1}$ such that the vector field $\mathbf{v}_M$ on U is of the form ∂_t. We may assume that $U \simeq I \times U_0$ where $U_0 \subseteq \mathbb{R}^{n-1}$ corresponds to $t = 0$ and $I \subseteq \mathbb{R}$ corresponds to $\vec{x} = 0$. We choose U_0 sufficiently small for there to exist a trivialization $\Phi\colon U_0 \times K \to \mathcal{K}|_{U_0}$, which we then extend to a trivialization $U \times K \simeq \mathcal{K}|_U$ over U by $(t,x,k) \mapsto \gamma(-t)\Phi(x,k)$. As $\frac{d}{dt}|_{t=0}\gamma(-t) = \mathbf{v}$, the vector field $\mathbf{v} \in \mathcal{V}(\mathcal{K})$ is horizontal in this trivialization. ■

We consider $\mathfrak{g}_U := C_c^\infty(U,\mathfrak{k})$ as a subalgebra of $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$ and wish to study the restriction $\mathtt{d}\rho_U$ of the representation $\mathtt{d}\rho$ to the subalgebra

$$\widehat{\mathfrak{g}}_U := \mathbb{R} \oplus_\omega (\mathfrak{g}_U \rtimes_{\partial_t} \mathbb{R}).$$

Note that the subalgebra $\widehat{\mathfrak{g}}_U \hookrightarrow \widehat{\mathfrak{g}}$ does not correspond to a Lie subgroup of $\widehat{G}$ unless U is γ-invariant, so we cannot work at the level of Lie groups.

If $A \in \Omega^1(U,\mathfrak{k})$ is the local connection 1-form corresponding to the Lie connection ∇, then up to coboundaries, by (4.3) and (4.4) the restriction ω_U of ω to $\mathfrak{g}_U \rtimes_{\partial_t} \mathbb{R}$ takes the form

$$\omega_U(fX, gY) \simeq \lambda_U(\kappa(fX, \mathtt{d}g \cdot Y + g[A,Y])) \tag{5.1}$$
$$\omega_U(\partial_t, fX) \simeq \lambda_U(\kappa(\partial_t A, fX)), \tag{5.2}$$

for some $\lambda_U \in \Omega^1_c(U, V(\mathfrak{k}))'$, where $f, g \in C_c^\infty(U,\mathbb{R})$ and $X, Y \in \mathfrak{k}$.

Proposition 5.3. *Let $m \in M$ be a point with $\mathbf{v}_M(m) \neq 0$ and let $U \simeq I \times U_0$ be a good flowbox (cf. Definition 5.1). Let $\iota\colon U_0 \hookrightarrow M$ be the corresponding inclusion. Then, the map $\Omega^1_c(U, V(\mathfrak{k})) \to \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k}))$, $\beta \mapsto \overline{\beta}$, defined by the integration*

$$\overline{\beta}(x_1, \ldots, x_{n-1}) := \int_{-\infty}^{\infty} \beta(t, x_1, \ldots, x_{n-1})dt,$$

yields a split exact sequence

$$0 \to L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k})) \hookrightarrow \Omega^1_c(U, V(\mathfrak{k})) \to \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k})) \to 0$$

of locally convex spaces. In particular, $\lambda_U\colon \Omega^1_c(U, V(\mathfrak{k})) \to \mathbb{R}$ factors through a continuous linear map $\overline{\lambda}_{U_0}\colon \Gamma_c(U_0, \iota^*T^*M \otimes V(\mathfrak{k})) \to \mathbb{R}$.

Proof. The second statement follows from the first because

$$\lambda_U(L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k}))) = \{0\}$$

by Theorem 4.16. The kernel of $\beta \mapsto \bar{\beta}$ is precisely $L_{\partial_t}\Omega^1_c(U, V(\mathfrak{k}))$ by the fundamental theorem of calculus. A bump function $\varphi \in C^\infty_c(I, \mathbb{R})$ of integral 1 yields the required continuous right inverse $\Gamma_c(U_0, \iota^* T^* M \otimes V(\mathfrak{k})) \to \Omega^1_c(U, V(\mathfrak{k}))$ for the integration map by sending $\vec{x} \mapsto \beta(\vec{x})$ to $(t, \vec{x}) \mapsto \varphi(t)\beta(x_1, \dots, x_{n-1})$. ∎

For $X = Y$ we obtain with (5.1) the relation

$$\omega(\partial_t f X, f X) = \lambda_U\big(\partial_t f \cdot \mathtt{d} f \cdot \kappa(X, X)\big).$$

Unlike (5.1), which holds only modulo coboundaries, this equation is exact because $(\partial_t f)X$ and fX commute. Lemma 3.10 (the Cauchy–Schwarz estimate) then yields

$$-\lambda_U\big(\partial_t f \cdot \mathtt{d} f \cdot \kappa(X, X)\big) \geq 0. \tag{5.3}$$

This allows us to characterize λ_U as follows.

Proposition 5.4. *Let $m \in M$ be a point with $\mathbf{v}_M(m) \neq 0$. Then, there exists an open neighborhood $U \subseteq M$ of m such that, for each $X \in \mathfrak{k}$, there exists a unique $\mathbf{v}_M$-invariant positive locally finite regular Borel measure $\mu_{U,X}$ on U such that the functional $\lambda_{U,X} \in \Omega^1_c(U, \mathbb{R})'$ defined by $\lambda_{U,X}(\beta) := -\lambda_U(\beta \cdot \kappa(X, X))$ takes the form*

$$\lambda_{U,X}(\beta) = \int_U (i_{\mathbf{v}_M}\beta) d\mu_{U,X}(m).$$

Proof. Introduce coordinates $x_0 := t$ and $\vec{x} := (x_1, \dots, x_{n-1})$ on $U \simeq I \times U_0$ as in Definition 5.1. Define $\lambda_{U,i} \in C^\infty_c(U, \mathbb{R})'$, $i = 0, \dots, n-1$, by $\lambda_{U,i}(f) := \lambda_{U,X}(f\, \mathtt{d}x_i)$ and let $\lambda_i \in C^\infty_c(U_0, \mathbb{R})'$ be the corresponding distribution on U_0 (cf. Proposition 5.3), so

$$\lambda_{U,i}(f) = \lambda_i(\bar{f})$$

with

$$\bar{f}(\vec{x}) := \int_I f(t, \vec{x}) dt.$$

Then,

$$\lambda_{U,X}(f \mathtt{d} g) = \sum_{i=0}^{n-1} \lambda_i(\overline{f \partial_i g}) \quad \text{for all } f, g \in C^\infty_c(U, \mathbb{R}).$$

Equation (5.3) then yields

$$\lambda_0\big(\overline{(\partial_t f)^2}\big) + \sum_{i=1}^{n-1} \lambda_i\big(\overline{\partial_t f \partial_i f}\big) \geq 0. \tag{5.4}$$

First, we show that $\lambda_0(h^2) \geq 0$ for any h in $C^\infty_c(U_0, \mathbb{R})$. Note that every element B of $C^\infty_c(I, \mathbb{R})$ satisfies

$$\int_I B \partial_t B dt = 0.$$

We choose $B \neq 0$, normalize it by

$$\int_I (\partial_t B)^2 dt = 1$$

and define

$$f(t, \vec{x}) := B(t)h(\vec{x}).$$

We then have

$$\overline{(\partial_t f)^2} = h^2 \quad \text{and} \quad \overline{\partial_t f \partial_i f} = h\partial_i h \int_I B\partial_t B dt = 0 \quad \text{for } i \geq 1.$$

Therefore, (5.4) yields $\lambda_0(h^2) \geq 0$ as required.

Since λ_0 extends[1] to a positive linear functional on $C_c(U_0, \mathbb{R})$, Riesz' representation theorem [96, Theorems 2.14 and 2.18] yields a unique locally finite regular Borel measure μ_0 on U_0 such that $\lambda_0(f) = \int_{U_0} f d\mu_0(x)$. This implies

$$\lambda_{U,0}(f) = \int_U f(u) d\mu_{U,X}(u),$$

with $\mu_{U,X}$ the product of μ_0 with the Lebesgue measure on I.

To finish the proof, we now prove that $\lambda_i = 0$ for $i > 0$. It suffices to show that $\lambda_i(h^2) = 0$ for all $h \in C_c^\infty(U_0, \mathbb{R})$. Choose $B_C, B_S \in C_c^\infty(I, \mathbb{R})$ so that

$$\int_I B_S(t) B_C'(t) dt = 1,$$

choose $C, S \in C_c^\infty(U_0, \mathbb{R})$ so that

$$C(x) = \cos\left(\sum_{i=1}^n k_i x^i\right) \quad \text{and} \quad S(x) = \sin\left(\sum_{i=1}^n k_i x^i\right)$$

for $x \in \mathrm{supp}(h)$, $k_i \in \mathbb{Z}$, and set

$$f(t, \vec{x}) := h(\vec{x})\big(B_C(t)C(\vec{x}) + B_S(t)S(\vec{x})\big).$$

Then, with

$$E := \int_I \big(|B_C'(t)| + |B_S'(t)|\big)^2 dt,$$

we have

$$0 \leq \overline{(\partial_t f)^2} = h^2(\vec{x}) \int_I \big(B_C'(t)C(\vec{x}) + B_S'(t)S(\vec{x})\big)^2 dt \leq Eh^2(x).$$

[1]For every compact $S \subseteq U_0$, there exists a $\varphi \in C_c^\infty(U_0, \mathbb{R})$ with $\varphi|_S > 1$. With $L_S = \lambda_0(\varphi^2)$, it then follows from the inequality $\lambda_0(\|f\|_\infty \varphi^2 \pm f) \geq 0$ that $|\lambda_0(f)| \leq L_S \|f\|_\infty$ for all f with support in S.

Making repeated use of

$$\int_I F(t,\vec{x})\partial_t F(t,\vec{x})dt = 0 \quad \text{and} \quad \int_I B'_C B_S + B'_S B_C \, dt = 0,$$

we find, for $i = 1, \ldots, n-1$,

$$\overline{\partial_t f \partial_i f} = k_i h^2.$$

Equation (5.4) then yields

$$\lambda_0\big(\overline{(\partial_t f)^2}\big) + \sum_{i=1}^{n-1} k_i \lambda_i(h^2) \geq 0 \quad \text{for all } k_i \in \mathbb{Z}, \tag{5.5}$$

where the function f depends on the k_i. As $\lambda_0(\overline{(\partial_t f)^2}) \leq E\lambda_0(h^2)$, the non-negative term $\lambda_0(\overline{(\partial_t f)^2})$ is bounded by a number that does not depend on k_i. It therefore follows from inequality (5.5) that $\lambda_i(h^2) = 0$ for all $i > 0$, as was to be proven. ∎

5.2 Infinitesimally free $\mathbb{R}$-actions

In Section 4.2, we saw that $\Gamma_c(M, \mathfrak{K})$ is isomorphic to the gauge algebra $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with *simple* fibers over a cover $\widehat{M} \to M$. The decomposition $\widehat{M} = \bigsqcup_{i=1}^r \widehat{M_i}$ in connected components therefore gives rise to a direct sum decomposition

$$\Gamma_c(M, \mathfrak{K}) = \bigoplus_{i=1}^r \Gamma_c(\widehat{M}_i, \widehat{\mathfrak{K}}), \tag{5.6}$$

where $\widehat{\mathfrak{K}} \to \widehat{M}_i$ is a Lie algebra bundle with simple fibers isomorphic to $\mathfrak{k}_i$.

5.2.1 Reduction to compact simple structure algebras

If $\mathbf{v}_M$ is non-vanishing, then we can restrict attention to the terms in (5.6) where $\mathfrak{k}_i$ is a compact simple Lie algebra.

Corollary 5.5. *Suppose that $\mathfrak{k}_i$ is not compact, and let $m \in \widehat{M}_i$ be a point such that $\pi_*\widehat{\mathbf{v}}_m \neq 0$. Let $U \subseteq \widehat{M}_i$ be as in Proposition 5.4 and let $\lambda_U \in \Omega^1_c(U, V(\mathfrak{k}_i))'$ be as in (5.1) and (5.2). Then, $\lambda_U \colon \Omega^1_c(U, V(\mathfrak{k})) \to \mathbb{R}$ is zero. Consequently, ω_U is cohomologous to zero on $\Gamma_c(U, \widehat{\mathfrak{K}})$.*

Proof. It suffices to show that $\mu_{U,X} = 0$ for all $X \in \mathfrak{k}_i$. If $X, Y \in \mathfrak{k}_i$ with $\kappa(X,X) = -\kappa(Y,Y)$, then $\mu_{U,X} = -\mu_{U,Y}$ implies $\mu_{U,X} = \mu_{U,Y} = 0$. If $\mathfrak{k}_i$ is a complex Lie algebra, i.e., $V(\mathfrak{k}_i) \simeq \mathbb{C}$ (cf. Section 4.3.1), then the previous argument with $Y = iX$ yields $\mu_{U,X} = 0$ for all $X \in \mathfrak{k}_i$. If $V(\mathfrak{k}_i) \simeq \mathbb{R}$, then $\mathfrak{k}_i$ is noncompact if and only if $\{\kappa(X, X); X \in \mathfrak{k}_i\} = \mathbb{R}$. Therefore, the same reasoning applies. ∎

Corollary 5.6. *If ρ is a positive energy representation of $\widehat{G}$ and $\mathbf{v}_M$ has no zeros, then ω is cohomologous to a cocycle that vanishes on the subalgebras $\Gamma_c(\widehat{M}_i, \widehat{\mathfrak{K}})$, where $\mathfrak{k}_i$ is noncompact.*

Proof. By Theorem 4.16 applied to $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, the class $[\omega] \in H^2(\mathfrak{g} \rtimes_D \mathbb{R}, \mathbb{R})$ is uniquely determined by a $\mathbb{V}$-valued current $\lambda\colon \Omega_c(\widehat{M}, \mathbb{V}) \to \mathbb{R}$. Since $\mathbf{v}_M$ is everywhere non-zero, the same holds for $\pi_*\widehat{\mathbf{v}}$. If $\mathfrak{k}_i$ is noncompact, by Corollary 5.5, $\widehat{M}_i$ can be covered with open sets U_{ij} such that λ vanishes on $\Omega_c(U_{ij}, \mathbb{V})$. As every element of $\Omega_c(\widehat{M}_i, \mathbb{V})$ can be written as a finite sum of elements of $\Omega_c(U_{ij}, \mathbb{V})$, the current λ vanishes on $\Omega_c(\widehat{M}_i, \mathbb{V})$. ∎

5.2.2 Reduction of currents to measures

Let $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$ be a positive energy representation, where $G = \widetilde{\Gamma}_c(M, \mathcal{K})_0$ is the simply connected Lie group with Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, which covers the identity component of the compactly supported gauge group. This gives rise to a Lie algebra cocycle ω on $\mathfrak{g} \rtimes_D \mathbb{R}$. Using the results of Section 4.2, we identify the gauge Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ with $\mathfrak{g} = \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ is a Lie algebra bundle with simple fibers. The cocycle ω can then be represented by a *measure* on $\widehat{M}$.

Theorem 5.7. *Suppose that $\mathbf{v}_M$ has no zeros, and that ω is a 2-cocycle on $\mathfrak{g} \rtimes_D \mathbb{R}$ induced by a positive energy representation $\rho\colon \widehat{G} \to \mathrm{U}(\mathcal{H})$. Then, there exists a positive, regular, locally finite Borel measure μ on $\widehat{M}$ invariant under the flow $\gamma_{\widehat{M}}$ on $\widehat{M}$ induced by $\gamma_{\mathcal{K}}$, such that ω is cohomologous to the 2-cocycle $\omega_{\mu,\nabla}$, given by*

$$\omega_{\mu,\nabla}(\xi, \eta) = -\int_{\widehat{M}} \kappa(\xi, \nabla_{\widehat{\mathbf{v}_M}}\eta)\,d\mu(m), \tag{5.7}$$

$$\omega_{\mu,\nabla}(D, \xi) = -\int_{\widehat{M}} \kappa(i_{\widehat{\mathbf{v}_M}}(L_{\widehat{\mathbf{v}}}\nabla), \xi)\,d\mu(m) \quad \text{for } \xi, \eta \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}}). \tag{5.8}$$

The support of μ is contained in the union of the connected components $\widehat{M}_i$ where the fibers of $\widehat{\mathfrak{K}}$ are compact simple Lie algebras. In (5.7) and (5.8), we identify κ with the positive definite invariant bilinear form normalized as in (4.2).

Proof. As $\mathbf{v}_M$ is nowhere zero, we can cover $\widehat{M}$ by good flowboxes $U \subseteq \widehat{M}$ in the sense of Definition 5.1. In the corresponding local trivialization $\Gamma_c(U, \widehat{\mathfrak{K}}) \simeq C^\infty_c(U, \mathfrak{k})$ (cf. Lemma 5.2), we may assume that $\mathfrak{k}$ is compact by Corollary 5.5. We normalize κ as in (4.2) and define μ_U as $\mu_{U,X}$ for any $X \in \mathfrak{k}$ with $\kappa(X, X) = 1$. If U and U' are two such open sets, then the measures μ_U and $\mu_{U'}$ from Proposition 5.4

coincide on the intersection $U \cap U'$, as both measures are uniquely determined by the cocycle ω. The measures μ_U thus splice together to form a positive regular locally finite Borel measure on $\widehat{M}$. Equations (5.7) and (5.8) then follow immediately from (4.3), (4.4) in Section 4.3.3, and (5.1), (5.2). ∎

Remark 5.8. As the cohomology class $[\omega_{\lambda,\nabla}]$ is independent of the choice of the Lie connection, we are free to choose ∇ so that $\widehat{\mathbf{v}}$ is horizontal. In that case, we have

$$i_{\widehat{\mathbf{v}_M}}(L_{\widehat{\mathbf{v}}}\nabla) = 0 \quad \text{and} \quad L_{\widehat{\mathbf{v}}}\xi = \nabla_{\pi_*\widehat{\mathbf{v}}}\xi$$

(cf. Remark 4.17). Equation (5.8) then becomes

$$\omega_{\mu,\nabla}(D,\xi) = 0.$$

From Examples 4.11-4.14 in Section 4.2, we obtain the following.

Example 5.9. If $\mathfrak{K} \to M$ has simple fibers, then $\widehat{M} = M$. The class $[\omega_{\mu,\nabla}]$ then corresponds to a measure μ on M. It vanishes on the connected components of M where the fibers of $\mathfrak{K} \to M$ are noncompact.

Example 5.10. Suppose that M is connected, and that the typical fiber $\mathfrak{k} = \bigoplus_{i=1}^{r} \mathfrak{k}_i$ is the direct sum of r mutually non-isomorphic simple ideals $\mathfrak{k}_i$. Then, $\widehat{M}$ is the disjoint union of r copies of M. The class $[\omega_{\mu,\nabla}]$ is then given by r measures μ_i on M, one for each simple ideal. The same holds if $\mathfrak{K} = M \times \mathfrak{k}$ is trivial, and the $\mathfrak{k}_i$ are not necessarily non-isomorphic.

Example 5.11 (Frame bundles of 4-manifolds). (cf. Examples 4.14). Suppose that M is a Riemannian 4-manifold, and $\mathfrak{K} = \mathrm{ad}(\mathrm{OF}(M))$ is the adjoint bundle of its orthogonal frame bundle. If M is orientable, then $\omega_\mu = \omega_{\mu_L} + \omega_{\mu_R}$ is the sum of two cocycles with measures μ_L and μ_R on M corresponding to the simple factors $\mathfrak{su}_L(2,\mathbb{C})$ and $\mathfrak{su}_R(2,\mathbb{C})$ of $\mathfrak{so}(4,\mathbb{R})$. If M is not orientable, then ω_μ is described by a single measure μ on the orientable cover $\widehat{M} \to M$.

5.3 Cauchy–Schwarz estimates revisited

Using the explicit form of the cocycles determined in Theorem 5.7, we revisit the Cauchy–Schwarz estimates of Section 3.4. In this section, we assume that $\mathfrak{K} \to M$ has semisimple fibers, and that the vector field $\mathbf{v}_M$ on M is nowhere vanishing. As before, we identify $\Gamma_c(M,\mathfrak{K})$ with $\Gamma_c(\widehat{M},\widehat{\mathfrak{K}})$, where $\widehat{\mathfrak{K}} \to \widehat{M}$ has simple fibers.

Define the positive semidefinite symmetric bilinear form on $\mathfrak{g} = \Gamma_c(\widehat{M},\widehat{\mathfrak{K}})$ by

$$\langle \xi,\eta\rangle_\mu := \int_{\widehat{M}} \kappa(\xi,\eta)d\mu(m). \tag{5.9}$$

Using Theorem 5.7 and Remark 5.8, we may replace ω by $\omega_{\mu,\nabla}$ for a Lie connection ∇ on $\widehat{\mathfrak{K}}$ that makes $\widehat{\mathbf{v}}$ horizontal. In that case, we have $i_{\widehat{\mathbf{v}_M}}(L_{\widehat{\mathbf{v}}}\nabla) = 0$ and $L_{\widehat{\mathbf{v}}}\xi = \nabla_{\pi_*\widehat{\mathbf{v}}}\xi$ (cf. Remark 4.17). We may thus assume, without loss of generality, that the cocycle associated to a positive energy representation takes the form

$$\omega(\xi,\eta) = -\langle \xi, L_{\widehat{\mathbf{v}}}\eta\rangle_\mu = \langle L_{\widehat{\mathbf{v}}}\xi, \eta\rangle_\mu, \quad \omega(D,\xi) = 0. \tag{5.10}$$

The Cauchy–Schwarz estimate (Lemma 3.10) can now be reformulated as follows.

Lemma 5.12 (Cauchy–Schwarz Estimate). *Let ρ be a positive energy representation of $\widehat{G}$, where $G = \widetilde{\Gamma}_c(M,\mathcal{K})_0$ is the simply connected gauge group. If the vector field $\mathbf{v}_M$ on M has no zeros, then, after replacing the linear lift $d\rho\colon \mathfrak{g} \to \mathrm{End}(\mathcal{H}^\infty)$ of the projective representation $\overline{d\rho}$ of $\mathfrak{g}$ by $d\rho + i\chi\mathbf{1}$ for some continuous linear functional $\chi\colon \mathfrak{g} \to \mathbb{R}$, we have*

$$\langle i\, d\rho(L_{\widehat{\mathbf{v}}}\xi)\rangle_\psi^2 \leq 2\langle H\rangle_\psi \|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2 \quad \text{for all } \xi \in \mathfrak{g} \text{ with } [L_{\widehat{\mathbf{v}}}\xi, \xi] = 0 \tag{5.11}$$

and every unit vector $\psi \in \mathcal{H}^\infty$.

Proof. First we observe that the passage from ω to an equivalent cocycle corresponds to replacing the subspace $\mathfrak{g} \subseteq \widehat{\mathfrak{g}}$ by the subspace $\chi(\xi)C + \xi$, $\xi \in \mathfrak{g}$, where $\chi\colon \mathfrak{g} \to \mathbb{R}$ is a continuous linear functional. For the representation $d\rho$ this changes the value of $d\rho(\xi)$ by adding $i\chi(\xi)$, so that we can achieve a cocycle of the form (5.10) by Theorem 5.7. Now we apply Lemma 3.10 with $i_D\omega_{\mu,\nabla} = 0$ and $\omega_{\mu,\nabla}(\xi, D\xi) = \|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2$. ■

In the same vein, the refined Cauchy–Schwarz estimate, Lemma 3.12, can be reformulated as follows.

Lemma 5.13. *Under the assumptions of Lemma 5.12, we have*

$$\begin{aligned}\Big(\big\langle i\, d\rho(e^{-s\,\mathrm{ad}_\eta}(L_{\widehat{\mathbf{v}}}\xi))\big\rangle_\psi - \Big\langle \frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(L_{\widehat{\mathbf{v}}}\xi), L_{\widehat{\mathbf{v}}}\eta\Big\rangle_\mu\Big)^2 \\ \leq 2\|L_{\widehat{\mathbf{v}}}\xi\|_\mu^2\Big(\langle H\rangle_\psi + s\langle i\, d\rho(L_{\widehat{\mathbf{v}}}\eta)\rangle_\psi + \frac{s^2}{2}\|L_{\widehat{\mathbf{v}}}\eta\|_\mu^2\Big)\end{aligned} \tag{5.12}$$

for all $s \in \mathbb{R}$, and for all $\xi, \eta \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ such that $[\xi, L_{\widehat{\mathbf{v}}}\xi] = 0$ and $[\eta, L_{\widehat{\mathbf{v}}}\eta] = 0$.