Chapter 6

Continuity properties

Having determined which cocycles are compatible with the Cauchy–Schwarz estimates, we now turn to the classification of positive energy representations for the central extensions that correspond to these cocycles. This will be achieved in Chapter 7, using continuity and extension results developed in the present chapter.

In this chapter, we assume that the flow $\mathbf{v}_M$ is nowhere vanishing. Further, we assume that the fibers of $\mathfrak{K} \to M$ are *simple* Lie algebras. This entails no loss of generality compared to semisimple fibers, as one can switch to the Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$ in that case by the results in Section 4.2.

In Section 6.1, we use the Cauchy–Schwarz estimate 5.12 to further reduce the problem to the case where $\mathfrak{K}$ has *compact* simple fibers. In Section 6.2, we use the refined Cauchy–Schwarz estimate of Lemma 5.13 to bound $i\,\mathrm{d}\rho(\xi)$ in terms of the Hamilton operator H, the L^2-norm $\|\xi\|_\mu$ with respect to the measure μ of Theorem 5.7, and the L^2-norm $\|\xi\|_{B\mu}$ with respect to the product of μ with a suitable upper semi-continuous function $B\colon M \to \mathbb{R}^+$. In Section 6.3, we interpret these estimates as a continuity property, and use this to define an extension of $\mathrm{d}\rho$ to a space $H^1_{B\mu}(M, \mathfrak{K})$ of sections that are differentiable in the direction of the orbits, but merely measurable in the transversal direction. In Section 6.4, we construct a subspace $H^1_\partial(M, \mathfrak{K})$ of bounded sections that is closed under the pointwise Lie bracket. Finally, in Section 6.5, we show that by extending to $H^1_{B\mu}(M, \mathfrak{K})$ and restricting to $H^1_\partial(M, \mathfrak{K})$, one obtains a representation of the latter Lie algebra that is compatible with the Hamiltonian H. On a subalgebra $H^2_\partial(M, \mathfrak{K})$ of sections that are twice differentiable in the orbit direction, we then show that there is a dense set of uniformly analytic vectors. In Chapter 7, this will be needed in order to integrate the Lie algebra representation to the group level.

6.1 Reduction to compact simple structure algebras

As a direct consequence of Lemma 5.12, we see that $\mathrm{d}\rho(L_{\mathbf{v}}\xi)$ vanishes for all $\xi \in \mathfrak{g}$ with $[\xi, L_{\mathbf{v}}\xi] = 0$ and $\|L_{\mathbf{v}}\xi\|_\mu = 0$. We use this to show that every positive energy representation factors through a gauge algebra with *compact* structure algebra.

Proposition 6.1. *For* $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ *with* $\mathbf{v}_M$ *without zeros, we have*

$$\mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$

Considered as subsets of $\widehat{\mathfrak{g}} = \mathbb{R} \oplus_\omega (\mathfrak{g} \rtimes_D \mathbb{R})$*, with* ω *as in Theorem* 5.7*, we have*

$$\mathbb{R} \oplus_\omega \mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$

Proof. By a partition of unity argument, it suffices to prove this for $\mathfrak{g} = C_c^\infty(U, \mathfrak{k})$, where $U = I \times U_0$ is a good flowbox (cf. Definition 5.1) and

$$D\xi = \frac{d}{dt}\xi$$

(cf. Lemma 5.2). If $f \in C_c^\infty(U, \mathfrak{k})$ and $X \in \mathfrak{k}$, then fX lies in $D\mathfrak{g} \subseteq \mathfrak{g}$ if and only if

$$f_0(x) := \int_{-\infty}^{\infty} f(t,x)dt$$

is zero in $C_c^\infty(U_0, \mathbb{R})$. Fix $\zeta \in C_c^\infty(I, \mathbb{R})$ with $\int_{-\infty}^{\infty} \zeta(t)dt = 0$ and $\int_{-\infty}^{\infty} \zeta^2(t)dt = 1$. Then,

$$fX = (f - \zeta^2 f_0)X + \zeta^2 f_0 X \quad \text{with } (f - \zeta^2 f_0)X \in D\mathfrak{g}.$$

To show that $\zeta^2 f_0 X \in [D\mathfrak{g}, D\mathfrak{g}]$, choose $\chi \in C_c^\infty(U_0, \mathfrak{k})$ such that $\chi|_{\mathrm{supp}(f_0)} = 1$, and choose $Y_i, Z_i \in \mathfrak{k}$ such that $X = \sum_{i=1}^r [Y_i, Z_i]$. Since

$$\sum_{i=1}^r [\zeta f_0 Y_i, \zeta\chi Z_i] = \zeta^2 f_0 X$$

with $\zeta f_0 Y_i, \zeta\chi Z_i \in D\mathfrak{g}$, we have

$$fX = (f - \zeta^2 f_0)X + \sum_{i=1}^r [\zeta f_0 Y_i, \zeta\chi Z_i] \in D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]. \tag{6.1}$$

This holds for the Lie bracket in $\mathfrak{g}$ as well as for the Lie bracket in $\widehat{\mathfrak{g}}$. The relation

$$\int_{-\infty}^{\infty} \zeta \frac{d}{dt}\zeta dt = 0$$

implies $\omega(\zeta f_0 Y_i, \zeta\chi Z_i) = 0$. This shows that $\mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ in $\mathfrak{g}$ and also $\mathfrak{g} \subseteq D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ in $\widehat{\mathfrak{g}}$. Since ω is not identically zero on $D\mathfrak{g} \times D\mathfrak{g}$, the subspace $D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$ of $\widehat{\mathfrak{g}}$ cannot be proper and thus $\mathbb{R}C \subseteq D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}]$. This shows that

$$\widehat{\mathfrak{g}} = \mathbb{R}C + \mathfrak{g} = D\mathfrak{g} + [D\mathfrak{g}, D\mathfrak{g}].$$ ■

Theorem 6.2 (Reduction to compact structure algebra). *Let $M_i \subseteq M$ be a connected component such that the (simple) fibers of $\mathfrak{K}|_{M_i}$ are not compact. Suppose that $\mathbf{v}_M$ is non-vanishing on M_i. Then, after twisting by a functional $\chi \in \Gamma_c(M_i, \mathfrak{K})'$ if necessary, every positive energy representation $d\rho$ of $\Gamma_c(M, \mathfrak{K})$ vanishes on the ideal $\Gamma_c(M_i, \mathfrak{K})$.*

Proof. By a partition of unity argument, it suffices to consider the restriction of $d\rho$ to $C_c^\infty(U, \mathfrak{k})$ for a good flowbox $U \subseteq M_i$ (cf. Definition 5.1). Every $\xi \in DC_c^\infty(U, \mathfrak{k})$ is a finite sum of elements of the form $f'X$, with $f \in C_c^\infty(U,\mathbb{R})$ and $X \in \mathfrak{k}$. Since $\mathfrak{k}$ is noncompact, μ vanishes on M_i by Theorem 5.7. Since

$$\|f'X\|_\mu = 0 \quad \text{and} \quad [fX_i, f'X_i] = 0,$$

it follows from Lemma 5.12 that, after twisting by χ so as to change ω to $\omega_{\mu,\nabla}$, we have $\mathtt{d}\rho(f'X_i) = 0$. Since $\mathtt{d}\rho(DC_c^\infty(U,\mathfrak{k})) = \{0\}$, Proposition 6.1 yields

$$\mathtt{d}\rho(C_c^\infty(U,\mathfrak{k})) = \{0\}.$$

Thus, $\mathtt{d}\rho(\Gamma_c(M_i,\mathfrak{K})) = \{0\}$, as required. ∎

This shows that we can restrict attention to bundles $\mathfrak{K} \to M$ with *compact* simple fibers. (Note that the result requires a non-zero vector field on M, so this is compatible with the unitary highest weight representations of $C^\infty(\mathbb{S}^1, \mathfrak{su}_{1,n-1}(\mathbb{C}))$ studied in [47].) In conjunction with Proposition 6.1, Lemma 5.12 can also be used to prove the following.

Corollary 6.3. *If $\mathfrak{g} = \Gamma_c(M,\mathfrak{K})$, where $\mathfrak{K} \to M$ has compact simple fibers, then, after twisting by $\chi \in \Gamma_c(M,\mathfrak{K})'$ if necessary, every positive energy representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}$ vanishes on the ideal*

$$I_\mu := \big\{\xi \in \mathfrak{g}; \mu(\{x \in M; \xi(x) \neq 0\}) = 0\big\}$$

of sections that vanish μ-almost everywhere.

Proof. By a partition of unity argument, we may assume that $\mathfrak{g} = C_c^\infty(U,\mathfrak{k})$, with $U \subseteq M$ a good flowbox (Definition 5.1). Let $\xi \in I_\mu$ and consider the open subset $\mathcal{O}_\xi := \{x \in M; \xi(x) \neq 0\}$, which is the "open support" of ξ. Since ξ is a linear combination of terms fX with smaller or equal open support, we may assume that $\xi = fX$ for $f \in C_c^\infty(U,\mathbb{R})$ and $X \in \mathfrak{k}$. If $fX \in D\mathfrak{g}$, then $fX \in I_\mu$ implies $\|fX\|_\mu = 0$ and hence $\mathtt{d}\rho(fX) = 0$ by Lemma 5.12. Decompose fX as in equation (6.1),

$$fX = (f - \zeta^2 f_0)X + \sum_{i=1}^{r} [\zeta f_0 Y_i, \zeta\chi Z_i].$$

As $\mathcal{O}_\xi$ is open and $\mu = dt \otimes \mu_0$, we have $\mu(\mathcal{O}_\xi) = 0$ if and only if $\mu_0(p(\mathcal{O}_\xi))$ vanishes, where $p\colon U \to U_0$ is the projection on the orbit space. Now $(f - \zeta^2 f_0)X$ and $\zeta f_0 Y_i$ are in $D\mathfrak{g}$ and vanish outside $p^{-1}p(\mathcal{O}_\xi)$, so that their images under $\mathtt{d}\rho$ vanish. Indeed, as these are both of the form $L_{\mathbf{v}}\eta$ with $\|L_{\mathbf{v}}\eta\|_\mu = 0$ and $[L_{\mathbf{v}}\eta, \eta] = 0$, this follows from Lemma 5.12. We conclude that $\mathtt{d}\rho(fX) = 0$, as required. ∎

6.2 Extending the estimates from $D\mathfrak{g}$ to $\mathfrak{g}$

To see that $\mathtt{d}\rho$ factors through a linear map on $\mathfrak{g}/I_\mu$, we used the Cauchy–Schwarz estimate of Lemma 5.12. Using the *refined* Cauchy–Schwarz estimate of Lemma 5.13, we then extend $\mathrm{d}\rho$ to a linear map on $\overline{\mathfrak{g}/I_\mu}$, the L^2-completion of $\mathfrak{g}/I_\mu$ with respect to the measure μ.

Note that an extension to the subspace $\overline{D\mathfrak{g}/I_\mu} \subseteq \overline{\mathfrak{g}/I_\mu}$ can already be achieved using the "ordinary" Cauchy–Schwarz estimate of Lemma 5.12. Indeed, for $\xi \in D\mathfrak{g}$, one can use (5.11) to show that $\mathrm{d}\rho(\xi)$ satisfies an operator inequality of the form

$$\pm\, i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_\mu(\alpha\mathbf{1} + \beta H) \tag{6.2}$$

for certain constants $\alpha, \beta > 0$. With this, one can prove that $\mathrm{d}\rho\colon D\mathfrak{g}/I_\mu \to \mathrm{End}(\mathcal{H}^\infty)$ is weakly continuous with respect to the L^2-topology on $D\mathfrak{g}/I_\mu$, and that it extends to the L^2-completion $\overline{D\mathfrak{g}/I_\mu}$.

In order to extend $\mathrm{d}\rho$ to the full space $\overline{\mathfrak{g}/I_\mu}$, however, we will need an analog of (6.2) that holds not just for $\xi \in D\mathfrak{g}$, but for all $\xi \in \mathfrak{g}$. This is Proposition 6.16, which we prove using the refined Cauchy–Schwarz estimate of Lemma 5.13.

6.2.1 The local gauge algebra with fibers $\mathfrak{k} = \mathfrak{su}(2, \mathbb{C})$

First, we restrict our attention to the compact structure algebra $\mathfrak{k} = \mathfrak{su}(2, \mathbb{C})$. We will later derive the general case from this example. Let $\kappa(a, b) = -\mathrm{tr}(ab)$ be the invariant bilinear form on $\mathfrak{k}$, normalized so that elements x with

$$\mathrm{Spec}(\mathrm{ad}\, x) = \{0, \pm 2i\} \;\text{ satisfy }\; \kappa(x, x) = 2.$$

Further, let $U' \subset U$ be a *good pair of flowboxes* in the sense of the following definition. We write $U \Subset V$ if the closure of U is contained in an open subset of V.

Definition 6.4 (Good pair of flowboxes). Let $U' \simeq I' \times U_0'$ and $U \simeq I \times U_0$ be good flowboxes in the sense of Definition 5.1, and let $U' \subset U$. We call $U' \subset U$ a *good pair of flowboxes* if $I' \Subset I$ and $U_0' \Subset U_0$.

Remark 6.5. Note that $U_0' = U_0 = \{0\}$ is allowed! Unless specified otherwise, we assume that $I' = (-T'/2, T'/2)$ and $I = (-T/2, T/2)$ with $0 < T' < T < \infty$.

Remark 6.6. Recall that M is equipped with a flow-invariant measure μ, which takes the form $dt \otimes \mu_0$ on $I \times U_0$. To a good pair of flowboxes, we can therefore assign the number

$$\frac{T}{T - T'}\frac{\mu_0(U_0)}{T'} = \frac{\mu(U)}{T'(T - T')},$$

which will play a significant role throughout this chapter. If this is not too large, we think of the flowboxes as "sufficiently quadratic".

In view of Lemma 5.2, we restrict attention to the case where the Lie algebra is $\mathfrak{g} = C_c^\infty(I \times U_0, \mathfrak{k})$, and $\mathbf{v} = \partial_t$. For $z \in C_c^\infty(U', \mathbb{C})$, we define $\xi(z) \in \mathfrak{g}$ by

$$\xi(z)(t, u) := \begin{pmatrix} 0 & z(t, u) \\ -\bar{z}(t, u) & 0 \end{pmatrix} \tag{6.3}$$

and note that $[\xi, \frac{\partial}{\partial t}\xi] = 0$. We also consider the element $\eta \in \mathfrak{g}$ defined by

$$\eta(t,u) := \chi(u)\begin{pmatrix} i\tau(t) & 0 \\ 0 & -i\tau(t)\end{pmatrix}, \tag{6.4}$$

where $\tau \in C_c^\infty(I,\mathbb{R})$ and $\chi \in C_c^\infty(U_0,\mathbb{R})$ are such that $\tau(t) = t$ for $t \in I'$ and $\chi(u) = 1$ for $u \in U_0'$. It also satisfies

$$\left[\tau, \frac{\partial}{\partial t}\tau\right] = 0.$$

Thus, $\chi(u)\tau(t)=t$ on U', hence, in particular, on the support of every $z\in C_c^\infty(U',\mathbb{C})$.

On $C_c^\infty(\mathbb{R},\mathbb{C})$, we define the usual scalar product

$$\langle f, g\rangle_{dt} := \int_{-\infty}^{\infty} \overline{f(t)}g(t)dt$$

and the Fourier transform

$$\hat{f}(k) := \int_{-\infty}^{\infty} f(t)e^{-ikt}dt, \quad k \in \mathbb{R}.$$

For $z \in C_c^\infty(U,\mathbb{C})$, we will denote by $\hat{z}(k,u)$ the "parallel" Fourier transform, i.e., the Fourier transform of $t \mapsto z(t,u)$ evaluated at k.

We can choose τ such that $\|\tau'\|_{dt}^2$ is arbitrarily close to $\frac{TT'}{T-T'}$, and we can choose $0 \le \chi \le 1$ so that $\|\chi\|_{\mu_0}^2 \le \mu_0(U_0)$. Thus, $\|\chi\tau'\|_\mu^2$ can be chosen arbitrarily close to $\frac{TT'}{T-T'}\mu_0(U_0)$. Therefore, for every $\varepsilon > 0$, there exists an $\eta \in C_c^\infty(U,\mathbb{C})$ as in (6.4) satisfying

$$\|L_{\mathrm{v}}\eta\|_\mu^2 = 2\frac{TT'}{T-T'}\mu_0(U_0) + \varepsilon. \tag{6.5}$$

For η as in (6.4), Lemma 5.13 yields the following estimate.

Proposition 6.7. *Let $z \in C_c^\infty(U',\mathbb{C})$, and let $k \in \mathbb{R}$ be such that $\hat{z}(k,u) = 0$ for all $u \in U_0'$. Then, we have*

$$\langle i\, d\rho(\xi(z))\rangle_\psi^2 \le 4\|z\|_\mu^2\left(\langle H\rangle_\psi - \frac{1}{2}k\langle i\, d\rho(L_{\mathrm{v}}\eta)\rangle_\psi + \frac{1}{8}k^2\|L_{\mathrm{v}}\eta\|_\mu^2\right).$$

Proof. Since $[\xi(z),\xi(z)'] = 0$ and $[\eta,\eta'] = 0$, we may apply Lemma 5.13. First, we evaluate the left-hand side of inequality (5.12). Since $\tau(t)\chi(u) = t$ on $\mathrm{supp}(\xi(z))$, we have $\mathrm{ad}_\eta(\xi(z)) = \xi(2tiz)$. Since $L_{\mathrm{v}}\xi(z) = \xi(z')$, we have

$$e^{-s\,\mathrm{ad}_\eta}(L_{\mathrm{v}}\xi(z)) = \xi(z'e^{-2its}).$$

As $\kappa(\xi(z), L_{\mathrm{v}}\eta) = 0$ for all $z \in C_c^\infty(U',\mathbb{C})$, we have

$$\left\langle \frac{e^{-s\,\mathrm{ad}_\eta} - \mathbf{1}}{\mathrm{ad}_\eta}(L_{\mathrm{v}}\xi(z)), L_{\mathrm{v}}\eta\right\rangle_\mu = \left\langle \xi\left(\frac{e^{-2tis}-1}{2it}z'\right), L_{\mathrm{v}}\eta\right\rangle_\mu = 0.$$

On the right-hand side of inequality (5.12), we have $\|L_{\mathrm{v}}\xi(z)\|_\mu^2 = 2\|z'\|_\mu^2$. We thus obtain

$$\big\langle i\,\mathrm{d}\rho(\xi(z'e^{-2ist}))\big\rangle_\psi^2 \leq 4\|z'\|_\mu^2\left(\langle H\rangle_\psi + s\langle i\,\mathrm{d}\rho(L_{\mathrm{v}}\eta)\rangle_\psi + \frac{s^2}{2}\|L_{\mathrm{v}}\eta\|_\mu^2\right)$$

for all $s \in \mathbb{R}$ and $z \in C_c^\infty(U',\mathbb{C})$. Note that $w \in C_c^\infty(U',\mathbb{C})$ is of the form $w = z'e^{-2ist}$ for some $z \in C_c^\infty(U,\mathbb{C})$ if and only if the parallel Fourier transform $\widehat{w}(k,u)$ vanishes for $k = -2s$. Since in that case $\|w\|_\mu^2 = \|z'\|_\mu^2$, the proposition follows. ∎

We thus obtain a 1-parameter family of inequalities indexed by $k \in \mathbb{R}$, the case $k = 0$ reducing to the Cauchy–Schwarz estimate because $\widehat{z}(0,u) = 0$ is equivalent to $\xi(z) \in D\mathfrak{g}$. The idea of the following proposition is to lift the requirement that the Fourier transform vanish by showing that every $z \in C_c^\infty(U',\mathbb{C})$ can be written, in a controlled way, as the sum of two functions whose parallel Fourier transform vanishes for some $k \in \mathbb{R}$.

Proposition 6.8. *There exist $a, b \in \mathbb{R}$ such that, for all $z \in C_c^\infty(U',\mathbb{C})$ for which $U' = I' \times U_0'$ contained in $U = I \times U_0$, we have*

$$\big\langle i\,d\rho(\xi(z))\big\rangle_\psi^2 \leq (a + b\langle H\rangle_\psi)\|\xi(z)\|_\mu^2 \tag{6.6}$$

for constants a and b that depend on the interval lengths $T = |I|$ and $T' = |I'|$ and on $\mu_0(U_0)$, but not on z or ψ.

Proof. Let k be an arbitrary real number not equal to zero, and choose a function $\zeta \in C_c^\infty(I',\mathbb{C})$ with $\widehat{\zeta}(0) \neq 0$ and $\widehat{\zeta}(k) = 0$. (Such functions certainly exist. For instance, one can choose $\zeta(t) = \alpha'(t)e^{ikt}$ for some $\alpha \in C_c^\infty(I',\mathbb{R})$ with $\widehat{\zeta}(0) = \widehat{\alpha'}(-k) = -ik\widehat{\alpha}(-k) \neq 0$.) If we split z into $z = z_0 + z_k$ with

$$z_k(t,u) := \widehat{z}(0,u)\widehat{\zeta}(0)^{-1}\zeta(t) \quad \text{and} \quad z_0 := z - z_k,$$

then $\widehat{z}_0(0,u) = 0$ and $\widehat{z}_k(k,u) = 0$. We apply Proposition 6.7 separately to both terms on the right-hand side of

$$|\langle i\,\mathrm{d}\rho(\xi(z))\rangle_\psi| \leq |\langle i\,\mathrm{d}\rho(\xi(z_0))\rangle_\psi| + |\langle i\,\mathrm{d}\rho(\xi(z_k))\rangle_\psi|$$

to obtain

$$|\langle i\,\mathrm{d}\rho(\xi(z))\rangle_\psi| \leq 2\|z_0\|_\mu\sqrt{\langle H\rangle_\psi} + 2\|z_k\|_\mu\sqrt{\langle H\rangle_\psi + \frac{k^2}{4}\frac{TT'}{T-T'}\mu_0(U_0)}. \tag{6.7}$$

Indeed, the term $k\langle i\,\mathrm{d}\rho(L_{\mathrm{v}}\eta)\rangle_\psi$ can be assumed non-positive by switching k with $-k$ and ζ with $\bar{\zeta}$ if necessary. The term $\|L_{\mathrm{v}}\eta\|_\mu^2$ is then estimated by (6.5), and we take the limit $\varepsilon \downarrow 0$.

Since $|\widehat{z}(0,u)|^2 \leq T'\|z(\,\cdot\,,u)\|_{dt}^2$, we have $\|\widehat{z}(0,\,\cdot\,)\|_{\mu_0}^2 \leq T'\|z\|_\mu^2$. It follows that $\|z_k\|_\mu$ can be estimated in terms of $\|z\|_\mu$ as

$$\|z_k\|_\mu = \|\widehat{z}(\,\cdot\,,0)\|_{\mu_0}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt} \leq \sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\|z\|_\mu.$$

Similarly, $\|z_0\|_\mu$ can be estimated in terms of $\|z\|_\mu$ by means of

$$\|z_0\|_\mu \leq \|z\|_\mu + \|z_k\|_\mu$$

and the above estimate on $\|z_k\|_\mu$. Substituting this into (6.7), we derive the estimate

$$\langle i\,\mathrm{d}\rho(\xi(z))\rangle_\psi^2 \leq 4\|z\|_\mu^2\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2\bigg(\langle H\rangle_\psi + \frac{k^2}{4}\frac{TT'}{T-T'}\mu_0(U_0)\bigg). \tag{6.8}$$

Since $\|\xi(z)\|_\mu^2 = 2\|z\|_\mu^2$, equation (6.8) is equivalent to (6.6) with the constants

$$a := 2\bigg(\frac{k^2}{4}\frac{TT'}{T-T'}\mu_0(U_0)\bigg)\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2, \tag{6.9}$$

$$b := 2\big(1+2\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\big)^2. \tag{6.10}$$

This completes the proof. ■

For $\xi(z)$ of the form (6.3) in a gauge algebra $\mathfrak{g} = C_c^\infty(U',\mathfrak{k})$ with $\mathfrak{k} = \mathfrak{su}(2,\mathbb{C})$, we can now prove an operator inequality of the form (6.2).

Proposition 6.9. *There exist constants $a, b \in \mathbb{R}$, depending on T, T' and $\mu_0(U_0)$, such that for all α, β with $\alpha^2 \geq a$ and $2\alpha\beta \geq b$, we have*

$$\pm\, i\,\mathrm{d}\rho(\xi(z)) \leq \|\xi(z)\|_\mu(\alpha\mathbf{1} + \beta H) \quad \text{for } z \in C_c^\infty(U',\mathbb{C}) \tag{6.11}$$

as an inequality of unbounded operators on $\mathcal{H}$ with domain containing $\mathcal{H}^\infty$.

Proof. Note that the inequality (6.11) is equivalent to

$$\langle\psi, i\,\mathrm{d}\rho(\xi(z))\psi\rangle^2 \leq \|\xi(z)\|_\mu^2\langle\psi,(\alpha\mathbf{1}+\beta H)\psi\rangle^2 \quad \text{for all } \psi \in \mathcal{H}^\infty.$$

As $\beta^2\langle\psi, H\psi\rangle^2 \geq 0$, this follows from Proposition 6.8 under the above conditions on α and β. ■

Remark 6.10. The estimate (6.11) is rather crude for large energies, in the sense that one expects $\mathrm{d}\rho(\xi) \sim \sqrt{H}$, not $\mathrm{d}\rho(\xi) \sim H$.

It will be convenient to gain more control over the constants a and b in Proposition 6.8, and the constants α, β in Proposition 6.9. For this, we need to remove the dependence on ζ in (6.9) and (6.10).

Proposition 6.11. *The constants a and b in Proposition 6.8 can be chosen as*

$$a = \frac{T}{T - T'}\left(\frac{\mu_0(U_0)}{T'}\right)\nu^2 b, \tag{6.12}$$

with

$$b = 2\left(1 + \frac{2}{\sqrt{1 - (\sin(\nu)/\nu)^2}}\right)^2. \tag{6.13}$$

Here, $\nu > 0$ can be chosen at will.

Remark 6.12. It will be convenient to choose $\nu = \pi$. Then, b attains its minimal value $b = 18$, and $a = 18\pi^2 \frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}$.

Proof. In (6.9) and (6.10), we need to minimize the expression $\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}$ over all $\zeta \in C_c^\infty(I', \mathbb{C})$ with $\widehat{\zeta}(k) = 0$ and $\widehat{\zeta}(0) \neq 0$, where $k \in \mathbb{R}^\times$ is arbitrary. Since $\widehat{\zeta}(k) = \langle e^{ikt}, \zeta\rangle_{dt}$ and $\widehat{\zeta}(0) = \langle 1, \zeta\rangle_{dt}$, this amounts to maximizing

$$F(\zeta) := \left(\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt}\right)^{-1} = \frac{|\langle 1, \zeta\rangle_{dt}|}{\|1\|_{dt}\|\zeta\|_{dt}}.$$

Since F is continuous on $L^2(I') \setminus \{0\}$, and $C_c^\infty(I', \mathbb{C})$ is dense in $L^2(I')$, $F(\zeta_{\max})$ is maximal on the projection $\zeta_{\max}$ of 1 on the orthogonal complement of the function $e^{ikt} \in L^2(I')$. This is essentially a two-dimensional problem in the space spanned by

$$e_0 := \frac{1}{\sqrt{T'}}1 \quad \text{and} \quad e_k := \frac{1}{\sqrt{T'}}e^{ikt},$$

with

$$\langle e_0, e_0\rangle = \langle e_k, e_k\rangle = 1 \quad \text{and} \quad \langle e_0, e_k\rangle = \frac{\sin(kT'/2)}{kT'/2}. \tag{6.14}$$

It follows that $\zeta_{\max} = e_0 - \langle e_k, e_0\rangle e_k$, and $F(\zeta_{\max}) = \sqrt{1 - |\langle e_0, e_k\rangle|^2}$. Using (6.14), we find

$$F(\zeta_{\max}) = \sqrt{1 - \left(\frac{\sin(kT'/2)}{kT'/2}\right)^2}. \tag{6.15}$$

Equations (6.12) and (6.13) are now obtained from (6.9) and (6.10) with $k = 2\nu/T'$ by substituting the maximal value (6.15) for $F(\zeta) = (\sqrt{T'}\|\widehat{\zeta}(0)^{-1}\zeta\|_{dt})^{-1}$. ∎

6.2.2 The local gauge algebra with compact simple fibers

We now extend Proposition 6.8 to the case where $\mathfrak{k}$ is an arbitrary compact simple Lie algebra. With $I' \times U_0'$ and $I \times U_0$ a good pair of flowboxes (cf. Definition 6.4), we consider $\mathfrak{g}_{U'} := C_c^\infty(I' \times U_0', \mathfrak{k})$ and $\mathfrak{g}_U := C_c^\infty(I \times U_0, \mathfrak{k})$ as subalgebras of the Lie algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$.

Lemma 6.13. *Let* $\mathrm{d}\rho$ *be a positive energy representation of* $\widehat{\mathfrak{g}}$, *and let* $\eta > 0$. *Then, we have*

$$\pm\, i\, \mathrm{d}\rho(\xi) \leq \|\xi\|_{\mu}\big(K(\eta)\mathbf{1} + \eta H\big) \quad \textit{for all } \xi \in \mathfrak{g}_{U'}, \tag{6.16}$$

where $K(\eta)$ *is a constant independent of* ξ. *More precisely,*

$$K(\eta) = \max\left(9d_{\mathfrak{k}}/\eta, 3\pi\sqrt{2d_{\mathfrak{k}}\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right), \tag{6.17}$$

where $d_{\mathfrak{k}}$ *is the dimension of* $\mathfrak{k}$.

Proof. Using the root decomposition of $\mathfrak{k}_{\mathbb{C}}$ with respect to the complexification $\mathfrak{t}_{\mathbb{C}}$ of a maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$, one obtains a basis $(X_1, \ldots, X_{d_{\mathfrak{k}}})$ of $\mathfrak{k}$ with $\kappa(X_i, X_j) = 2\delta_{ij}$, where $-\kappa$ is the Killing form of $\mathfrak{k}$ and such that every X_j is contained in some $\mathfrak{su}(2,\mathbb{C})$-triple in $\mathfrak{k}$ [42, Proposition 6.45]. Every $\xi \in \mathfrak{g}_{U'}$ can then be written as $\xi = \sum_i f_i X_i$ for $f_i \in C_c^{\infty}(U_0' \times I', \mathbb{R})$. Since every X_i is contained in an $\mathfrak{su}(2,\mathbb{C})$-triple, we can apply Proposition 6.9 to $f_i X_i \in \mathfrak{g}_{U'}$ with $z = f_i$. We obtain

$$\pm i\,\mathrm{d}\rho(fX_i) \leq \|fX_i\|_{\mu}(\alpha\mathbf{1} + \beta H),$$

and thus

$$\pm i\,\mathrm{d}\rho(\xi) \leq \left(\sum_{i=1}^{d_{\mathfrak{k}}} \|fX_i\|_{\mu}\right)(\alpha\mathbf{1} + \beta H).$$

As the different terms $f_i X_i$ are orthogonal, we have $\sum_{i=1}^{d_{\mathfrak{k}}} \|f_i X_i\|_{\mu} \leq \sqrt{d_{\mathfrak{k}}}\|\xi\|_{\mu}$, and we obtain

$$\pm\, i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_{\mu}\big(\sqrt{d_{\mathfrak{k}}}\alpha\mathbf{1} + \sqrt{d_{\mathfrak{k}}}\beta H\big). \tag{6.18}$$

By Proposition 6.9, we are allowed to choose any α and β with $\alpha^2 \geq a$ and $2\alpha\beta \geq b$. Following Remark 6.12, we take $a = 18\pi^2\lambda(\mu_0(U_0)/T')$ and $b = 18$. The inequality (6.18) therefore holds for any value of $\beta > 0$ if we set

$$\alpha = \max\left(9/\beta, 3\pi\sqrt{2\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right). \tag{6.19}$$

Inequality (6.16) now follows from (6.18) with $\beta = \eta/\sqrt{d_{\mathfrak{k}}}$ and $K(\eta) = \sqrt{d_{\mathfrak{k}}}\alpha$. ■

Proposition 6.14. *For all* $\xi \in \mathfrak{g}_{U'}$ *and* $t > 0$, *the spectrum of* $tH \pm i\,\mathrm{d}\rho(\xi)$ *is bounded below. More precisely,*

$$-\max\left(9d_{\mathfrak{k}}\|\xi\|_{\mu}^2/t, 3\pi\|\xi\|_{\mu}\sqrt{2d_{\mathfrak{k}}\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}}\right) \leq \inf\big(\mathrm{Spec}(tH \pm i\,\mathrm{d}\rho(\xi))\big). \tag{6.20}$$

Proof. If $\|\xi\|_{\mu} = 0$, then $\mathrm{d}\rho(\xi) = 0$ by Corollary 6.3. In that case, (6.20) simply follows from the fact that H has non-negative spectrum. If $\|\xi\|_{\mu} \neq 0$, we apply Lemma 6.13 with $\eta = t/\|\xi\|_{\mu}$. ■

6.2.3 Global estimates and the bounding function

We need to derive suitable estimates of the type (6.16) globally, on the full Lie algebra $\Gamma_c(M, \mathfrak{K})$ rather than merely on $C_c^\infty(U, \mathfrak{k})$. In this section, we show how to do this for compact as well as noncompact manifolds M, under the assumption that $\mathbf{v}_M$ is nowhere vanishing.

For compact manifolds, we will derive an estimate of the form (6.16), albeit with a larger constant $K(\eta)$. In the noncompact case, however, the expression $\|\xi\|_\mu K(\eta)$ in (6.16) will have to be replaced by $\|\xi\|_{B_\varepsilon\mu}$, where $B_\varepsilon\colon M \to \mathbb{R}^+$ is a suitable upper semi-continuous function on M that is invariant under the flow, and $\|\xi\|_{B_\varepsilon\mu}$ is the L^2-norm of $\xi \in \Gamma_c(M, \mathfrak{K})$ with respect to the measure $B_\varepsilon\mu$,

$$\|\xi\|^2_{B_\varepsilon\mu} = \langle \xi, \xi\rangle_{B_\varepsilon\mu}, \quad \langle \xi, \eta\rangle_{B_\varepsilon\mu} = \int_M \kappa(\xi,\eta) B_\varepsilon(m) d\mu(m). \tag{6.21}$$

In this setting, we will prove that

$$\pm i\, \mathrm{d}\rho(\xi) \le \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \quad \text{for all } \xi \in \Gamma_c(M, \mathfrak{K}).$$

Note that, since $\mathbf{v}_M$ is nowhere vanishing on M, every $m \in M$ is contained in a good pair of flow boxes in the sense of Definition 6.4.

Definition 6.15. For $m \in M$, define $b(m)$ as the infimum of the set of numbers $\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'}$, ranging over all good pairs of flowboxes $U' \Subset U$ containing m.

Proposition 6.16. *The function $b\colon M \to \mathbb{R}^+$ is invariant under the flow $(\gamma_{M,t})_{t\in\mathbb{R}}$. Further, it is upper semi-continuous, hence, in particular, measurable.*

Proof. The invariance under the flow follows from the fact that $U' \subset U$ is a good pair of flow boxes around m if and only if $\gamma_{M,t}(U') \subset \gamma_{M,t}(U)$ is a good pair of flow boxes around $\gamma_{M,t}(m)$. For the upper semi-continuity, note that for every $\varepsilon > 0$, there is a good pair of flowboxes $U' \subset U$ around m such that

$$\frac{T}{T-T'}\frac{\mu_0(U_0)}{T'} < b(m) + \varepsilon.$$

For every m' in the open neighborhood U' of m, we thus have $b(m') \le b(m) + \varepsilon$. ∎

Theorem 6.17. *Let $d\rho$ be a positive energy representation of $\widehat{\mathfrak{g}}$, and let $\varepsilon > 0$. Then, we have*

$$\pm\, i\, d\rho(\xi) \le \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \quad \textit{for every } \xi \in \Gamma_c(M, \mathfrak{K}). \tag{6.22}$$

Here, $B_\varepsilon\colon M \to \mathbb{R}^+$ is the upper semi-continuous function defined by

$$B_\varepsilon(m) := \max\big(81 d_\mathfrak{k}^2(d_M+1)^4/\varepsilon^2, 18\pi^2 d_\mathfrak{k}(d_M+1)^2 b(m)\big), \tag{6.23}$$

with $b(m)$ as in Definition 6.15. It is invariant under the flow $(\gamma_{M,t})_{t\in\mathbb{R}}$.

Proof. Let d be a Riemannian metric on M for which M is complete, so that closed bounded subsets of M are compact by the Hopf–Rinow theorem. Let $V \subseteq M$ be the compact support of $\xi \in \Gamma_c(M, \mathfrak{K})$. Since b is upper semi-continuous, the functions $\beta_n : M \to \mathbb{R}^+$ defined by

$$\beta_n(m) := \sup\{b(m'); d(m, m') \leq 1/n\}$$

constitute a decreasing sequence converging pointwise to b as $n \to \infty$. We now show that the functions β_n are upper semi-continuous. To see this, note that, for every $m_0 \in M$ and every $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ with $b(m) < b(m_0) + \varepsilon/2$ for m in the closed ball $\overline{W}_{2/n}(m_0)$ with radius $2/n$ around m_0. Since this ball is compact, it contains finitely many m_i such that it is covered by open neighborhoods $\mathcal{O}_i$ of m_i such that $b(m) \leq b(m_i) + \varepsilon/2$ for all $m \in \mathcal{O}_i$. If $d(m, m_0) < \frac{1}{n}$, then $W_{1/n}(m) \subseteq \bigcup_i \mathcal{O}_i$, so that $\beta_n(m) < \beta_n(m_0) + \varepsilon$. In particular, β_n is measurable, and bounded on the compact set V.

For every $n \in \mathbb{N}$, choose a cover of V by finitely many open balls $W_{r_i}(m_i)$ of radius $r_i \leq 1/n$ around m_i, with the property that $W_{r_i}(m_i) \subseteq U' \Subset U$ for a good pair $U' \Subset U$ of flow boxes with $\frac{T}{T-T'} \frac{\mu_0(U_0)}{T} \leq b(m_i) + 1/n$. Since $b(m_i) \leq \beta_n(m)$ for all $m \in W_{r_i}(m_i)$, it follows that

$$\frac{T}{T-T'} \frac{\mu_0(U_0)}{T} \leq \beta_n(m) + 1/n \quad \text{for all } m \in W_{r_i}(m_i). \tag{6.24}$$

By the Brouwer–Lebesgue paving principle [44, Theorem V1], there exists a finite subcover $(W_j)_{j \in J}$ with the property that every point $m \in V$ is contained in at most $d_M + 1$ sets.

Let φ_j be a partition of unity with respect to $(W_j)_{j \in J}$. By Lemma 6.13, applied to $\eta := \varepsilon/(d_M + 1)$, we obtain $\pm i\,\mathrm{d}\rho(\varphi_j \xi) \leq \|\varphi_j \xi\|_\mu (K_j(\eta)\mathbf{1} + \eta H)$, where $K_j(\eta)$ is given by (6.17) for a good pair of flowboxes $U' \Subset U$ containing W_j. From (6.17) and (6.24), we find that

$$B_{n,\eta}(m) := \max\left((9d_{\mathfrak{k}}/\eta)^2, 18\pi^2 d_{\mathfrak{k}}(\beta_n(m) + 1/n)\right) \geq K_j(\eta)^2 \quad \text{for all } m \in W_j.$$

As $\|\varphi_j \xi\|_\mu K(\eta) \leq \|\varphi_j \xi\|_{B_{n,\eta}\mu}$, we have $\pm i\,\mathrm{d}\rho(\varphi_j \xi) \leq \|\varphi_j \xi\|_{B_{n,\eta}\mu}\mathbf{1} + \eta \|\varphi_j \xi\|_\mu H$ for all $j \in J$, and thus

$$\pm i\,\mathrm{d}\rho(\xi) \leq \Big(\sum_{j \in J} \|\varphi_j \xi\|_{B_{n,\eta}\mu}\Big)\mathbf{1} + \eta\Big(\sum_{j \in J} \|\varphi_j \xi\|_\mu\Big) H.$$

Since $\|(\varphi_j \xi)(m)\|_\kappa \leq \|\xi(m)\|_\kappa$, and since at most $d_M + 1$ of the values $\varphi_j(m)$ are non-zero, it follows that

$$\sum_{j \in J} \|\varphi_j \xi\|_\mu \leq (d_M + 1)\|\xi\|_\mu \quad \text{and} \quad \sum_{j \in J} \|\varphi_j \xi\|_{B_{n,\eta}\mu} \leq (d_M + 1)\|\xi\|_{B_{n,\eta}\mu},$$

so that

$$\pm i\,\mathrm{d}\rho(\xi) \leq (d_M + 1)(\|\xi\|_{B_{n,\eta}\mu}\mathbf{1} + \eta\|\xi\|_\mu H). \tag{6.25}$$

To obtain (6.22) from (6.25), recall that $\eta := \varepsilon/(d_M + 1)$. The second term on the right is thus $(d_M+1)\eta\|\xi\|_\mu = \varepsilon\|\xi\|_\mu$, as required. For the first term, note that $\beta_n + 1/n$ is a bounded, decreasing sequence converging pointwise to b on V. The bounded, decreasing sequence $(d_M + 1)^2 B_{n,\eta}(m)$ thus converges to $B_\varepsilon(m)$ in (6.22), where $\varepsilon = (d_M + 1)\eta$. By the dominated convergence theorem, we find that, for $n \to \infty$, the squared norm $((d_M + 1)\|\xi\|_{B_{n,\eta}})^2$ approaches

$$\int_V \|\xi\|_\kappa^2 (d_M + 1)^2 B_{n,\eta} d\mu(m) \to \int_V \|\xi\|_\kappa^2 B_\varepsilon d\mu(m) = \|\xi\|^2_{B_\varepsilon\mu}.$$

Since (6.25) holds for every $n \in \mathbb{N}$, the proposition follows. ■

Note that if the function $b\colon M \to \mathbb{R}^+$ of Definition 6.15 is bounded, then so is B_ε. If we define $K(\varepsilon)^2 := \|B_\varepsilon\|_\infty$, then we recover the inequality

$$\pm i\,\mathrm{d}\rho(\xi) \leq \|\xi\|_\mu(K(\varepsilon)\mathbf{1} + \varepsilon H), \tag{6.26}$$

since $\|\xi\|_{B_\varepsilon\mu} \leq K(\varepsilon)\|\xi\|_\mu$. This happens, in particular, if M is compact because the upper semi-continuous function B_ε is then automatically bounded.

Corollary 6.18. *Suppose that M is compact and $\mathbf{v}_M$ is nowhere vanishing on M. Then, for every $\varepsilon > 0$, there exists a constant $K(\varepsilon) > 0$ such that (6.26) holds for all $\xi \in \Gamma(M, \mathfrak{K})$.*

Another important situation in which B_ε is bounded is for product manifolds of the form $M = \mathbb{R} \times \Sigma$.

Corollary 6.19. *Suppose that $M \simeq \mathbb{R} \times \Sigma$ with $\mathbf{v}_M = \frac{\partial}{\partial t}$. Then, the inequality (6.26) holds for the compactly supported gauge algebra $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$, with constant $K(\varepsilon) = 9d_{\mathfrak{k}}(d_M + 1)^2/\varepsilon$ depending on M and $\mathfrak{K}$ only through the dimension.*

Proof. For $(t, x) \in \mathbb{R} \times \Sigma$, choose $U_0' \Subset U_0 \subseteq \Sigma$ with $U_0 \subseteq \Sigma$ relatively compact, and $x \in U_0'$. For T' sufficiently large, (t, x) is contained in the good pair of flowboxes $U' = U_0' \times (-T'/2, T'/2)$, and $U = U_0 \times (-T/2, T/2)$ for $T = 2T'$. Since

$$\frac{T}{T - T'}\frac{\mu_0(U_0)}{T'} = 2\mu_0(U_0)/T'$$

approaches 0 for $T' \to \infty$, it follows that $b(t, x) = 0$. In particular,

$$B_\varepsilon(m) = 81 d_{\mathfrak{k}}^2 (d_M + 1)^4/\varepsilon^2$$

is constant, and the result follows. ■

6.3 Extending representations to Sobolev spaces

In this section, we extend the map $\mathrm{d}\rho$ to the Hilbert completion $L^2_{B\mu}(M,\mathfrak{K})$ of $\mathfrak{g}/I_\mu$ with respect to the inner product (6.21) corresponding to $B_\varepsilon\mu$.

Note that since $\|\xi\|_\mu$ is dominated by a multiple of $\|\xi\|_{B_\varepsilon\mu}$, the inner product $\langle\xi,\eta\rangle_\mu$ is continuous on $L^2_{B\mu}(M,\mathfrak{K})$. As the difference between $\|\xi\|_{B_\varepsilon\mu}$ and $\|\xi\|_{B_{\widetilde{\varepsilon}}\mu}$ for $\varepsilon,\widetilde{\varepsilon}>0$ is a multiple of $\|\xi\|_\mu$, the space $L^2_{B\mu}(M,\mathfrak{K})$ and its topology are independent of ε. (This is why we omit ε from the notation in $L^2_{B\mu}(M,\mathcal{K})$.)

6.3.1 The completion $L^2_{B\mu}(M,\mathfrak{K})$ in L^2-norm

We use Theorem 6.17 to extend $\mathrm{d}\rho$ from $\mathfrak{g}$ to $L^2_{B\mu}(M,\mathfrak{g})$. Define

$$\Gamma_\xi = \|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H,$$

and note that its domain $\mathcal{D}(\Gamma_\xi)$ is contained in the domain $\mathcal{D}(H)$ of H. With this notation, (6.22) becomes

$$0 \le \Gamma_\xi \pm i\,\mathrm{d}\rho(\xi), \tag{6.27}$$

as an inequality of unbounded operators on $\mathcal{H}^\infty$. Further, define

$$A := \mathbf{1} + H \quad \text{with } \mathcal{D}(A) = \mathcal{D}(H). \tag{6.28}$$

Proposition 6.20. *Let $0<\varepsilon\le 1$. There exists a map r from $L^2_{B\mu}(M,\mathfrak{K})$ to the unbounded, skew-symmetric operators on $\mathcal{H}$ such that $\mathcal{D}(r(\xi))$ contains $\mathcal{D}(H)$ for all $\xi\in L^2_{B\mu}(M,\mathfrak{K})$, $r(\xi)|_{\mathcal{H}^\infty} = \mathrm{d}\rho(\xi)$ for all $\xi\in\mathfrak{g}$, and, for all $\psi\in\mathcal{D}(H)$, the functional*

$$L^2_{B\mu}(M,\mathfrak{K}) \to \mathbb{C} \ \textit{ defined by } \ \xi \mapsto \langle r(\xi)\rangle_\psi$$

is continuous. Furthermore, there exists a continuous map

$$\lambda\colon L^2_{B\mu}(M,\mathfrak{K}) \to B(\mathcal{H})$$

into the bounded operators such that $\|\lambda(\xi)\|\le\|\xi\|_{B_\varepsilon\mu}$, $\lambda(\xi)$ is skew-hermitian, $\lambda(\xi)$ leaves the domain of $A^{1/2}$ invariant, and

$$r(\xi) = A^{1/2}\lambda(\xi)A^{1/2},$$

as an equality of unbounded operators on $\mathcal{D}(H)$.

Proof. Let ξ_n be a sequence in $\mathfrak{g}/I_\mu$ for which $\|\xi-\xi_n\|_{B_\varepsilon\mu}\to 0$, and hence also $\|\xi-\xi_n\|_\mu\to 0$. Without loss of generality, we assume that $\|\xi-\xi_n\|_{B_\varepsilon\mu}\le\frac{1}{2}$ and $\varepsilon\|\xi-\xi_n\|_\mu\le\frac{1}{2}$ for all n, so that

$$\Gamma_\xi - \Gamma_{\xi_n} + A \ge \frac{1}{2}A. \tag{6.29}$$

Define the sesquilinear forms

$$B_n^{\pm}\colon \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}, \quad B_n^{\pm}(\psi,\chi) := \big\langle \psi, ((\Gamma_\xi + A) \pm i\,\mathrm{d}\rho(\xi_n))\chi \big\rangle.$$

The forms $B_n^{\pm}$ are positive definite; combining (6.29) with inequality (6.27) applied to ξ_n, we find

$$B_n^{\pm}(\psi,\psi) \geq \langle \psi, (\Gamma_\xi - \Gamma_{\xi_n} + A)\psi \rangle \geq \frac{1}{2}\langle \psi, A\psi \rangle \geq \frac{1}{2}\|\psi\|^2. \tag{6.30}$$

By (6.27) and the convergence of ξ_n, we find that $B_n^{+}(\psi,\psi)$ is a Cauchy sequence for every $\psi \in \mathcal{H}^\infty$,

$$|B_n^{+}(\psi,\psi) - B_m^{+}(\psi,\psi)| = |\langle \psi, i\,\mathrm{d}\rho(\xi_n - \xi_m)\psi \rangle| \leq \langle \psi, \Gamma_{\xi_n - \xi_m}\psi \rangle \to 0.$$

It follows that $B^{+}(\psi,\chi) := \lim_{n\to\infty} B_n^{+}(\psi,\chi)$ defines a positive definite, sesquilinear form $\mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathbb{C}$. Here we use that the estimate (6.30) is independent of n. The same argument applies to $B^{-}(\psi,\chi) := \lim_{n\to\infty} B_n^{-}(\psi,\chi)$. Note that

$$\frac{1}{2}\langle \psi, A\psi \rangle \leq B^{\pm}(\psi,\psi) \leq \langle \psi, (2\Gamma_\xi + A)\psi \rangle \leq c_\xi \langle \psi, A\psi \rangle \tag{6.31}$$

for some $c_\xi > 0$. The forms $B^{\pm}$ therefore extend uniquely to closed, sesquilinear forms $\overline{B}^{\pm}\colon \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2}) \to \mathbb{C}$. In turn, the forms $\overline{B}^{\pm}$ define a Friedrichs extension; a closed, possibly unbounded positive operator $b^{\pm}(\xi)\colon \mathcal{D}(H) \to \mathcal{H}$, such that $\overline{B}^{\pm}(\psi,\chi) = \langle \psi, b^{\pm}(\xi)\chi \rangle$ for all $\psi,\chi \in \mathcal{D}(H)$ (cf. [25, Appendix I.A.2]). Set

$$r(\xi) := \frac{1}{2i}(b^{+}(\xi) - b^{-}(\xi)).$$

Since $b^{+}(\xi)$ and $b^{-}(\xi)$ are selfadjoint, $r(\xi)$ is skew-symmetric. If $\xi \in \mathfrak{g}$, then

$$\langle \psi, r(\xi)\chi \rangle = \langle \psi, \mathrm{d}\rho(\xi)\chi \rangle \quad \text{for all } \psi,\chi \in \mathcal{H}^\infty,$$

so $r(\xi)$ is an extension of $\mathrm{d}\rho(\xi)$.

Define

$$\lambda(\xi)\colon D(A^{1/2}) \to D(A^{1/2}), \quad \lambda(\xi) := A^{-1/2} r(\xi) A^{-1/2}.$$

Then, for $\psi,\chi \in D(A^{1/2})$, we have $A^{-1/2}\psi, A^{-1/2}\chi \in \mathcal{D}(H)$. Therefore,

$$\langle \psi, \lambda(\xi)\chi \rangle = -\langle \lambda(\xi)\psi, \chi \rangle = \frac{1}{2i}(\overline{B}^{+} - \overline{B}^{-})(A^{-1/2}\psi, A^{-1/2}\chi). \tag{6.32}$$

By (6.31) and Cauchy–Schwarz, we have

$$|\overline{B}^{\pm}(\psi,\chi)| \leq c_\xi \|A^{1/2}\psi\| \|A^{1/2}\chi\|,$$

so $|\langle\psi,\lambda(\xi)\chi\rangle| \le c_\xi\|\psi\|\|\chi\|$ by (6.32). Therefore, $\lambda(\xi)$ extends to a hermitian operator on $\mathcal{H}$. As such, the operator norm $\|\lambda(\xi)\|$ is the supremum of $|\langle\psi,\lambda(\xi)\psi\rangle|$ over all ψ in the unit sphere of $\mathcal{H}$. For $\psi\in A^{1/2}\mathcal{H}^\infty$, (6.27) yields

$$|\langle\psi,\lambda(\xi)\psi\rangle| = \lim_{n\to\infty}|\langle A^{-1/2}\psi, \mathrm{d}\rho(\xi_n)A^{-1/2}\psi\rangle| \le \langle\psi, A^{-1/2}\Gamma_\xi A^{-1/2}\psi\rangle.$$

We claim that

$$A^{-1/2}\Gamma_\xi A^{-1/2} \le \|\xi\|_{B_\varepsilon\mu} \quad \text{for } 0<\varepsilon<1. \tag{6.33}$$

In fact, since Γ_ξ and A commute, this is equivalent to $\Gamma_\xi \le A\|\xi\|_{B\mu}$, which in turn is equivalent to

$$\|\xi\|_{B_\varepsilon\mu}\mathbf{1} + \varepsilon\|\xi\|_\mu H \le \|\xi\|_{B_\varepsilon\mu}(\mathbf{1}+H)$$

and this to $\varepsilon\|\xi\|_\mu \le \|\xi\|_{B_\varepsilon\mu}$, which, for $\varepsilon<1$, follows from the estimate

$$B_\varepsilon \ge 81 d_{\mathfrak{k}}^2(d_M+1)^4/\varepsilon^2 > 1.$$

With (6.33), we find

$$|\langle\psi,\lambda(\xi)\psi\rangle| \le \|\xi\|_{B_\varepsilon\mu}\|\psi\|^2 \quad \text{for } \psi\in A^{1/2}\mathcal{H}^\infty.$$

To prove that $\|\lambda(\xi)\| \le \|\xi\|_{B_\varepsilon\mu}$, it therefore suffices to show that $A^{1/2}\mathcal{H}^\infty$ is dense in $\mathcal{H}$. First, we show that $A\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Since $\exp(itA) = e^{it}\exp(itH)$ leaves the space $\mathcal{H}^\infty$ of smooth vectors invariant, the restriction A_0 of A to $\mathcal{H}^\infty$ is essentially selfadjoint [95, Section VIII.4]. Suppose that $\psi \perp A_0\mathcal{H}^\infty$. Then, $\psi\in\mathcal{D}(A_0^*) = \mathcal{D}(A)$, and $A_0^*\psi = A\psi = 0$. Since A is injective, $\psi = 0$ and $A\mathcal{H}^\infty$ is dense in $\mathcal{H}$. Applying the contraction $A^{-1/2}$, we find that $A^{1/2}\mathcal{H}^\infty$ is dense in $A^{-1/2}\mathcal{H}$. Since $A^{-1/2}\mathcal{H} = \mathcal{D}(A^{1/2})$ is dense in $\mathcal{H}$, we conclude that $A^{1/2}\mathcal{H}^\infty$ is dense in $\mathcal{H}$, as required. ∎

For $s\in\mathbb{R}$, denote by $\mathcal{H}_s$ the Hilbert space completion of $\mathcal{D}(A^s)$ with respect to the inner product

$$\langle\psi,\chi\rangle_s = \langle A^s\psi, A^s\chi\rangle.$$

Denote the corresponding norm by $\|\psi\|_s = \|A^s\psi\|$, and denote the norm of a continuous operator $A\colon\mathcal{H}_s\to\mathcal{H}_t$ by $\|A\|_{s,t}$. As

$$r(\xi) = A^{1/2}\lambda(\xi)A^{1/2}$$

with $\|\lambda(\xi)\| \le \|\xi\|_{B_\varepsilon\mu}$, the operator $r(\xi)\colon\mathcal{D}(A)\to\mathcal{H}$ extends to a bounded operator $r(\xi)\colon\mathcal{H}_{1/2}\to\mathcal{H}_{-1/2}$, with

$$\|r(\xi)\psi\|_{-1/2} \le \|\xi\|_{B_\varepsilon\mu}\|\psi\|_{1/2}. \tag{6.34}$$

We thus have

$$\|r(\xi)\|_{1/2,-1/2} \le \|\xi\|_{B_\varepsilon\mu}.$$

6.3.2 The completion in Sobolev norm

Note that convergence of ξ_n to ξ in $L^2_{B\mu}(M,\mathfrak{K})$ only implies *weak* operator convergence of $r(\xi_n)$ to $r(\xi)$, as operators on the pre-Hilbert space $\mathcal{D}(H)$. In this section, we define a subspace $H^1_{B\mu}(M,\mathfrak{K})$ of $L^2_{B\mu}(M,\mathfrak{K})$ where convergence to ξ implies *strong* convergence to $r(\xi)$.

Definition 6.21 (Parallel Sobolov spaces). For $k \geq 0$, the *parallel Sobolev norm* q_k is defined by

$$q_k(\xi) := \sum_{n=0}^{k} \|\xi\|_n, \quad \text{where } \|\xi\|_n := \|D^n\xi\|_{B_\varepsilon\mu}.$$

The *parallel Sobolev space* $H^k_{B\mu}(M,\mathfrak{K}) \subseteq L^2_{B\mu}(M,\mathfrak{K})$ is the Banach completion of $\mathfrak{g}/I_\mu$ with respect to the norm q_k.

Proposition 6.22. *Let r be as in Proposition 6.20. If $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, then $r(\xi)$ maps $\mathcal{D}(H^{k+1})$ into $\mathcal{D}(H^k)$. For $k=1$, we have*

$$[H, r(\xi)] = ir(D\xi) \tag{6.35}$$

as an equality of unbounded operators on $\mathcal{D}(H^2)$. Furthermore, if $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, then $r(\xi)$ extends to a continuous operator $\mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ with

$$\|r(\xi)\psi\|_{k-1/2} \leq \sum_{j=0}^{k} \binom{k}{j} \|\xi\|_j \|\psi\|_{k-j+1/2}. \tag{6.36}$$

Finally, for all $\xi \in H^1_{B\mu}(M,\mathfrak{K})$, the skew-symmetric operator $r(\xi)$ is essentially skew-adjoint.

Proof. We prove that for $\xi \in H^k_{B\mu}(M,\mathfrak{K})$, $r(\xi)$ maps $\mathcal{D}(H^{k+1})$ into $\mathcal{D}(H^k)$. We proceed by induction on k. Since $H^0_{B\mu}(M,\mathfrak{K}) = L^2_{B\mu}(M,\mathfrak{K})$, the case $k=0$ follows from Proposition 6.20. Suppose that the statement holds for all $\xi \in H^k_{B\mu}(M,\mathfrak{K})$. For $\xi \in H^{k+1}_{B\mu}(M,\mathfrak{K})$ and $\psi \in \mathcal{D}(H^{k+2})$, we show that $r(\xi)\psi \in \mathcal{D}(H^{k+1})$. Since H^{k+1} is selfadjoint, it suffices to show that $\chi \mapsto \langle r(\xi)\psi, H^{k+1}\chi\rangle$ is a continuous, linear functional on $\mathcal{H}^\infty$ with respect to the subspace topology induced by the inclusion in $\mathcal{H}$.

Let $\xi_n \in \mathfrak{g}/I_\mu$ be a sequence such that $\xi_n \to \xi$ and $D\xi_n \to D\xi$ in $L^2_{B\mu}(M,\mathfrak{K})$. Since $Hr(\xi_n) = r(\xi_n)H + ir(D\xi_n)$ on $\mathcal{H}^\infty$, we have

$$\begin{aligned}
\langle r(\xi)\psi, H^{k+1}\chi\rangle &= -\lim_{n\to\infty}\langle \psi, r(\xi_n)H^{k+1}\chi\rangle \\
&= -\lim_{n\to\infty}\langle H\psi, r(\xi_n)H^k\chi\rangle + \lim_{n\to\infty}\langle \psi, ir(D\xi_n)H^k\chi\rangle \\
&= \langle r(\xi)H\psi + ir(D\xi)\psi, H^k\chi\rangle.
\end{aligned} \tag{6.37}$$

As $\psi \in \mathcal{D}(H^{k+2})$, both $H\psi$ and ψ are in $\mathcal{D}(H^{k+1})$. Since $\xi \in H^{k+1}_{B\mu}(M, \mathfrak{K})$, we have $\xi, D\xi \in H^k_{B\mu}(M, \mathfrak{K})$, so that $r(\xi)H\psi + ir(D\xi)\psi \in \mathcal{D}(H^k)$ by the induction hypothesis. From (6.37), we thus find that

$$\langle r(\xi)\psi, H^{k+1}\chi\rangle = \big\langle H^k(r(\xi)H\psi + ir(D\xi)\psi), \chi\big\rangle,$$

which is manifestly continuous in the variable χ. We conclude that $r(\xi)$ maps the domain $\mathcal{D}(H^{k+2})$ to $\mathcal{D}(H^{k+1})$. Moreover, for $k = 0$, we find that

$$Hr(\xi) - r(\xi)H - ir(D\xi)$$

vanishes on $\mathcal{D}(H^2)$.

The inequality (6.36) is proven in a similar fashion. Assume by induction that (6.36) holds for all $\xi \in H^k_{B\mu}(M, \mathfrak{K})$ and $\psi \in \mathcal{H}_{k+1/2}$, the case $k = 0$ being (6.34). We recall that $\|\psi\|_s = \|A^s\psi\|$ with $A = \mathbf{1} + H$ (see (6.28)). For $\xi \in H^{k+1}_{B\mu}(M, \mathfrak{K})$ and $\psi \in \mathcal{H}_{k+3/2}$, we use $Ar(\xi)\psi = r(\xi)A\psi + ir(D\xi)\psi$ to see that

$$\|r(\xi)\psi\|_{k+1/2} = \|Ar(\xi)\psi\|_{k-1/2} \leq \|r(\xi)A\psi\|_{k-1/2} + \|r(D\xi)\|_{k-1/2}.$$

By the induction hypothesis with $\|A\psi\|_{k-j+1/2} = \|\psi\|_{(k+1)-j+1/2}$ (for the first term) and $\|D\xi\|_j = \|\xi\|_{j+1}$ (for the second), we find that $\|r(\xi)\psi\|_{k+1/2}$ is bounded by

$$\sum_{j=0}^{k} \binom{k}{j} \|\xi\|_j \|\psi\|_{(k+1)-j+1/2} + \sum_{j=0}^{k} \binom{k}{j} \|\xi\|_{j+1} \|\psi\|_{k-j+1/2}.$$

Since $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$, we have

$$\|r(\xi)\psi\|_{k+1/2} \leq \sum_{j=0}^{k+1} \binom{k+1}{j} \|\xi\|_j \|\psi\|_{k+1-j+1/2},$$

as required.

Finally, if $\xi \in H^1_{B\mu}(M, \mathfrak{K})$, then $\xi, D\xi \in L^2_{B\mu}(M, \mathfrak{K})$. By (6.34), the operators $r(\xi)$ and $[A, r(\xi)] = ir(D\xi)$ from $\mathcal{D}(H)$ to $\mathcal{H}$ extend continuously to bounded operators $\mathcal{H}_{1/2} \to \mathcal{H}_{-1/2}$. It then follows from a result of Nelson [87, Proposition 2] that $r(\xi)$ is essentially skew-adjoint. ∎

If we estimate $\|\xi\|_j \leq q_k(\xi)$ and $\|\psi\|_{k-j+1/2} \leq \|\psi\|_{k+1/2}$ in (6.36), we find that $r(\xi)\colon \mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ satisfies

$$\|r(\xi)\psi\|_{k-1/2} \leq 2^k q_k(\xi)\|\psi\|_{k+1/2},$$

so the linear map $H^k_{B\mu}(M, \mathfrak{K}) \times \mathcal{H}_{k+1/2} \to \mathcal{H}_{k-1/2}$ defined by $(\xi, \psi) \mapsto r(\xi)\psi$ is jointly continuous. For $k = 1$, we find from (6.36) the slightly stronger estimate

$$\|r(\xi)\psi\| \leq \|r(\xi)\psi\|_{1/2} \leq q_1(\xi)\|A^{3/2}\psi\|. \tag{6.38}$$

In particular, convergence of ξ_n to ξ in $H^1_{B\mu}(M,\mathfrak{K})$ implies *strong* convergence of $r(\xi_n)$ to $r(\xi)$ on $\mathcal{D}(A^{3/2})$.

6.4 Sobolev–Lie algebras

Having established that the positive energy representation $\mathrm{d}\rho$ extends to a continuous map r on $H^k_{B\mu}(M,\mathfrak{K})$, we would like to determine whether r gives rise to a Lie algebra representation. Since the spaces $H^k_{B\mu}(M,\mathfrak{K})$ do not inherit the Lie algebra structure from $\mathfrak{g}/I_\mu$, we introduce two spaces of *bounded* Sobolev sections of $\mathfrak{K}\to M$, both equipped with the pointwise Lie bracket.

For an open subset $N\subseteq M$, we define the Lie algebra $H^k_b(N,\mathfrak{K})$ of bounded parallel Sobolev sections, and a certain subalgebra $H^k_\partial(N,\mathfrak{K})$ of sections that vanish to order k at the boundary of the 1-point compactification of N. As before, the underlying measure is the restriction to N of the flow-invariant measure $B_\varepsilon\mu$ on M. For convenience of notation, we will denote this measure by $\nu = B_\varepsilon\mu$.

6.4.1 The Lie algebra $L^2_b(N,\mathfrak{K})$ of bounded L^2-sections

Let N be an open subset of M, and let ξ be a measurable section of $\mathfrak{K}\to N$. Then,

$$\|\xi\|_\kappa = \sqrt{\kappa(\xi,\xi)}$$

is a measurable function on N. We define $\|\xi\|_\infty$ to be the essential supremum of $\|\xi\|_\kappa$ with respect to ν, and we define $L^\infty(N,\mathfrak{K})$ to be the Lie algebra of equivalence classes of essentially bounded, measurable sections of $\mathfrak{K}\to N$. This is a Banach–Lie algebra with respect to the norm $\|\xi\|_\infty$, and the Lie bracket coming from the pointwise bracket of sections. We define $L^2_b(N,\mathfrak{K})$ to be the space of equivalence classes of sections which are both essentially bounded and square integrable with respect to ν. Since both $L^2(N,\mathfrak{K})$ and $L^\infty(N,\mathfrak{K})$ are complete, it follows that $L^2_b(N,\mathfrak{K})$ is a Banach space with respect to the norm $\|\xi\|_\nu + \|\xi\|_\infty$.

Let $c_\mathfrak{k}$ be a constant such that

$$\|[X,Y]\|_\kappa \le c_\mathfrak{k}\|X\|_\kappa\|Y\|_\kappa \tag{6.39}$$

for all $X,Y\in\mathfrak{k}$. Then, we find

$$\|[\xi,\eta]\|_\nu \le c_\mathfrak{k}\|\xi\|_\infty\|\eta\|_\nu, \tag{6.40}$$

$$\|[\xi,\eta]\|_\infty \le c_\mathfrak{k}\|\xi\|_\infty\|\eta\|_\infty. \tag{6.41}$$

It follows that the Lie bracket $[\cdot,\cdot]\colon L^2_b(N,\mathfrak{K})\times L^2_b(N,\mathfrak{K})\to L^\infty(N,\mathfrak{K})$ takes values in $L^2_b(N,\mathfrak{K})$ and is continuous with respect to the norm $p_0(\xi) := \|\xi\|_\nu + \|\xi\|_\infty$. In particular, $L^2_b(N,\mathfrak{K})$ is a Banach–Lie algebra, and the inclusion $L^2_b(N,\mathfrak{K})\hookrightarrow L^\infty(N,\mathfrak{K})$ is a continuous homomorphism of Banach–Lie algebras. If $N\subseteq N'$, then $L^2_b(N,\mathfrak{K})$ is a subalgebra of $L^2_b(N',\mathfrak{K})$ in the natural fashion.

6.4.2 The "parallel" Sobolev–Lie algebras $H_b^k(N, \mathfrak{K})$

Recall from Definition 4.6 that a one-parameter group $(\gamma_t)_{t\in\mathbb{R}}$ of automorphisms of $\mathfrak{K} \to M$ gives rise to a one-parameter group $(\alpha_t)_{t\in\mathbb{R}}$ of automorphisms of

$$\mathfrak{g} = \Gamma_c(M, \mathfrak{K}).$$

In the same way, we obtain a one-parameter group of automorphisms of $L_b^2(M, \mathfrak{K})$.

Indeed, since the Killing form is invariant under automorphisms, $\|\alpha_t(\xi)\|_\kappa = \|\xi\|_\kappa \circ \gamma_{M,t}$, so that, in particular, $\|\alpha_t(\xi)\|_\infty = \|\xi\|_\infty$. Further, since the measure $\nu = B_\varepsilon\mu$ is invariant under the flow $\gamma_{M,t}$ (Theorem 5.7), we find $\|\alpha_t(\xi)\|_\nu = \|\xi\|_\nu$.

Since α_t is a one-parameter group of unitary transformations of the Hilbert space $L_\nu^2(M, \mathfrak{K})$, it is generated by a skew-adjoint operator D. We define $H_\nu^1(N, \mathfrak{K})$ to be the intersection of its domain with $L_\nu^2(N, \mathfrak{K})$, and we define $H_b^1(N, \mathfrak{K})$ to be the space of all $\xi \in H_\nu^1(N, \mathfrak{K})$, where both $\|\xi\|_\infty$ and $\|D\xi\|_\infty$ are finite. In other words, $H_b^1(N, \mathfrak{K})$ is the space of equivalence classes of essentially bounded, square integrable sections ξ of $\mathfrak{K} \to N$ such that the L^2-limit

$$D(\xi) := L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t(\xi) - \xi)$$

exists, and $\|D(\xi)\|_\infty$ is finite.

Proposition 6.23. *The space $H_b^1(N, \mathfrak{K})$ is a Lie subalgebra of $L_b^2(N, \mathfrak{K})$, and the generator $D\colon H_b^1(N, \mathfrak{K}) \to L_b^2(N, \mathfrak{K})$ satisfies*

$$D([\xi, \eta]) = [D(\xi), \eta] + [\xi, D(\eta)] \quad \textit{for all } \xi, \eta \in H_b^1(N, \mathfrak{K}). \tag{6.42}$$

Proof. Let $\xi, \eta \in H_b^1(N, \mathfrak{K})$, and denote by L^2-lim the limit with respect to the norm $\|\xi\|_\nu$. First, we show that $[\xi, \eta]$ is in the domain of D:

$$\begin{aligned} D([\xi, \eta]) &= L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t([\xi, \eta]) - [\xi, \eta]) \\ &= L^2\text{-}\lim_{t\to 0}[D\xi, \alpha_t(\eta)] + L^2\text{-}\lim_{t\to 0}\left[\frac{1}{t}(\alpha_t(\xi) - \xi) - D(\xi), \alpha_t(\eta)\right] \\ &\quad + L^2\text{-}\lim_{t\to 0}[\xi, \frac{1}{t}(\alpha_t(\eta) - \eta)] = [D\xi, \eta] + [\xi, D\eta]. \end{aligned}$$

In the last step, we used the inequality (6.40) three times. Since $\|D\xi\|_\infty$ is bounded and $L^2\text{-}\lim_{t\to 0} \alpha_t(\eta) = \eta$, it follows from (6.40) that the first term is given by

$$L^2\text{-}\lim_{t\to 0}[D\xi, \alpha_t(\eta)] = [D\xi, \eta].$$

Similarly, since $\|\xi\|_\infty$ is bounded and $L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t(\eta) - \eta) = D(\eta)$, the third term equals $[\xi, D(\eta)]$. To see that the second term is zero, note that $\|\alpha_t(\eta)\|_\infty = \|\eta\|_\infty$. It then follows from (6.40) and the fact that $L^2\text{-}\lim_{t\to 0} \frac{1}{t}(\alpha_t(\xi) - \xi) - D(\xi) = 0$.

This shows not only that $[\xi, \eta]$ is in the domain of D, but also that (6.42) holds. By (6.41), it follows that $\|D([\xi,\eta])\|_\infty \leq c_{\mathfrak{k}}(\|D\xi\|_\infty\|\eta\|_\infty + \|\xi\|_\infty\|D\eta\|_\infty)$ is finite, so that $[\xi, \eta] \in H^1_b(N, \mathfrak{K})$. ∎

This allows us to define parallel Sobolev–Lie algebras of order $k \in \mathbb{N}$. We set

$$H^0_b(N, \mathfrak{K}) := L^2_b(N, \mathfrak{K}),$$

and define $H^1_b(N, \mathfrak{K})$ as above. For $k \geq 2$, we define $H^k_b(N, \mathfrak{K})$ as

$$H^{k-1}_b(N, \mathfrak{K}) \cap D^{-1}(H^{k-1}_b(N, \mathfrak{K})).$$

In other words, ξ is in $H^k_b(N, \mathfrak{K})$ if both ξ and $D\xi$ are in $H^{k-1}_b(N, \mathfrak{K})$.

Proposition 6.24. *The space $H^k_b(N, \mathfrak{K})$ is a Lie subalgebra of $H^{k-1}_b(N, \mathfrak{K})$.*

Proof. The proof is by induction on k, where $k = 1$ is Proposition 6.23. If $\xi, \eta \in H^k_b(N, \mathfrak{K})$, then $\xi, D(\xi), \eta, D(\eta) \in H^{k-1}_b(N, \mathfrak{K})$. Since $H^{k-1}_b(N, \mathfrak{K})$ is a Lie algebra, it follows that $D([\xi, \eta]) = [D(\xi), \eta] + [\xi, D(\eta)]$ is in $H^{k-1}_b(N, \mathfrak{K})$. Thus, $[\xi, \eta] \in H^k_b(N, \mathfrak{K})$, as required. ∎

On $H^k_b(N, \mathfrak{K})$, we define for every $n \in \{0, \ldots, k\}$ the derived norms

$$\|\xi\|_{n,\infty} := \|D^n\xi\|_\infty \quad \text{and} \quad \|\xi\|_n := \|D^n\xi\|_\nu.$$

The parallel C^k-norm q_{C^k} and the parallel Sobolev norm q_k are defined by

$$q_{C^k}(\xi) := \sum_{n=0}^{k} \|\xi\|_{n,\infty} \quad \text{and} \quad q_k(\xi) := \sum_{n=0}^{k} \|\xi\|_n, \tag{6.43}$$

respectively. We equip $H^k_b(N, \mathfrak{K})$ with the topology derived from the combined norm

$$p_k(\xi) := \sum_{n=0}^{k} \|\xi\|_{n,\infty} + \|\xi\|_n. \tag{6.44}$$

Note that for $\xi \in H^k_b(N, \xi)$, we have $p_{k-1}(\xi) \leq p_k(\xi)$, but also $p_{k-1}(D(\xi)) \leq p_k(\xi)$. It follows that both the inclusion $\iota\colon H^{k+1}_b(N, \mathfrak{K}) \hookrightarrow H^k_b(N, \mathfrak{K})$ and the derivative $D\colon H^{k+1}_b(N, \mathfrak{K}) \to H^k_b(N, \mathfrak{K})$ are continuous.

Proposition 6.25. *For every $k \geq 0$, $H^k_b(N, \mathfrak{K})$ is a Banach–Lie algebra with respect to the norm p_k. The Lie bracket is separately continuous with respect to the Sobolev norm q_k.*

Proof. By the derivation property and (6.39), we have

$$\|D^n([\xi, \eta])\|_\kappa \leq c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|D^j\xi\|_\kappa \|D^{n-j}\eta\|_\kappa.$$

Since $\|[\xi,\eta]\|_n = \|D^n([\xi,\eta])\|_\nu$ and $\|[\xi,\eta]\|_{n,\infty} = \|D^n([\xi,\eta])\|_\infty$, it follows that

$$\|[\xi,\eta]\|_n \le c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|\xi\|_{j,\infty}\|\eta\|_{n-j},$$

$$\|[\xi,\eta]\|_{n,\infty} \le c_{\mathfrak{k}} \sum_{j=0}^{n} \binom{n}{j} \|\xi\|_{j,\infty}\|\eta\|_{n-j,\infty}.$$

Taking $n = k$ and estimating the binomial coefficients by 2^k, it follows that

$$q_k([\xi,\eta]) \le 2^k c_{\mathfrak{k}} q_{C^k}(\xi) q_k(\eta), \tag{6.45}$$

$$q_{C^k}([\xi,\eta]) \le 2^k c_{\mathfrak{k}} q_{C^k}(\xi) q_{C^k}(\eta). \tag{6.46}$$

This shows that the Lie bracket is continuous for the norm p_k, and separately continuous for the Sobolev norm q_k.

To show that $H^k_b(N,\mathfrak{K})$ is complete, we note that $H^0_b(N,\mathfrak{K}) = L^2_b(N,\mathfrak{K})$ is a Banach space, and proceed by induction on k. Let $\xi_n \in H^k_b(N,\mathfrak{K})$ be a sequence with $p_k(\xi_n - \xi_m) \to 0$. Then, $p_{k-1}(\xi_n - \xi_m) \to 0$ and $p_{k-1}(D(\xi_n) - D(\xi_m)) \to 0$, so by induction, there exist $\xi, \Xi \in H^{k-1}_b(N,\mathfrak{K})$ with

$$p_{k-1}(\xi - \xi_n) \to 0 \quad \text{and} \quad p_{k-1}(\Xi - D(\xi_n)) \to 0.$$

Since $D\colon H^1_\nu(M,\mathfrak{K}) \to L^2(M,\mathfrak{K})$ is the generator of a strongly continuous 1-parameter group of unitary operators, Stone's theorem implies that it is selfadjoint, and hence, in particular, closed. It follows that ξ lies in the domain of D, and $D(\xi) = \Xi$ lies in $H^{k-1}_b(N,\mathfrak{K})$. Thus, $\xi \in H^k_b(N,\mathfrak{K})$, and

$$p_k(\xi - \xi_n) \le p_{k-1}(\xi - \xi_n) + p_{k-1}(D(\xi) - D(\xi_n)) \to 0. \qquad \blacksquare$$

We denote by $H^\infty_b(N,\mathfrak{K})$ the Fréchet–Lie algebra arising from the inverse limit of the Banach–Lie algebras $H^k_b(N,\mathfrak{K})$ with respect to the natural inclusions

$$\iota\colon H^{k+1}_b(N,\mathfrak{K}) \hookrightarrow H^k_b(N,\mathfrak{K}).$$

The derivative $D\colon H^\infty_b(N,\mathfrak{K}) \to H^\infty_b(N,\mathfrak{K})$ is a continuous derivation, giving rise to the Fréchet–Lie algebra $H^\infty_b(N,\mathfrak{K}) \rtimes \mathbb{R}D$.

6.4.3 Boundary conditions and the Lie algebras $H^k_\partial(N,\mathfrak{K})$

Let $H^1_\partial(N,\mathfrak{K})$ be the closure of $\Gamma_c(N,\mathfrak{K})$ in $H^1_b(N,\mathfrak{K})$ with respect to the parallel Sobolev norm $q_1(\xi) = \|\xi\|_\nu + \|\xi\|_{1,\nu}$.

Proposition 6.26. *The space $H^1_\partial(N,\mathfrak{K})$ is a closed Lie subalgebra of $H^1_b(N,\mathfrak{K})$. In particular, it is a Banach–Lie algebra with respect to the subspace topology, induced by the norm $p_1(\xi)$ of* (6.44).

Proof. Since $H^1_\partial(N,\mathfrak{K})$ is by definition closed with respect to the Sobolev norm $q_1(\xi)$, it is a fortiori closed with respect to the larger norm $p_1(\xi)$ that defines the Banach space topology on $H^1_b(N,\mathfrak{K})$. As $H^1_\partial(N,\mathfrak{K})$ is a closed subspace of a Banach space, it is a Banach space itself.

It remains to show that $H^1_\partial(N,\mathfrak{K})$ is closed under the Lie bracket. For every $\xi \in H^1_b(N,\mathfrak{K})$, the linear operator $\mathrm{ad}_\xi\colon H^1_b(N,\mathfrak{K}) \to H^1_b(N,\mathfrak{K})$ is continuous with respect to the norm $q_1(\xi)$, as

$$q_1(\mathrm{ad}_\xi(\eta)) \leq 2c_{\mathfrak{k}}q_{C^1}(\xi)q_1(\eta)$$

by (6.45). If $\xi \in \Gamma_c(N,\mathfrak{K})$, then $\mathrm{ad}(\xi)$ maps $\Gamma_c(N,\mathfrak{K})$ to $\Gamma_c(N,\mathfrak{K})$. As ad_ξ is continuous for the norm q_1, it also maps $H^1_\partial(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$. It follows that, for all $\eta \in H^1_\partial(N,\mathfrak{K})$, ad_η maps $\Gamma_c(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$. By continuity with respect to q_1, it therefore maps $H^1_\partial(N,\mathfrak{K})$ to $H^1_\partial(N,\mathfrak{K})$, and we conclude that $H^1_\partial(N,\mathfrak{K})$ is closed under the Lie bracket. ∎

For $k \geq 2$, we define $H^k_\partial(N,\mathfrak{K})$ as the space of all $\xi \in H^k_b(N,\mathfrak{K})$ such that both ξ and $D(\xi)$ lie in $H^{k-1}_\partial(N,\mathfrak{K})$.

Proposition 6.27. *The space $H^k_\partial(N,\mathfrak{K})$ is a closed Lie subalgebra of $H^k_b(N,\mathfrak{K})$. In particular, it is a Banach–Lie algebra with respect to the subspace topology, induced by the norm $p_k(\xi)$ of* (6.44).

Proof. We proceed by induction on k, the case $k = 1$ being Proposition 6.26. Recall that both the inclusion $\iota\colon H^k_b(N,\mathfrak{K}) \hookrightarrow H^{k-1}_b(N,\mathfrak{K})$ and the derivative

$$D\colon H^k_b(N,\mathfrak{K}) \to H^{k-1}_b(N,\mathfrak{K})$$

are continuous. Since

$$H^k_\partial(N,\mathfrak{K}) = \iota^{-1}(H^{k-1}_\partial(N,\mathfrak{K})) \cap D^{-1}(H^{k-1}_\partial(N,\mathfrak{K}))$$

is the intersection of two closed subspaces, it is a closed subspace of $H^k_b(N,\mathfrak{K})$ itself. To show that it is closed under the Lie bracket, suppose that $\xi,\eta \in H^k_\partial(N,\mathfrak{K})$, so that $\xi,\eta,D\xi,D\eta \in H^{k-1}_\partial(N,\mathfrak{K})$. As $H^{k-1}_\partial(N,\mathfrak{K})$ is a Lie algebra, it follows that $[\xi,\eta]$ and $D([\xi,\eta]) = [D(\xi),\eta] + [\xi,D(\eta)]$ are both in $H^{k-1}_\partial(N,\mathfrak{K})$. From this, one sees that also $[\xi,\eta] \in H^k_\partial(N,\mathfrak{K})$. ∎

Note that the 2-cocycle $\omega(\xi,\eta) = \langle D\xi,\eta\rangle_\mu$ on $\mathfrak{g}$ is continuous for the Sobolev norm $q_1(\xi)$ and hence extends uniquely to $H^k_\partial(N,\mathfrak{K})$. This defines a continuous central extension of $H^k_\partial(N,\mathfrak{K})$,

$$\mathbb{R}C \oplus_\omega H^k_\partial(N,\mathfrak{K}).$$

Define the Fréchet–Lie algebra $H^\infty_\partial(N,\mathfrak{K})$ as the inverse limit of the Banach–Lie algebras $H^k_\partial(N,\mathfrak{K})$ under the natural inclusions $H^k_\partial(N,\mathfrak{K}) \to H^{k-1}_\partial(N,\mathfrak{K})$. Since $D\colon H^\infty_\partial(N,\mathfrak{K}) \to H^\infty_\partial(N,\mathfrak{K})$ is a continuous derivation, we obtain the double extension of Fréchet–Lie algebras

$$\big(\mathbb{R}C \oplus_\omega H^\infty_\partial(N,\mathfrak{K})\big) \rtimes \mathbb{R}D.$$

6.4.4 Intervals and blocks

Suppose that $N \simeq \Sigma \times I$, where $I \subseteq \mathbb{R}$ is an open, not necessarily finite interval with the Lebesgue measure dt, and Σ is a $(d_M - 1)$-dimensional manifold with locally finite measure ν_0. The bundle $\mathfrak{K}|_N \simeq N \times \mathfrak{k}$ is trivial, and the translation by t' sends (x,t) to $(x, t - t')$ wherever it is defined. In this cartesian product situation, it will be useful to separate the variables in Σ from those in I.

Define a bilinear map

$$T\colon L^2_b(\Sigma,\mathbb{R}) \times L^2_b(I,\mathfrak{k}) \to L^2_b(N,\mathfrak{k}), \quad T(f,\xi)(x,t) = f(x)\xi(t).$$

It is continuous since $\|f\xi\|_\nu = \|f\|_{\nu_0}\|\xi\|_{dt}$ and $\|f\xi\|_\infty = \|f\|_\infty\|\xi\|_\infty$.

Proposition 6.28. *The product $T(f,\xi) = f\xi$ defines a continuous bilinear map*

$$T\colon L^2_b(\Sigma,\mathbb{R}) \times H^k_\partial(I,\mathfrak{k}) \to H^k_\partial(N,\mathfrak{k}).$$

Proof. Since $\|f\xi\|_\nu = \|f\|_{\nu_0}\|\xi\|_{dt}$, and since time translation acts only on ξ, it follows that $f\xi \in \mathcal{D}(D)$ if and only if $\xi \in \mathcal{D}(D)$, and $D(f\xi) = fD(\xi)$. From this, one derives that T maps $L^2_b(\Sigma,\mathbb{R}) \times H^k_b(I,\mathfrak{k})$ to $H^k_b(N,\mathfrak{k})$, with $\|f\xi\|_n = \|f\|_{\nu_0}\|\xi\|_n$ and $\|f\xi\|_{n,\infty} = \|f\|_\infty\|\xi\|_{n,\infty}$.

Suppose that $\xi \in H^1_\partial(I,\mathfrak{k})$, so that there exists a sequence $\xi_n \in C^\infty_c(I,\mathfrak{k})$ with $\|\xi - \xi_n\|_{dt} \to 0$ and $\|D(\xi) - D(\xi_n)\|_{dt} \to 0$. For every $f \in L^2_b(\Sigma,\mathbb{R})$, it is possible to find a sequence $f_n \in C^\infty_c(\Sigma,\mathbb{R})$ with $\|f - f_n\|_{\nu_0} \to 0$. Then

$$\|f\xi - f_n\xi_n\|_\nu \le \|f - f_n\|_{\nu_0}\|\xi\|_{dt} + \|f_n\|_{\nu_0}\|\xi - \xi_n\|_{dt} \to 0.$$

Similarly, one finds that $\|D(f\xi) - D(f_n\xi_n)\|_\nu = \|fD(\xi) - f_nD(\xi_n)\| \to 0$. It follows that T maps $L^2_b(\Sigma,\mathbb{R}) \times H^1_\partial(I,\mathfrak{k})$ to $H^1_\partial(N,\mathfrak{k})$. From $D(f\xi) = fD(\xi)$, one then finds that it maps $L^2_b(\Sigma,\mathbb{R}) \times H^k_\partial(I,\mathfrak{k})$ to $H^k_\partial(N,\mathfrak{k})$. ∎

In Lemma 7.10, we will need the above result in the following form.

Corollary 6.29. *Let $E \subseteq \Sigma$ be a subset of finite measure, and let χ_E be the corresponding indicator function. Then, the map $\iota_E\colon H^k_\partial(I,\mathfrak{k}) \to H^k_\partial(N,\mathfrak{K})$ defined by $\iota_E(\xi)(x,t) = \chi_E(x)\xi(t)$ is a continuous Lie algebra homomorphism.*

6.5 The continuous extension theorem

It follows from Proposition 6.20 that the Lie algebra representation $\mathtt{d}\rho$ extends from $\mathfrak{g} = \Gamma_c(M, \mathfrak{K})$ to $L^2_{B\mu}(M, \mathfrak{K})$. In the following theorem, we show that this extension yields a Lie algebra representation of $\mathbb{R}C \oplus_\omega H^1_\partial(M, \mathfrak{K})$, which is compatible with the derivation $D \colon H^1_\partial(M, \mathfrak{K}) \to L^2_b(M, \mathfrak{K})$.

Theorem 6.30 (Continuous extension). *Let ρ be a positive energy representation of $\widehat{G}$ with derived representation $\mathtt{d}\rho$, and let $N \subseteq M$ be an open subset.*

(a) *There exists a linear map r from $L^2_b(N, \mathfrak{K})$ to the unbounded, skew-symmetric operators on $\mathcal{H}$ with domain $\mathcal{D}(H)$ such that $r(\xi)\psi$ coincides with $\mathtt{d}\rho(\xi)\psi$ for all $\xi \in \Gamma_c(N, \mathfrak{K})$ and $\psi \in \mathcal{H}^\infty$.*

(b) *This defines a representation of the Banach–Lie algebra $\mathbb{R}C \oplus_\omega H^1_\partial(N, \mathfrak{K})$ by essentially skew-adjoint operators. For $\xi, \eta \in H^1_\partial(N, \mathfrak{K})$, the operators $r(\xi)$ and $r(\eta)$ map $\mathcal{D}(H^2)$ to $\mathcal{D}(H)$. On $\mathcal{D}(H^2)$, we have the commutator relation*

$$[r(\xi), r(\eta)] = r([\xi, \eta]) + i\omega(\xi, \eta)\mathbf{1}, \tag{6.47}$$

where $\omega(\xi, \eta) = \langle D\xi, \eta\rangle_\mu$.

(c) *If $\xi \in H^1_\partial(N, \mathfrak{K})$, then $D\xi \in L^2_b(N, \mathfrak{K})$ and*

$$[\mathtt{d}\rho(D), r(\xi)] = r(D\xi).$$

In particular, we obtain a positive energy representation of the Fréchet–Lie algebra $(\mathbb{R}C \oplus_\omega H^\infty_\partial(N, \mathfrak{K})) \rtimes \mathbb{R}D$.

Proof. The derived representation $\mathtt{d}\rho$ is defined on the Lie algebra

$$\widehat{\mathfrak{g}} = \big(\mathbb{R}C \oplus_\omega \mathfrak{g}\big) \rtimes \mathbb{R}D.$$

By Proposition 6.20, we obtain an extension r of $\mathtt{d}\rho$ to $L^2_{B\mu}(M, \mathfrak{K})$, hence, in particular, to $L^2_b(N, \mathfrak{K})$. From Proposition 6.22, we see that $r(\xi)$ is essentially skew-adjoint for ξ in the smaller space $H^1_{B\mu}(M, \mathfrak{K}) \subseteq L^2_{B\mu}(M, \mathfrak{K})$, and that $[\mathtt{d}\rho(D), r(\xi)] = r(\xi')$ for all $\xi \in H^1_{B\mu}(M, \mathfrak{K})$, hence, in particular, for $\xi \in H^1_\partial(N, \mathfrak{K}) \subseteq H^1_{B\mu}(M, \mathfrak{K})$.

By Cauchy–Schwarz and the inequality (6.38), we have

$$|\langle r(\xi)\psi, r(\eta)\chi\rangle| \leq \|A^{3/2}\psi\| \|A^{3/2}\chi\| q_1(\xi) q_1(\chi) \tag{6.48}$$

for all $\psi, \chi \in \mathcal{D}(H^2)$ and $\xi, \eta \in H^1_{B\mu}(M, \mathfrak{g})$, where $A := \mathbf{1} + H$ and q_1 is the parallel Sobolev norm of (6.43). Further, by Proposition 6.22, the products $r(\xi)r(\eta)$ and $r(\eta)r(\xi)$ are well defined on $\mathcal{D}(H^2)$. Since

$$\langle \psi, [r(\xi), r(\eta)]\chi\rangle = -\langle r(\xi)\psi, r(\eta)\chi\rangle + \langle r(\eta)\psi, r(\xi)\chi\rangle,$$

it follows that the bilinear form

$$H^1_{B\mu}(M,\mathfrak{K}) \times H^1_{B\mu}(M,\mathfrak{K}) \to \mathbb{C}, \quad (\xi,\eta) \mapsto \langle \psi, [r(\xi), r(\eta)]\chi \rangle$$

is continuous with respect to the parallel Sobolev norm q_1. In particular, its restriction to

$$H^1_\partial(N,\mathfrak{K}) \subseteq H^1_{B\mu}(M,\mathfrak{K})$$

is continuous with respect to q_1.

Similarly, using Cauchy–Schwarz and (6.38), we find for $\xi,\eta \in H^1_\partial(N,\mathfrak{K})$ that

$$|\langle \chi, r([\xi,\eta])\psi\rangle| \leq \|\chi\| \|A^{3/2}\psi\| q_1([\xi,\eta]).$$

Since the Lie bracket on $H^1_\partial(N,\mathfrak{K})$ is *separately* continuous for the norm q_1 by Proposition 6.25, it follows that the bilinear form $H^1_\partial(N,\mathfrak{K}) \times H^1_\partial(N,\mathfrak{K}) \to \mathbb{C}$ defined by

$$(\xi,\eta) \mapsto \langle \chi, r([\xi,\eta])\psi \rangle$$

is *separately* continuous with respect to q_1.

As the cocycle $\omega(\xi,\eta) = \langle D\xi, \eta\rangle_\mu$ extends to a bilinear map on $H^1_\partial(N,\mathfrak{K})$ that is continuous with respect to q_1, the bilinear form

$$(\xi,\eta) \mapsto \big\langle \chi, \big([r(\xi), r(\eta)] - r([\xi,\eta]) - i\omega(\xi,\eta)\big)\psi \big\rangle$$

is *separately* continuous for the q_1-topology. Since it vanishes on the dense subset $\Gamma_c(N,\mathfrak{K}) \subseteq H^1_\partial(N,\mathfrak{K})$, it is identically zero. It follows that

$$[r(\xi), r(\eta)]\psi = r([\xi,\eta])\psi + i\omega(\xi,\eta)\psi$$

for all $\psi \in \mathcal{D}(H^2)$. The operator $r([\xi,\eta]) + i\omega(\xi,\eta)\mathbf{1}$ with domain containing $\mathcal{D}(H)$ is thus an essentially skew-adjoint extension of the operator $[r(\xi), r(\eta)]$ with domain $\mathcal{D}(H^2)$. ■

6.5.1 Semibounded representations

The concept of a semibounded representation, introduced in [73,75], is much stronger than that of a positive energy condition. As results in [81] show, it provides enough regularity to lead to a sufficient supply of C^*-algebraic tools to decompose representations as direct integrals.

Definition 6.31 (Semibounded representations). A smooth representation $(\rho, \mathcal{H})$ of a locally convex Lie group G is called *semibounded* if the function

$$s_\rho \colon \mathfrak{g} \to \mathbb{R} \cup \{\infty\}, \quad s_\rho(x) := \sup(\operatorname{Spec}(i\,\mathrm{d}\rho(x))) \tag{6.49}$$

is bounded on a neighborhood of some point $x_0 \in \mathfrak{g}$. Then, the set W_ρ of all such points x_0 is an open $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}$.

For Lie groups G which are locally exponential or whose Lie algebra $\mathfrak{g}$ is barrelled[1], a semibounded representation is bounded if and only if $W_\rho = \mathfrak{g}$ [73, Theorem 3.1 and Proposition 3.5]. The positive energy representation r of $H^1_\partial(N, \mathfrak{K})$ fulfills the following semiboundedness condition.

Proposition 6.32. *Let $\xi \in L^2_{B\mu}(M, \mathfrak{K})$, and let $t > 0$. Then*

$$-\|\xi\|_{B_1\mu} - 9\|\xi\|^2_\mu d_{\mathfrak{k}}(d_M + 1)^2/t \leq \inf(\mathrm{Spec}(ir(tD \oplus \xi))).$$

In particular, the spectrum of $tH \pm ir(\xi)$ is bounded below for every $t > 0$, and this bound is uniform on an open neighborhood of D in $L^2_{B\mu}(M, \mathfrak{K}) \rtimes \mathbb{R}D$.

Proof. Using Proposition 6.20, one finds that the map $L^2_{B\mu}(M, \mathfrak{K}) \to \mathbb{C}$ defined by

$$\xi \mapsto \langle \Gamma_\xi \pm ir(\xi)\rangle_\psi$$

is continuous for every $\psi \in \mathcal{D}(H)$, and every $\varepsilon > 0$. It is non-negative on the dense subspace $\Gamma_c(M, \mathfrak{K})$ by Theorem 6.17, and hence on all of $L^2_{B\mu}(M, \mathfrak{K})$ by continuity. If $\|\xi\|_\mu = 0$, then $r(\xi) = 0$, and the proposition holds trivially. If $\|\xi\|_\mu \neq 0$ and $\varepsilon := t/\|\xi\|_\mu$, then

$$\Gamma_\xi = tH + \|\xi\|_{B_\varepsilon\mu}\mathbf{1}$$

satisfies $0 \leq \langle \Gamma_\xi \pm ir(\xi)\rangle_\psi$, and thus

$$-\|\xi\|_{B_\varepsilon\mu}\|\psi\|^2 \leq \langle \psi, tH \pm ir(\xi), \psi\rangle.$$

Since

$$\|\xi\|_{B_\varepsilon\mu} \leq \|\xi\|_{B_1\mu} + 9\|\xi\|_\mu d_{\mathfrak{k}}(d_M + 1)^2/\varepsilon,$$

the result follows by substituting $\varepsilon = t/\|\xi\|_\mu$. ∎

Corollary 6.33. *The positive energy representation $\mathtt{d}\rho$ of the Lie algebra*

$$\big(\mathbb{R}C \oplus_\omega \Gamma_c(M, \mathfrak{K})\big) \rtimes \mathbb{R}D$$

is semibounded and the cone W_ρ contains the open half space

$$\big(\mathbb{R}C \oplus_\omega \Gamma_c(M, \mathfrak{K})\big) - \mathbb{R}_+ D.$$

Proof. This follows from Proposition 6.32 because $\mathtt{d}\rho$ comes from a group representation, the central element C is represented by a constant, and the inclusion $\Gamma_c(M, \mathfrak{K}) \hookrightarrow L^2_{B\mu}(M, \mathfrak{K})$ is continuous. ∎

[1] These are the locally convex spaces for which the Uniform Boundedness Principle holds. All Fréchet spaces and locally convex direct limits of Fréchet spaces are barrelled, which includes, in particular, LF spaces of test functions on noncompact manifolds.

6.5.2 Analytic vectors

A vector ψ in a Banach space $\mathcal{X}$ is called *analytic* for an unbounded operator A on $\mathcal{X}$ if $\psi \in \bigcap_{n\in\mathbb{N}} \mathcal{D}(A^n)$, and the series $\sum_{n=0}^{\infty} \frac{s^n}{n!}\|A^n\psi\|$ has positive radius of convergence $R_A > 0$.

Lemma 6.34. *Let $\xi \in H^2_\partial(N, \mathfrak{K})$, and consider H and $r(\xi)$ as unbounded operators on $\mathcal{H}$. If $\psi \in \mathcal{H}$ is an analytic vector for H, then it is also analytic for $r(\xi)$. If ψ has radius of convergence R_H for H, then the exponential series*

$$\exp(r(\xi))\psi = \sum_{n=0}^{\infty} \frac{1}{n!} r(\xi)^n \psi$$

is absolutely convergent on the ball defined by

$$p_2(\xi) < -\frac{1}{2c_{\mathfrak{k}}} \log\left(1 - \frac{(2c_{\mathfrak{k}})^2}{(c_{\mathfrak{k}}+1)^2}\left(1 - \exp\left(-\frac{(c_{\mathfrak{k}}+1)^2}{2c_{\mathfrak{k}}} R_H\right)\right)\right). \tag{6.50}$$

Proof. We apply [86, Theorem 1] to $r(\xi)$ and $A = \mathbf{1} + H$, considered as unbounded operators on the Banach space $\mathcal{H}_{1/2}$. For $\xi \in H^1_{B\mu}(M, \mathfrak{K})$ and $\psi \in \mathcal{D}(H^2) \subseteq \mathcal{H}_{1/2}$, the inequality (6.38) yields

$$\|r(\xi)\psi\|_{1/2} \leq q_1(\xi)\|A\psi\|_{1/2}. \tag{6.51}$$

By (6.35), we have $\mathrm{ad}_{r(\xi)} A = -ir(D\xi)$. If $\xi \in H^2_\partial(N, \mathfrak{K})$, then by definition, both ξ and $D\xi$ are in $H^1_\partial(N, \mathfrak{K})$. It follows that also $\mathrm{ad}^{n-1}_\xi(D\xi) \in H^1_\partial(N, \mathfrak{K})$ for $n \geq 1$. By (6.47) and induction, we find

$$\mathrm{ad}^n_{r(\xi)}(A) = -i\,\mathrm{ad}^{n-1}_{r(\xi)}(r(D\xi)) = -i\,r(\mathrm{ad}^{n-1}_\xi(D\xi)) + \omega(\xi, \mathrm{ad}^{n-2}_\xi(D\xi))\mathbf{1} \tag{6.52}$$

as an equality of unbounded operators from $\mathcal{D}(H^2)$ to $\mathcal{H}_{1/2}$. From (6.40) and (6.45), we infer that

$$\|\mathrm{ad}^n_\xi(D\xi)\|_{B_\varepsilon\mu} \leq (c_{\mathfrak{k}}\|\xi\|_\infty)^n \|D\xi\|_{B_\varepsilon\mu}, \tag{6.53}$$

$$q_1\big(\mathrm{ad}^n_\xi(D\xi)\big) \leq (2c_{\mathfrak{k}} q_{C^1}(\xi))^n q_1(D\xi). \tag{6.54}$$

Next we estimate $\|\mathrm{ad}^n_{r(\xi)}(A)\psi\|_{1/2}$. Applying (6.52) and noting that

$$|\omega(\xi, \eta)| = |\langle D\xi, \eta\rangle_\mu| \leq \|D\xi\|_\mu \|\eta\|_\mu \quad \text{and} \quad \|D\xi\|_\mu \leq \|D\xi\|_{B_\varepsilon\mu},$$

the second term on the right-hand side of (6.52) satisfies

$$\|\omega(\xi, \mathrm{ad}^{n-2}_\xi(D\xi))\psi\|_{1/2} \leq (c_{\mathfrak{k}}\|\xi\|_\infty)^{n-2} \|D\xi\|^2_{B_\varepsilon\mu} \|\psi\|_{1/2}. \tag{6.55}$$

Applying (6.51) and (6.54) to the first term on the right-hand side of (6.52), we find

$$\|r(\mathrm{ad}^{n-1}_\xi(D\xi))\psi\|_{1/2} \leq (2c_{\mathfrak{k}} q_{C^1}(\xi))^{n-1} q_1(D\xi)\|A\psi\|_{1/2}. \tag{6.56}$$

Combining (6.55) and (6.56) with (6.52), and using that $\|\psi\|_{1/2} \le \|A\psi\|_{1/2}$, we find

$$\|\mathrm{ad}^n_{r(\xi)}(A)\psi\|_{1/2} \le c_n \|A\psi\|_{1/2},$$

with

$$c_n = q_1(D\xi)\big(\|D\xi\|_{B_\varepsilon\mu} + (2c_{\mathfrak{k}} q_{C^1}(\xi))\big)(2c_{\mathfrak{k}} q_{C^1}(\xi))^{n-2}. \tag{6.57}$$

Since the series

$$\nu(s) := \sum_{n=1}^{\infty} \frac{c_n}{n!} s^n$$

has positive radius of convergence, we may now fix some $t_0 > 0$ with $\nu(t_0) < 1$ and assume that $0 \le s, t \le t_0$. Applying [86, Theorem 1] to $\mathcal{H}_{1/2}$ guarantees that for

$$\varpi(s) := \int_0^s (1 - \nu(t))^{-1} dt,$$

we have

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|r(\xi)^n \psi\|_{1/2} \le \sum_{n=0}^{\infty} \frac{(c \cdot \varpi(s))^n}{n!} \|A^n \psi\|_{1/2}, \quad \text{with } c := q_1(\xi)$$

as in (6.51). Since $\|r(\xi)^n\psi\| \le \|r(\xi)^n\psi\|_{1/2}$ and $\|A^n\psi\|_{1/2} \le \|A^{n+1}\psi\|$, this yields

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \|r(\xi)^n \psi\| \le \sum_{n=0}^{\infty} \frac{(c \cdot \varpi(s))^n}{n!} \|A^{n+1} \psi\|. \tag{6.58}$$

To get an explicit estimate on the radius of convergence, note that all norms of (derivatives of) ξ occurring in (6.57) are dominated by $p_2(\xi)$ (cf. (6.44)). The estimate $c_n \le ab^n$ with

$$a := (1 + 2c_{\mathfrak{k}})/(2c_{\mathfrak{k}})^2 \quad \text{and} \quad b := 2c_{\mathfrak{k}} p_2(\xi)$$

yields $\nu(s) \le a(e^{bs} - 1)$. Accordingly, $\nu(t_0) < 1$ is ensured if

$$bt_0 < \log\left(1 + \frac{1}{a}\right) = -\log\left(1 - \frac{1}{1+a}\right).$$

In particular, $s = 1$ is allowed if $p_2(\xi) < \frac{1}{2c_{\mathfrak{k}}} \log(1 + \frac{1}{a})$. Substituting this in

$$\varpi(s) = \int_0^s (1 - \nu(t))^{-1} dt$$

and integrating, we obtain

$$\varpi(s) \le -\frac{1}{(1+a)b} \log((1+a)e^{-bs} - a). \tag{6.59}$$

If ψ is an analytic vector for H, it is analytic for $A = \mathbf{1} + H$ with the same radius of convergence R_H. The right-hand side of (6.58) therefore converges absolutely if $c \cdot \varpi(s) < R_H$, where $c = q_1(\xi)$. Since $q_1(\xi) \leq p_2(\xi)$, we find $\frac{c}{b} \leq \frac{1}{2c_{\mathfrak{k}}}$, and hence $\frac{c}{(1+a)b} \leq 2c_{\mathfrak{k}}/(c_{\mathfrak{k}} + 1)^2$. Substituting this in (6.59), we find that $c \cdot \varpi(s) \leq R_H$ if

$$bs \leq -\log\left(1 - \frac{1}{1+a}\left(1 - \exp\left(-\frac{(c_{\mathfrak{k}}+1)^2}{2c_{\mathfrak{k}}} R_H\right)\right)\right) < -\log\left(1 - \frac{1}{1+a}\right).$$

Putting $s = 1$, and substituting a and b in the above equation, we find that (6.58) converges if $p_2(\xi)$ satisfies (6.50). ■