Chapter 7

The localization theorem

In this section, we use the continuity and analyticity results from Chapter 6 to prove a *localization theorem*. Our main result reduces the classification of positive energy representations of the identity component $\Gamma_c(M, \mathcal{K})_0$ to the case where the base manifold M is one-dimensional. We start in the setting of a fixed point free $\mathbb{R}$-action on the manifold M, and extend this to more general Lie group actions in Section 7.5.

7.1 Statement and discussion of the theorem

Theorem 7.1 (Localization theorem). *Let $\pi\colon \mathcal{K} \to M$ be a Lie group bundle whose fibers are 1-connected semisimple. Let $\gamma_{\mathcal{K}}\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ be a homomorphism that defines a smooth action on $\mathcal{K}$, and induces a fixed-point free flow γ_M on M. Then, for every projective positive energy representation*

$$\bar{\rho}\colon \Gamma_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$$

of the connected gauge group $\Gamma_c(M, \mathcal{K})_0$, there exists a one-dimensional, closed, embedded, flow-invariant submanifold $S \subseteq M$ such that $\bar{\rho}$ factors through a projective positive energy representation $\bar{\rho}_S$ of the connected Lie group $\Gamma_c(S, \mathcal{K})$. The diagram

$$\begin{array}{ccc} \Gamma_c(M, \mathcal{K})_0 & \xrightarrow{\ \bar{\rho}\ } & \mathrm{PU}(\mathcal{H}) \\ \big\downarrow{\scriptstyle r_S} & \nearrow{\scriptstyle \bar{\rho}_S} & \\ \Gamma_c(S, \mathcal{K}) & & \end{array}$$

commutes, where $r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ is the restriction homomorphism.

Remark 7.2. It is convenient to define $\Gamma_c(\emptyset, \mathcal{K}) := \{\mathbf{1}\}$, so that the above theorem holds for the trivial representation with $S = \emptyset$.

Remark 7.3 (Localization for the simply connected cover). In fact, we will prove a slightly stronger result: every projective positive energy representation

$$\bar{\rho}\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$$

of the simply connected cover of $\Gamma_c(M, \mathcal{K})_0$ factors through $\widetilde{r}_S := r_S \circ q_\Gamma$, where $q_\Gamma\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(M, \mathcal{K})$ is the covering map and

$$r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$$

the restriction. This strengthening of Theorem 7.1 is needed in Part II, where we handle localization for gauge groups on manifolds with boundary.

Note that M is not required to be compact or connected, and that the fibers of $\mathcal{K} \to M$ are not required to be compact. The result for noncompact M is a major feature, which we will use extensively later on (see Chapter 9 and Part II). Allowing noncompact fibers, however, is not a big step. Indeed, noncompact simple fibers result in trivial representations by Theorem 6.2, so we already know that the theorem holds with $S = \emptyset$ in that case. Before proceeding with the proof in Section 7.2, we show that the assumption of 1-connectedness of the fibers is not essential.

Remark 7.4 (Non-simply connected fibers). Suppose that the typical fibers of the bundle $\mathcal{K} \to M$ are connected, but not necessarily simply connected. Let K_i be the typical fiber over the connected component M_i of M, and let $\widetilde{K}_i$ be its 1-connected universal cover. The kernel $\pi_1(K_i)$ of the covering map $\widetilde{K}_i \twoheadrightarrow K_i$ is a finite, central subgroup, yielding a central extension

$$\pi_1(K_i) \hookrightarrow \widetilde{K}_i \twoheadrightarrow K_i. \tag{7.1}$$

The natural inclusion $\mathrm{Aut}(K_i) \hookrightarrow \mathrm{Aut}(\widetilde{K}_i)$, obtained by the canonical lift of automorphisms, yields a Lie group bundle $\widetilde{\mathcal{K}}_i \to M_i$ with fiber $\widetilde{K}_i$ over each M_i, and hence a Lie group bundle $\widetilde{\mathcal{K}} \to M$ over M. It comes with a natural bundle map $\widetilde{\mathcal{K}} \to \mathcal{K}$ over the identity of M, which restricts to the universal covering map on every fiber. The kernel $\mathcal{Z} \subseteq \widetilde{\mathcal{K}}$ of this map is a bundle of discrete, abelian groups, whose fibers over M_i are isomorphic to $\pi_1(K_i)$. Analogous to (7.1), we thus obtain an exact sequence of Lie group bundles

$$\mathcal{Z} \hookrightarrow \widetilde{\mathcal{K}} \twoheadrightarrow \mathcal{K}.$$

The 1-parameter group $\gamma_{\mathcal{K}}\colon \mathbb{R} \to \mathrm{Aut}(\mathcal{K})$ lifts to $\gamma_{\widetilde{\mathcal{K}}}\colon \mathbb{R} \to \mathrm{Aut}(\widetilde{\mathcal{K}})$ with the same infinitesimal generator $\mathbf{v} \in \Gamma(M, \mathfrak{a}(\mathfrak{K}))$ (cf. Remark 4.8). As every smooth section $\xi \in \Gamma_c(M, \mathcal{K})_0$ lifts to a section of $\widetilde{\mathcal{K}}$ because the natural map $\Gamma_c(M, \widetilde{\mathcal{K}}) \to \Gamma_c(M, \mathcal{K})$ is a covering morphism of Lie groups, the projection $\widetilde{\mathcal{K}} \to \mathcal{K}$ yields a surjective Lie group homomorphism, and hence an exact sequence

$$\Gamma_c(M, \mathcal{Z}) \hookrightarrow \Gamma_c(M, \widetilde{\mathcal{K}}) \to \Gamma_c(M, \mathcal{K}). \tag{7.2}$$

Since the fibers of $\mathcal{Z}$ are discrete, the group $\Gamma_c(M_i, \mathcal{Z}_i)$ of compactly supported sections of $\mathcal{Z}_i \to M_i$ is trivial if M_i is noncompact. If M_i is compact, $\Gamma_c(M_i, \mathcal{Z}_i)$ can be identified with $\pi_1(K_i)^{\pi_1(M_i)}$, the fixed point subgroup of $\pi_1(K_i)$ under the monodromy action $\pi_1(M_i) \to \mathrm{Aut}(\pi_1(K_i))$. We thus obtain an isomorphism

$$\Gamma_c(M, \mathcal{Z}) \simeq {\prod_{i\in I}}' \pi_1(K_i)^{\pi_1(M_i)} \tag{7.3}$$

of discrete groups where $\prod'_{i\in I} \pi_1(K_i)^{\pi_1(M_i)}$ denotes the weak direct product of the finite abelian groups $\pi_1(K_i)^{\pi_1(M_i)}$ (all tuples with finitely many non-zero entries), running over all i for which the connected component M_i is compact. In particular, it follows from (7.2) and (7.3) that projective positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ correspond to projective positive energy representations of $\Gamma_c(M, \widetilde{\mathcal{K}})_0$ that are trivial on

$$Z_{[M]} := \Gamma_c(M, \mathcal{Z}) \cap \Gamma_c(M, \widetilde{\mathcal{K}})_0.$$

Note that the embedding $S \hookrightarrow M$ yields a "diagonal" morphism $Z_{[M]} \to Z_{[S]}$. The term "diagonal" is justified by the special case where $\mathcal{K}$ is a trivial bundle over a compact, connected manifold M. Then, the embedded 1-dimensional submanifold $\emptyset \neq S \subseteq M$ is the disjoint union of N circles, and $Z_{[M]} \simeq \pi_1(K)$ can literally be identified with the diagonal subgroup of $Z_{[S]} \simeq \pi_1(K)^N$.

Combining Theorem 7.1 with Remark 7.4, we obtain a localization theorem for bundles whose fibers are not necessarily simply connected.

Corollary 7.5 (Localization theorem for non-simply connected fibers). *Suppose that the fibers of $\mathcal{K} \to M$ are connected, but not necessarily simply connected. Then, $\bar{\rho}$ arises by factorization from a projective positive energy representation of $\Gamma_c(S, \widetilde{\mathcal{K}})$ that is trivial on the image of $Z_{[M]}$ in $Z_{[S]}$.*

Remark 7.6 (Abelian groups). In the localization Theorem 7.1 we have assumed that the fiber Lie group K is semisimple. We now explain why this is crucial and that there is no localization for abelian target groups, so that the localization theorem does not extend to bundles with general compact fiber Lie algebras. To this end, let $K = (\mathfrak{k}, +)$ be a finite-dimensional real vector space and fix a positive definite scalar product κ on $\mathfrak{k}$. Further, let M be a smooth manifold and consider the Lie group $G := \mathfrak{g} := C^\infty_c(M, \mathfrak{k})$, which can be identified with the group of compactly supported sections of the trivial bundle $\mathcal{K} = M \times K$. We also fix a smooth flow $\gamma_M : \mathbb{R} \to \mathrm{Diff}(M)$, its generator $\mathbf{v}_M \in \mathcal{V}(M)$, and a γ_M-invariant positive Radon measure μ on M. Then

$$\kappa_{\mathfrak{g}}(\xi, \eta) := \int_M \kappa(\xi, \eta) d\mu$$

defines a positive semidefinite scalar product on $\mathfrak{g}$, invariant under the $\mathbb{R}$-action on $\mathfrak{g}$ given by

$$\alpha_t \xi := \xi \circ \gamma_M(t),$$

whose infinitesimal generator is $D\xi = \mathcal{L}_{\mathbf{v}_M} \xi$. Then

$$\omega(\xi, \eta) := \kappa_{\mathfrak{g}}(D\xi, \eta) = \int_M \kappa(\mathcal{L}_{\mathbf{v}_M} \xi, \eta) d\mu$$

is an $\mathbb{R}$-invariant skew-symmetric form on the abelian Lie algebra $\mathfrak{g}$, hence a Lie algebra 2-cocycle. Combining in [85, Theorems 3.2 and 5.9], it now follows that all these cocycles are obtained from projective positive energy representations of the groups $G \rtimes_\alpha \mathbb{R}$. This shows that, for abelian fibers, no restrictions on the measure μ exist.

Example 7.7. We consider the Lie algebra $\mathfrak{g} = C^\infty(\mathbb{T}^d, \mathfrak{k})$, $\mathfrak{k}$ compact simple and $\alpha_t(\xi) = \xi \circ \gamma_t$, where

$$\gamma_t(z_1, \ldots, z_d) = \left(e^{2\pi i t\theta_1} z_1, \ldots, e^{2\pi i t\theta_{d-1}} z_{d-1}, e^{2\pi i t} z_d\right).$$

This means that $\mathbf{v}_M$ is the invariant vector field on the Lie group $\mathbb{T}^d \cong \mathbb{R}^d/\mathbb{Z}^d$ with exponential function

$$\exp(x_1, \ldots, x_d) = \left(e^{2\pi i x_1}, \ldots, e^{2\pi i x_d}\right)$$

whose value in $\mathbf{1}$ is given by $x = (\theta_1, \ldots, \theta_{d-1}, 1)$. This action has a closed orbit if and only if the one-parameter group $A := \exp(\mathbb{R}x)$ is closed, which is equivalent to $\theta_j \in \mathbb{Q}$ for all j.

If this condition is satisfied, then $A \cong \mathbb{T}$ and the α-orbits are the A-cosets in the group $\mathbb{T}^d$. This situation is also studied by Torresani in [105]. If this condition is not satisfied, then the localization theorem implies that there are no non-trivial projective positive energy representations.

Remark 7.8. The localization theorem also yields partial information for flows with fixed points, and for manifolds with boundary.

(a) If the vector field $\mathbf{v}_M$ has zeros, then

$$M^\times := \left\{x \in M : \mathbf{v}_M(x) \neq 0\right\}$$

is an open flow-invariant submanifold of M and the localization theorem applies to the bundle $\mathfrak{K}|_{M^\times}$. In this context, this theorem does not provide a complete reduction to the one-dimensional case for two reasons. One is that the representations of $\Gamma_c(M^\times, \mathfrak{K})$ do not uniquely determine those of $\Gamma_c(M, \mathfrak{K})$ and the other reason is that the 1-dimensional submanifold S of $M^\times$ need not be closed in M, so that the extendability of the representation of $\Gamma_c(M^\times, \mathfrak{K})$ to the Lie algebra $\Gamma_c(M, \mathfrak{K})$ provides "boundary conditions at infinity" for the corresponding representations of $\Gamma_c(S, \mathfrak{K})$. We will further explore these boundary conditions in future work.

(b) Similarly, if $\bar{M}$ is a manifold with boundary, then both its interior $M = \bar{M} \setminus \partial M$ and its boundary ∂M are invariant under the flow. In Part II of this series of papers, we apply the localization theorem to M and ∂M separately, and combine the information to obtain classification results for positive energy representations of the gauge group $\Gamma(\bar{M}, \mathcal{K})$. The main challenge here is that although every projective unitary representation of $\Gamma(\bar{M}, \mathcal{K})$ automatically restricts to $\Gamma_c(M, \mathcal{K})$, we heavily rely on the positive energy condition to obtain a representation of $\Gamma_c(\partial M, \mathcal{K})$.

Example 7.9. A typical example of a flow with fixed points is the 2-sphere $M = \mathbb{S}^2$, where

$$\gamma_{M,t}(x, y, z) = \begin{pmatrix} \cos(t) & \sin(t) & 0 \\ -\sin(t) & \cos(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{7.4}$$

is the rotation around the z-axis with unit angular velocity, and $P = \mathbb{S}^2 \times \mathfrak{k}$ is the trivial bundle. The lift of the infinitesimal action is then given by

$$\mathbf{v}(x, y, z) = (y\partial_x - x\partial_y) + A(x, y, z), \tag{7.5}$$

where the first part is the horizontal lift of the infinitesimal action corresponding to (7.4), and the second part is the vertical vector field corresponding to a smooth function $A: \mathbb{S}^2 \to \mathfrak{k}$.

Then, $M^\times = \mathbb{S}^2 \setminus \{(0, 0, \pm 1)\}$, and the integral curves on $\mathbb{S}^2$ are precisely the circles of latitude. Therefore, S is either compact and a finite union of circles, or it is noncompact and an infinite union of circles. More precisely,

$$S = \{(x, y, z) \in \mathbb{S}^2 : z \in J\},$$

where $J \subset (-1, 1)$ is a discrete set that has at most two accumulation points ± 1, corresponding to the two fixed points of the circle action. We return to this example in Section 9.3.

7.2 Localization at the Lie algebra level

The remainder of this chapter is devoted to the proof of Theorem 7.1. We start by proving the statement at the level of Lie algebras. This proceeds through several lemmas. In the first one, relying heavily on Theorem 6.30 and Lemma 6.34, we derive integrality results for the flow-invariant measure μ of Section 5.2.2.

Lemma 7.10. *Suppose that the fibers of $\mathfrak{K} \to M$ are simple Lie algebras. Consider a good flow box $U \simeq U_0 \times I \subseteq M$ around $x \in M$ in the sense of Definition 5.1, so that the restriction of the invariant measure μ to $U \simeq U_0 \times I$ takes the form*

$$\mu|_U = \mu_0 \otimes dt.$$

Then, for every measurable subset $E \subseteq U_0$,

$$\mu_0(E) \in \frac{1}{2\pi}\mathbb{N}_0.$$

Proof. We may assume without loss of generality that the fibers of $\mathfrak{K}$ over U are compact, as $\mu_0(E)$ would otherwise be zero by Corollary 5.5. Let $\chi_E: U_0 \to \{0, 1\}$ be the indicator function of E. Consider the Lie algebra homomorphism

$$\iota_E: \mathbb{R}C \oplus_\omega H^2_\partial(I, \mathfrak{k}) \to \mathbb{R}C \oplus_\omega H^2_\partial(U, \mathfrak{k}), \quad zC \oplus \xi \mapsto zC \oplus \chi_E \xi$$

whose continuity follows from Corollary 6.29. If we pull back the representation r of $\mathbb{R}C \oplus_\omega H^2_\partial(U, \mathfrak{k})$ of Theorem 6.30 along ι_E, we obtain a projective $*$-representation of the Banach–Lie algebra $\mathfrak{h} := H^2_\partial(I, \mathfrak{k})$. By Lemma 6.34, its space of analytic vectors is dense in $\mathcal{H}$.

Since $\mathfrak{h}$ consists of functions $I \to \mathfrak{k}$ and it contains $C^\infty_c(I, \mathfrak{k})$, the fact that $\mathfrak{z}(\mathfrak{k}) = \{0\}$ implies that the center of $\mathfrak{h}$ is trivial. As $\mathfrak{h}$ is a Banach–Lie algebra, it is in particular locally exponential, so there exists a 1-connected Lie group H with Lie algebra $\mathfrak{h}$ by [71, Theorem IV.3.8] (see [30] for a complete proof).

Now Theorem 2.18 provides a smooth, projective, unitary representation

$$\pi \colon H \to \mathrm{PU}(\mathcal{H}).$$

By Theorem 5.7, the corresponding Lie algebra cocycle is given by

$$\begin{aligned}\omega(\xi, \eta) &= -\int_U \kappa(\chi_E \xi, \nabla_{\mathbf{v}_M}(\chi_E \eta)) d\mu = -\int_{U_0 \times I} \kappa(\chi_E \xi, \chi_E \eta') d\mu_0 dt \\ &= -\mu_0(E) \int_I \kappa(\xi, \eta') dt = \mu_0(E) \int_I \kappa(\xi', \eta) dt.\end{aligned}$$

Theorem 2.13 now implies the existence of a central Lie group extension $H^\sharp$ of H by $\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ with Lie algebra $\mathfrak{h}^\sharp_\omega = \mathbb{R}C \oplus_\omega \mathfrak{h}$.

This in turn implies integrality conditions on the values of $\mu_0(E)$. To see how these can be obtained, we associate to ω the corresponding left invariant 2-form Ω on H with $\Omega_\mathbf{1} = \omega$. This form defines a *period homomorphism*

$$\mathrm{per}_\omega \colon \pi_2(H) \to \mathbb{R}, \quad [\sigma] \mapsto \int_\sigma \Omega$$

(cf. [69, Definition 5.8]) and [69, Lemma 5.11] implies that

$$\mathrm{im}(\mathrm{per}_\omega) \subseteq 2\pi\mathbb{Z}.$$

Since the rescaling map

$$\gamma \colon C^\infty_c(I, \mathfrak{k}) \to C^\infty_c((-\pi, \pi), \mathfrak{k}), \quad \gamma(\xi)(\theta) = \xi\left(\frac{T}{2\pi}\theta\right)$$

from the interval $I = (-T/2, T/2)$ to the interval $(-\pi, \pi)$ is an isomorphism of Lie algebras, the cocycle $\int_I \kappa(\xi', \eta) dt$ on $C^\infty_c(I, \mathfrak{k})$ has the same period group as the cocycle $\int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ on $C^\infty_c((-\pi, \pi), \mathfrak{k})$. In [70, Lemma V.11], it was shown that this, in turn, has the same period group as the cocycle $\int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ on $C^\infty_c(\mathbb{S}^1, \mathfrak{k})$. By [68, Theorem II.5], the period group of $\frac{1}{2\pi} \int_{-\pi}^{\pi} \kappa(\xi', \eta) d\theta$ is $2\pi\mathbb{Z}$, provided that κ is normalized as in (4.2). Combining all this, we conclude that $\mu_0(E) \in \frac{1}{2\pi}\mathbb{Z}$. ∎

As the measure $2\pi\mu_0$ takes integral values, the following proposition shows that it is automatically discrete.

Proposition 7.11. *Let ζ be a locally finite, regular Borel measure on a locally compact space Σ. If ζ takes values in $\mathbb{N}_0 \cup \{\infty\}$, then there exists a locally finite subset $\Lambda \subseteq \Sigma$ and natural numbers $c_x = \zeta(\{x\})$ such that*

$$\zeta = \sum_{x \in \Lambda} c_x \delta_x.$$

Proof. By regularity, ζ is determined by its values on compact subsets, so it suffices to assume that Σ is compact and to show that, in this case, ζ is a finite sum of Dirac measures.

Let $\mathcal{F}$ be the family of compact subsets of full measure. For $F_1, F_2 \in \mathcal{F}$, we have

$$\mu(F_1 \setminus F_2) = \mu(F_2 \setminus F_1) = 0,$$

so that $F_1 \cap F_2$ also has full measure. This shows that $\mathcal{F}$ is closed under finite intersections. We show that

$$C := \bigcap_{F \in \mathcal{F}} F$$

has full measure. Let V be an open set containing C. Since the open complements F^c cover the compact set V^c, there exist finitely many F_i such that $F_1^c \cup \cdots \cup F_k^c \supseteq V^c$, and hence $F_1 \cap \cdots \cap F_k \subseteq V$. Since $\mathcal{F}$ is closed under finite intersections, every open set V containing C has full measure. By regularity, we conclude that C has full measure itself.

Pick $x \in C$. For any open neighborhood U of x in C, the minimality of C implies that $\zeta(C \setminus U) < \zeta(C)$, so that $\zeta(U) > 0$. Let U be an open neighborhood of x in C for which $\zeta(U)$ is minimal; here we use that the values of ζ are contained in $\mathbb{N}_0$. For any smaller open neighborhood $V \subseteq U$ of x in C we then have $\zeta(V) = \zeta(U)$ and therefore $\zeta(U \setminus V) = 0$. This implies that $\zeta(K) = 0$ for any compact subset $K \subseteq U \setminus \{x\}$ and hence that $\zeta(U \setminus \{x\}) = 0$ by the regularity of ζ. Now the minimality of C entails that $C = \{x\} \cup (C \setminus U)$. Since $x \in C$ was arbitrary, it follows that C is discrete, hence finite: $C = \{x_1, \ldots, x_k\}$. Accordingly, the restriction of ζ to a compact subset is the finite sum

$$\zeta = \sum_{j=1}^{k} \zeta(\{x_j\}) \delta_{x_j}$$

of Dirac measures. ∎

Recall from Theorem 4.9 that the bundle $\mathfrak{K} \to M$ of semisimple Lie algebras gives rise to a bundle $\widehat{\mathfrak{K}} \to \widehat{M}$ of simple Lie algebras with $\Gamma_c(M, \mathfrak{K}) \simeq \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$. By Remark 4.10, it inherits the 1-parameter group of automorphisms.

Lemma 7.12. *If the flow on M has no fixed points, then the support $\widehat{S}$ of μ is a one-dimensional, flow-invariant, closed embedded submanifold of $\widehat{M}_{\mathrm{cpt}}$, the part of $\widehat{M}$ over which the fibers of $\widehat{\mathfrak{K}}$ are compact.*

Proof. Since the flow on M has no fixed points, the vector field $\mathbf{v}_M$ on M has no zeros. As the same holds for its lift to $\widehat{M}$, every point $x \in \widehat{M}$ is contained in a good flow box $U \cong U_0 \times I$ in the sense of Definition 5.1. In any such flow box, the measure μ is of the form $\mu_0 \otimes dt$, where μ_0 is a regular measure on U_0. From Lemma 7.10 and Proposition 7.11, we conclude that μ_0 has finite support in U_0, so that $\widehat{S} \cap U \cong F \times I$, where $F \subseteq U_0$ is a finite subset. This implies that $\widehat{S}$ is a one-dimensional, closed embedded submanifold invariant under the flow on $\widehat{M}$. The final statement follows from Theorem 6.2. ∎

Combined with Corollary 6.3, this shows that Theorem 7.1 holds at the level of Lie algebras.

Lemma 7.13. *There exists a 1-dimensional, closed, embedded, flow-invariant submanifold $S \subseteq M$ such that the projective positive energy representation $d\rho$ of the Lie algebra $\Gamma_c(M, \mathfrak{K})$ factors through the restriction map $r_S^{\mathfrak{k}} \colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K})$.*

Proof. Combining Lemma 7.12 with Corollary 6.3 and Theorem 6.2, we conclude that the projective Lie algebra representation $d\rho$ of $\Gamma_c(\widehat{M}, \widehat{\mathfrak{K}})$ vanishes on the ideal

$$J_{\widehat{S}} := \{\xi \in \Gamma_c(\widehat{M}, \widehat{\mathfrak{K}}) : \xi|_{\widehat{S}} = 0\}.$$

It follows that the projective positive energy representation of $\Gamma_c(M, \mathfrak{K})$ vanishes on $J_S := \{\xi \in \Gamma_c(M, \mathfrak{K}) : \xi|_S = 0\}$, where $S \subseteq M$ is the image of $\widehat{S}$ under the finite, $\mathbb{R}$-equivariant covering map $\widehat{M} \to M$. Since $\widehat{S} \subseteq \widehat{M}$ is a 1-dimensional, closed, embedded, flow-invariant submanifold, the same holds for $S \subseteq M$. This implies that the projective representation factors through the restriction map

$$r_S^{\mathfrak{k}} \colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}),$$

which is a quotient map of locally convex spaces. ∎

7.3 Twisted loop groups

Let S be a one-dimensional, embedded, flow-invariant submanifold of M. Then, it is the disjoint union $S = \bigsqcup_{j \in J} S_j$ of its connected components S_j, which are either diffeomorphic to $\mathbb{R}$ (for a non-periodic orbit), or to $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (for a periodic orbit).

Fix $j \in J$ and let $K = K_j$ denote the fiber of $\mathcal{K}|_{S_j}$. If $S_j \cong \mathbb{R}$, then the bundle $\mathcal{K}|_{S_j}$ is trivial, i.e., equivalent to

$$S_j \times K \cong \mathbb{R} \times K.$$

This trivialization can be achieved $\mathbb{R}$-equivariantly, using an integral curve in the corresponding frame bundle $\mathrm{Aut}(\mathcal{K}) \to \mathbb{R}$, a principal bundle with fiber $\mathrm{Aut}(K)$.

The action of $\mathbb{R}$ on $\mathcal{K}$ is then simply given by

$$\gamma_t(x,k) = (x+t,k) \quad \text{for } t, x \in \mathbb{R} \text{ and } k \in K. \tag{7.6}$$

If $S_j \cong \mathbb{S}^1$ is a periodic orbit, then the universal covering map $q_j \colon \widetilde{S}_j \to S_j$ can be identified with the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$. If the period of the orbit S_j is T, then we scale the $\mathbb{R}$-action on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ by $1/T$, yielding

$$\gamma_{\mathbb{S}^1,t}([x]) = [x + t/T].$$

We have seen above that the pullback $q_j^*(\mathcal{K}|_{S_j})$ is equivariantly equivalent to the trivial bundle $\mathbb{R} \times K$ on which $\mathbb{R}$ acts by translation in the first factor. The action of the fundamental group $\pi_1(S_j) \cong \mathbb{Z}$ on $\mathbb{R} \times K$ is given by bundle automorphisms that commute with the $\mathbb{R}$-action; there exists an automorphism $\Phi \in \mathrm{Aut}(K)$ such that

$$n \cdot (x,k) = (x+n, \Phi^{-n}(k)) \quad \text{for all } n \in \mathbb{Z}.$$

Accordingly, we have an equivariant isomorphism

$$\mathcal{K}|_{S_j} \cong (\mathbb{R} \times K)/\sim,$$

where

$$(x,k) \sim (x+n, \Phi^{-n}(k))$$

for all $x \in \mathbb{R}, k \in K$ and $n \in \mathbb{Z}$. We write the equivalence classes as $[x,k]$, and we denote the K-bundle over $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ obtained in this way by

$$\mathcal{K}_\Phi := (\mathbb{R} \times K)/\sim, \quad \text{with } \mathcal{K}_\Phi \to \mathbb{R}/\mathbb{Z}$$

given by

$$[x,k] \mapsto [x] = x + \mathbb{Z}.$$

The $\mathbb{R}$-action is given in these terms by

$$\gamma_t([x,k]) = [x + t/T, k].$$

Note that

$$\gamma_T([x,k]) = [x+1,k] = [x, \Phi(k)],$$

so that Φ can be interpreted as a *holonomy*.

Recall that, for two automorphisms $\Phi, \Psi \in \mathrm{Aut}(K)$, the corresponding K-bundles $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ are equivalent if and only if the classes $[\Phi]$ and $[\Psi]$ are conjugate in the component group $\pi_0(\mathrm{Aut}(K))$, and they are $\mathbb{R}$-equivariantly isomorphic if and only if Φ and Ψ are conjugate in $\mathrm{Aut}(K)$. Indeed, any isomorphism $\Gamma_{\Psi,\Phi} \colon \mathcal{K}_\Phi \to \mathcal{K}_\Psi$ inducing the identity on the base is of the form

$$\Gamma_{\Psi,\Phi}([x,k]) = [x, \zeta_x(k)],$$

where $\zeta\colon \mathbb{R} \to \mathrm{Aut}(K)$ is smooth and satisfies

$$\zeta_{x+1} = \Psi^{-1} \circ \zeta_x \circ \Phi \quad \text{for all } x \in \mathbb{R}. \tag{7.7}$$

Such a smooth curve ζ exists if and only if $[\Phi]$ and $[\Psi]$ are conjugate in the finite group $\pi_0(\mathrm{Aut}(K))$. In particular, the set of equivalence classes of group bundles with fiber K over $\mathbb{S}^1$ corresponds to the set of conjugacy classes in the group $\pi_0(\mathrm{Aut}(K))$, which is finite for a semisimple compact Lie group K. This follows from the compactness of the group $\mathrm{Aut}(K) \subseteq \mathrm{Aut}(\widetilde{K}) \cong \mathrm{Aut}(\mathfrak{k})$ as a subgroup of $\mathrm{GL}(\mathfrak{k})$ preserving the scalar product κ.

The bundle isomorphism $\Gamma_{\Psi,\Phi}$ is $\mathbb{R}$-equivariant if and only if the function ζ is constant. Accordingly, the two bundles $\mathcal{K}_\Phi$ and $\mathcal{K}_\Psi$ are $\mathbb{R}$-equivariantly isomorphic if and only if Φ and Ψ are conjugate in $\mathrm{Aut}(K)$, so that equivariant isomorphism classes of principal K-bundles over $\mathbb{S}^1$ correspond to conjugacy classes in the group $\mathrm{Aut}(K)$ (cf. [94, Section 4.4] and [20, Section 9]).

The group $\Gamma_c(\mathbb{R}/\mathbb{Z}, \mathcal{K}_\Phi)$ is isomorphic to the twisted loop group

$$\mathcal{L}_\Phi(K) := \left\{\xi \in C^\infty(\mathbb{R}, K) : (\forall x \in \mathbb{R})\xi(x+1) = \Phi^{-1}(\xi(x))\right\} \tag{7.8}$$

with Lie algebra

$$\mathcal{L}_\varphi(\mathfrak{k}) := \left\{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall x \in \mathbb{R})\, \xi(x+1) = \varphi^{-1}(\xi(x))\right\}, \tag{7.9}$$

where $\varphi \in \mathrm{Aut}(\mathfrak{k})$ is the automorphism of $\mathfrak{k}$ induced by Φ. The $\mathbb{R}$-action on $\mathcal{L}_\varphi(\mathfrak{k})$ is given by

$$\alpha_t(\xi)(x) = \xi(x + t/T) \quad \text{and} \quad D\xi = \frac{1}{T}\xi'.$$

In some situations it is convenient to use a slightly different normalization for which Φ is of finite order, but then the $\mathbb{R}$-action becomes more complicated. If K is compact, then $\mathrm{Aut}(K)$ is compact as well. In this case, there exists a finite subgroup $F \subseteq \mathrm{Aut}(K)$ with $\mathrm{Aut}(K) = F\,\mathrm{Aut}(K)_0$ (see [42, Theorem 6.36]) and we may choose a representative Φ_0 of $[\Phi] \in \pi_0(\mathrm{Aut}(K))$ in such a way that $\Phi_0 \in F$.

If $\Gamma_{\Phi,\Phi_0}\colon \mathcal{K}_{\Phi_0} \to \mathcal{K}_\Phi$ is a group bundle isomorphism specified by the smooth curve $\zeta\colon \mathbb{R} \to \mathrm{Aut}(K)$ satisfying

$$\zeta_{x+1} = \Phi^{-1}\zeta_x\Phi_0 \quad \text{for } x \in \mathbb{R}$$

(see (7.7)), then the $\mathbb{R}$-action on $\Gamma_c(\mathbb{R}/\mathbb{Z}, \mathcal{K}_{\Phi_0}) \cong \mathcal{L}_{\Phi_0}(K)$ takes the form

$$\widetilde{\alpha}_t(\xi)(x) = \zeta_x^{-1}\zeta_{x+t/T}\xi(x + t/T) \quad \text{for } \xi \in \mathcal{L}_{\Phi_0}(K).$$

On the Lie algebra level we obtain the corresponding derivation given by

$$\widetilde{D}\xi = \frac{1}{T}(\xi' + \delta^l(\zeta)\xi),$$

where

$$\delta^l(\zeta)\colon \mathbb{R} \to \mathbf{L}(\mathrm{Aut}(K)) = \mathrm{der}(\mathfrak{k}), \quad \delta^l(\zeta)_x = \frac{d}{dt}\Big|_{t=0} \zeta_x^{-1}\zeta_{t+x}$$

is the left logarithmic derivative of ζ. Identifying $\mathfrak{k}$ via the adjoint representation with $\mathrm{der}(\mathfrak{k})$, we obtain a smooth curve $A\colon \mathbb{R} \to \mathfrak{k}$ with $\mathrm{ad} \circ A = \delta^l(\zeta)$ for which

$$\widetilde{D}\xi = \frac{1}{T}(\xi' + [A, \xi]). \tag{7.10}$$

Note that A belongs to the twisted loop algebra $\mathcal{L}_{\varphi_0}(\mathfrak{k})$; since $\zeta_{x+1} = \Phi^{-1}\zeta_x\Phi_0$, we have

$$\zeta_{x+1}^{-1}\zeta_{x+1+t} = \Phi_0^{-1}(\zeta_x^{-1}\zeta_{x+t})\Phi_0,$$

and hence

$$\delta^l(\zeta)_{x+1} = \varphi_0^{-1}\delta^l(\zeta)_x\varphi_0.$$

It follows that the curve A satisfies

$$A_{x+1} = \varphi_0^{-1}A_x,$$

so that $A \in \mathcal{L}_{\varphi_0}(\mathfrak{k})$.

Remark 7.14. We denote by $\mathcal{L}_\Phi^\sharp(K)_c$ the central $\mathbb{T}$-extension of $\mathcal{L}_\Phi(K)$ corresponding to the Lie algebra cocycle

$$\omega(\xi, \eta) = \frac{c}{2\pi}\int_0^1 \kappa(\xi', \eta)\,dt, \quad c \in \mathbb{Z}$$

with period group $2\pi c\mathbb{Z}$ (see the discussion in Section 7.2). If the central charge c is 1, we omit the subscript and simply write $\mathcal{L}_\Phi^\sharp(K)$. Since the Lie algebra $\mathcal{L}_\varphi(\mathfrak{k})$ of $\mathcal{L}_\Phi(K)$ is perfect [62, Theorem VI.3] implies that the $\mathbb{R}$-action α on $\mathcal{L}_\Phi(K)$ lifts to a smooth $\mathbb{R}$-action $\alpha^\sharp$ on $\mathcal{L}_\Phi^\sharp(K)_c$, and we obtain a double extension of the form

$$\widehat{\mathcal{L}}_\Phi(K)_c \cong \mathcal{L}_\Phi^\sharp(K)_c \rtimes_{\alpha^\sharp} \mathbb{R}.$$

The c-fold cover $\mathbb{T} \twoheadrightarrow \mathbb{T}\colon z \mapsto z^c$ extends to a c-fold cover $\mathcal{L}_\Phi^\sharp(K) \twoheadrightarrow \mathcal{L}_\Phi^\sharp(K)_c$, for which the following diagram commutes:

$$\begin{array}{ccccc}
\mathbb{T} & \longrightarrow & \mathcal{L}_\Phi^\sharp(K) & \longrightarrow & \mathcal{L}_\Phi(K) \\
{\scriptstyle z^c}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\
\mathbb{T} & \longrightarrow & \mathcal{L}_\Phi^\sharp(K)_c & \longrightarrow & \mathcal{L}_\Phi(K).
\end{array}$$

Using this covering map, we can identify the representations of $\mathcal{L}_\Phi(K)_c$ with those representations of $\mathcal{L}_\Phi(K)$ for which the roots $\{z \in \mathbb{T}; z^c = 1\} \subseteq \mathbb{T}$ of order c act trivially.

7.4 Localization at the group level

To obtain the localization result at the group level, we need the following factorization lemma.

Lemma 7.15. *Let $r: G \to H$ be an open, surjective morphism of locally exponential Lie groups, and let $R: G \to U$ be a continuous homomorphism of topological groups such that*

$$\mathbf{L}(\ker R) := \{x \in \mathfrak{g} : \exp(\mathbb{R}x) \subseteq \ker R\} \supseteq \ker \mathbf{L}(r) = \mathbf{L}(\ker r).$$

Then, R factors through a continuous homomorphism $\overline{R}: G/(\ker r)_0 \to U$ and r induces a covering morphism $G/(\ker r)_0 \to H$ of Lie groups.

Proof. In view of [71, Proposition IV.3.4] (see [30] for a complete proof), $N := \ker r$ is a closed, locally exponential Lie subgroup of G. In particular, its identity component N_0 is open in N, so that the isomorphism $G/N \to H$ of locally exponential Lie groups leads to a covering morphism $G/N_0 \to H$ ([71, Theorem IV.3.5]). For every $x \in \mathbf{L}(N)$, we have $\exp(\mathbb{R}x) \subseteq \ker R$, so that $N_0 = \langle \exp \mathbf{L}(N) \rangle \subseteq \ker R$. Therefore, R factors through G/N_0. ■

Lemma 7.16. *Let $S \subseteq M$ be a closed, 1-dimensional submanifold and suppose that the fibers of $\mathcal{K}|_S \to S$ are 1-connected, semisimple Lie groups. Then, $\Gamma_c(S, \mathcal{K})$ is 1-connected.*

For $S \cong \mathbb{R}/T\mathbb{Z} \cong \mathbb{S}^1$, it follows in particular that, for a 1-connected Lie group K and an automorphism $\Phi \in \mathrm{Aut}(K)$, the twisted loop group.

$$\mathcal{L}_\Phi^T(K) := \{\xi \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R})\ \xi(t+T) = \Phi^{-1}(\xi(t))\} \tag{7.11}$$

is 1-connected.

Proof. If S has connected components $(S_j)_{j \in J}$ with typical fiber K_j of $\mathcal{K}|_{S_j}$, then

$$\Gamma_c(S, \mathcal{K}) \cong \prod\nolimits'_{j \in J} \Gamma_c(S_j, \mathcal{K}). \tag{7.12}$$

(We refer to [26, Proposition 7.3] for a discussion of weak direct products of Lie groups.)

If $S_j \simeq \mathbb{S}^1$, then $\Gamma_c(S_j, \mathcal{K})$ is isomorphic to the twisted loop group $\mathcal{L}_{\Phi_j}^T(K_j)$, where Φ_j is an automorphism of K_j. Since $\pi_0(K_j)$, $\pi_1(K_j)$ vanish, $\pi_2(K_j)$ vanishes as well[1]. The long exact sequence of homotopy groups corresponding to the

[1] Since K_j is homotopy equivalent to a maximal compact subgroup, this follows from Cartan's theorem [64, Theorem 3.7].

Serre fibration $\mathrm{ev}_0\colon \mathcal{L}^T_{\Phi_j}(K_j) \to K_j$ thus yields an isomorphism between the homotopy groups π_0 and π_1 of $\mathcal{L}^T_{\Phi_j}(K_j)$ and $\mathcal{L}^T_{\Phi_j}(K_j)_* := \ker(\mathrm{ev}_0)$. Since the inclusion $\mathcal{L}^T_{\Phi_j}(K_j)_* \hookrightarrow \mathcal{L}^T_{\Phi_j}(K_j)_{*,\mathrm{ct}}$ into the group of continuous, based, twisted loops is a homotopy equivalence by [84, Corollary 3.4], and since $\pi_m(\mathcal{L}^T_{\Phi_j}(K_j)_{*,\mathrm{ct}}) \simeq \pi_m(\Omega K_j) \simeq \pi_{m+1}(K_j)$ for $m \in \mathbb{N}_0$ (cf. [84, page 391]), we conclude that $\mathcal{L}^T_{\Phi_j}(K_j)$ is 1-connected.

If $S_j \simeq \mathbb{R}$, then $\Gamma_c(S_j, \mathcal{K}) \simeq C^\infty_c(\mathbb{R}, K_j)$ is 1-connected by [70, Theorem A.10]. From [28, Proposition 3.3], we then conclude that the locally exponential Lie group (7.12) is 1-connected. ■

With these topological considerations out of the way, we now complete the proof of the localization theorem.

Proof of Theorem 7.1. In Lemma 7.13, we showed that the projective positive energy representation $\mathrm{d}\rho$ of $\Gamma_c(M, \mathfrak{K})$ factors through the restriction map

$$r^{\mathfrak{k}}_S\colon \Gamma_c(M, \mathfrak{K}) \to \Gamma_c(S, \mathfrak{K}),$$

so it remains to prove the corresponding factorization on the group level. For this, apply Lemma 7.15 to the locally exponential Lie groups $G = \widetilde{\Gamma}_c(M, \mathcal{K})_0$ and $H = \Gamma_c(S, \mathcal{K})$ (which are both 1-connected by Lemma 7.16), and the topological group

$$U = \mathrm{PU}(\mathcal{H}).$$

The homomorphism r is the homomorphism $\tilde{r}_S\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$, induced by the restriction $r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$, and R is the projective representation $\bar{\rho}\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$. We conclude that $\bar{\rho}$ factors through a projective positive energy representation of the 1-connected Lie group $\Gamma_c(S, \mathcal{K})$.

Since every representation of $\Gamma_c(M, \mathcal{K})_0$ defines by pullback a representation of its simply connected covering, the assertion also follows for representations of this group. This concludes the proof of the theorem. ■

7.5 Localization for equivariant representations

In this section we extend the localization Theorem 7.1 to the *equivariant* setting, where the action of $\mathbb{R}$ on M is replaced by a smooth action of a Lie group P on M. The positive energy condition (cf. Section 3.2) then refers not to an $\mathbb{R}$-action, but to the *positive energy cone* $\mathcal{C} \subseteq \mathfrak{p}$ inside the Lie algebra $\mathfrak{p}$ of P.

Let M be a manifold, let P be a Lie group acting smoothly on M, and let $\mathcal{K} \to M$ be a bundle of 1-connected, semisimple Lie groups that is equipped with a lift of this action. We denote the P-action on M by $\gamma_M\colon P \to \mathrm{Diff}(M)$, its lift to $\mathcal{K}$ by

$\gamma\colon P \to \mathrm{Aut}(\mathcal{K})$, and the corresponding action on the compactly supported gauge group by $\alpha\colon P \to \mathrm{Aut}(\Gamma_c(M, \mathcal{K}))$. On the infinitesimal level, the action of P on M gives rise to the action $\mathbf{v}_M\colon \mathfrak{p} \to \mathcal{V}(M)$, $p \mapsto \mathbf{v}_M^p$ of the Lie algebra $\mathfrak{p} := \mathbf{L}(P)$.

Let $(\bar{\rho}, \mathcal{H})$ be a smooth, projective, positive energy representation of the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes_\alpha P$ (cf. Definition 3.5), with positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$.

Definition 7.17. The *fixed point set* $\Sigma \subseteq M$ of the positive energy cone $\mathcal{C} \subseteq \mathfrak{p}$ (a closed convex invariant cone in $\mathfrak{p}$) is defined as

$$\Sigma := \left\{m \in M : (\forall p \in \mathcal{C})\mathbf{v}_M^p(m) = 0\right\}.$$

Since the positive energy cone $\mathcal{C}$ is Ad_P-invariant, its fixed point set Σ is a closed, P-invariant subset of M. In the following we first consider the fixed-point-free scenario $\Sigma = \emptyset$, and return to the general case in [49, Part II].

Definition 7.18. Let $\bar{\rho}$ be a smooth, projective, unitary representation of $\Gamma_c(M, \mathcal{K})$. The *support* of $\bar{\rho}$, denoted $\mathrm{supp}(\bar{\rho})$, is defined as the complement of the union of all open subsets $U \subseteq M$ for which the kernel of $\bar{\rho}$ contains the normal subgroup $\Gamma_c(U, \mathcal{K})$. Similarly, the *support* of $\mathtt{d}\rho$ is the complement of the union of all open sets $U \subseteq M$ such that the kernel of $\mathtt{d}\rho$ contains $\Gamma_c(U, \mathfrak{K})$.

Note that the support is a closed subset of M. If the representation $\bar{\rho}$ extends to the semidirect product $\Gamma_c(M, \mathcal{K}) \rtimes_\alpha P$, then the support of $\bar{\rho}$ is invariant under the action of P on M. This leads to severe restrictions for positive energy representations.

Theorem 7.19 (Equivariant localization theorem). *Let $(\bar{\rho}, \mathcal{H})$ be a smooth, projective, positive energy representation of $\Gamma_c(M, \mathcal{K})_0 \rtimes_\alpha P$, and suppose that $\mathcal{C}$ has no fixed points. Then, there exists a 1-dimensional, P-equivariantly embedded submanifold $S \subseteq M$ such that $\bar{\rho}$ factors through the restriction homomorphism*

$$r_S\colon \Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K}).$$

Remark 7.20 (Equivariant localization for the simply connected cover). Since the P-action on $\Gamma_c(M, \mathcal{K})$ preserves the identity component $\Gamma_c(M, \mathcal{K})_0$, it lifts to the simply connected cover $\widetilde{\Gamma}_c(M, \mathcal{K})_0$. In this context the same result remains valid: every smooth, projective, positive energy representation $\bar{\rho}$ of $\widetilde{\Gamma}_c(M, \mathcal{K}) \rtimes_\alpha P$ factors through the homomorphism $\tilde{r}_S\colon \widetilde{\Gamma}_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ obtained by composing the restriction r_S with the covering map.

Proof. For every $p \in \mathcal{C}$, let $U_p \subseteq M$ be the open set of points in M where $\mathbf{v}_M^p$ is non-vanishing. Applying Lemma 7.13 to the manifold U_p, with the gauge group $\Gamma_c(U_p, \mathcal{K})$ and the $\mathbb{R}$-action $\alpha_p(t) := \alpha(\exp(tp))$, one finds an embedded, 1-dimensional submanifold $S_p \subseteq U_p$ such that the projective Lie algebra representation $\mathtt{d}\rho$ factors through the restriction map $r_{S_p}^{\mathfrak{k}}\colon \Gamma_c(U_p, \mathfrak{K}) \to \Gamma_c(S_p, \mathfrak{K})$. The support of

$\mathtt{d}\rho|_{\Gamma_c(U_p,\mathfrak{K})}$ is thus contained in S_p. It actually equals S_p because the cocycle on $\Gamma_c(U_p, \mathfrak{K})$ is given by a measure with support S_p. The sets S_p and $S_{p'}$ therefore coincide on $U_p \cap U_{p'}$, so the union $S = \bigcup_{p \in \mathcal{C}} S_p$ is a 1-dimensional, closed embedded submanifold of M. Here we use that the U_p cover M because $\mathcal{C}$ has no common fixed point. Since $gS_p = S_{\mathrm{Ad}_g(p)}$ for every $g \in P$, the union S is P-invariant.

Let $I_S := \{\xi \in \Gamma_c(M, \mathfrak{K}); \xi|_S = 0\}$ be the vanishing ideal of S in $\Gamma_c(M, \mathfrak{K})$. Since any $\xi \in I_S$ can be written as a finite sum of $\xi_p \in I_{S_p} \subseteq \Gamma_c(U_p, \mathfrak{K})$, and since the restriction of $\mathtt{d}\rho$ to $\Gamma_c(U_p, \mathfrak{K})$ vanishes on I_{S_p}, we conclude that $\mathtt{d}\rho$ vanishes on I_S. From Lemma 7.15 and Lemma 7.16, we then find (as in the proof of Theorem 7.1) that $\bar{\rho}$ factors through the restriction $\Gamma_c(M, \mathcal{K})_0 \to \Gamma_c(S, \mathcal{K})$ and that the corresponding assertion holds for representations of the covering group $\widetilde{\Gamma}_c(M, \mathcal{K})_0$. ■

The building blocks for the positive energy representations therefore come from actions of P on 1-dimensional manifolds on which $\mathcal{C}$ has no fixed point. According to the classification of hyperplane subalgebras of finite-dimensional Lie algebras [40, 41], an effective action of a connected finite-dimensional Lie group P on a simply connected one-dimensional manifold is of one of the following 3 types:

- the action of $P = \mathbb{R}$ on the line $\mathbb{R}$,
- the action of the affine group $P = \mathrm{Aff}(\mathbb{R})_0$ on the real line $\mathbb{R}$,
- the action of $P = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ on the real line $\mathbb{R}$, considered as the simply connected cover of $\mathbb{P}_1(\mathbb{R}) \cong \mathbb{S}^1$.

In the infinite-dimensional context, the action of the simply connected covering group $P = \widetilde{\mathrm{Diff}}_+(\mathbb{S}^1)$ on $\mathbb{R} \cong \widetilde{\mathbb{S}}^1$ is a natural example.