Chapter 8

The classification for M compact

If the flow γ_M on M has no fixed points, the localization Theorem 7.1 reduces the classification of projective positive energy representations of the identity component $\Gamma_c(M, \mathcal{K})_0$ of the compactly supported gauge group to the situation where the base manifold is a closed, embedded, flow-invariant submanifold $S \subseteq M$ of dimension one.

The connected components of S are either diffeomorphic to $\mathbb{R}$ (for a non-periodic orbit), or to $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$ (for a periodic orbit). Since a gauge group on $\mathbb{R}$ is equivariantly isomorphic to $C^\infty_c(\mathbb{R}, K)$ (with $\mathbb{R}$ acting by translation), and a gauge group on $\mathbb{S}^1$ is equivariantly isomorphic to a twisted loop group (with $\mathbb{R}$ acting by rotation), the gauge group on S is a product of twisted loop groups and groups of the form $C^\infty_c(\mathbb{R}, K)$.

In this chapter, we describe the complete classification of positive energy representations for twisted loop groups. This leads to a classification of the positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ for which the one-dimensional submanifold S is compact. Since this is automatically the case if M is compact, we arrive at a complete classification in this setting.

8.1 Positive energy representation of twisted loop groups

We now describe the complete classification of projective positive energy representations for twisted loop groups.

In this section K denotes a 1-connected compact (hence semisimple) Lie group, $\Phi \in \mathrm{Aut}(K)$ is an automorphism of *finite order* $\Phi^N = \mathrm{id}_K$, and $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$ is the corresponding automorphism of $\mathfrak{k}$. We further assume that the invariant form κ on $\mathfrak{k}$ is normalized in such a way that

$$\kappa(i\alpha^\vee, i\alpha^\vee) = 2$$

for all long roots α. We denote the (twisted) loop groups and algebras by $\mathcal{L}_\Phi(K)$ and $\mathcal{L}_\varphi(\mathfrak{k})$ respectively, as in (7.8) and (7.9). The (double) extensions with $c = 1$ are denoted by $\mathcal{L}^\sharp_\Phi(K)$ and $\widehat{\mathcal{L}}_\Phi(K)$, cf. Remark 7.14.

Definition 8.1. We call a positive energy representation $(\rho, \mathcal{H})$ of $\widehat{\mathcal{L}}_\Phi(K)$

(i) *basic* if $U_t := \rho(\exp tD) \subseteq \rho(\mathcal{L}^\sharp_\Phi(K))''$ for every $t \in \mathbb{R}$,

(ii) *periodic* if $U_T = \mathbf{1}$ for some $T > 0$.

Note that if ρ is minimal (Definition 3.8), then it is in particular basic.

Remark 8.2. If $(\rho, \mathcal{H})$ is periodic with $U_T = \mathbf{1}$, then [79, Lemma 5.1] implies that the space $\mathcal{H}^\infty$ of smooth vectors is invariant under the operators

$$p_n(v) := \frac{1}{T}\int_0^T e^{-2\pi int/T} U_t v\,dt.$$

These are orthogonal projections onto the eigenvectors of $H = i\,\mathtt{d}\rho(D)$ for the eigenvalues $-2\pi n/T$, $n \in \mathbb{Z}$.

Recall from Section 7.3 that with the identification $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$ and with $\Phi^N = \mathrm{id}_K$, we have

$$D(\xi) = \frac{1}{T}(\xi' + [A, \xi]).$$

It will be convenient to introduce the derivative

$$\mathbf{d}(\xi) = \frac{d}{dx}\xi, \quad \text{so that } D = \frac{1}{T}(\mathbf{d} + \mathrm{ad}_A). \tag{8.1}$$

Remark 8.3 (Independence of positive energy condition from lift of $\mathbb{R}$-action). From Proposition 6.32, applied to $M = \mathbb{R}/\mathbb{Z}$, it follows that a smooth representation of $\mathcal{L}_\Phi^\sharp(K)$ is of positive energy with respect to the derivation D if and only if it is of positive energy with respect to the derivation $\mathbf{d}$. Then, the representation is semibounded in the sense of Definition 6.31. As this holds for $D = \frac{1}{T}(\mathbf{d} + \mathrm{ad}_L)$ with any $T > 0$ and $L \in \mathcal{L}_\varphi(\mathfrak{k})$, the positive energy condition does not depend on the choice of the vector field $\mathbf{v}$ on $\mathcal{K}_\Phi = \mathbb{R} \times_\Phi K$ lifting the vector field $\mathbf{v}_M = \frac{1}{T}\frac{d}{dt}$ on $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$.

From $\Phi^N = \mathrm{id}_K$, we immediately derive that $\varphi^N = \mathrm{id}_{\mathfrak{k}}$. For $\widehat{\mathfrak{g}} = \widehat{\mathcal{L}}_\varphi(\mathfrak{k})$, we define the canonical triangular decomposition by

$$\widehat{\mathfrak{g}}_{\mathbb{C}} = \widehat{\mathfrak{g}}_{\mathbb{C}}^+ \oplus \widehat{\mathfrak{g}}_{\mathbb{C}}^0 \oplus \widehat{\mathfrak{g}}_{\mathbb{C}}^-$$

with

$$\widehat{\mathfrak{g}}_{\mathbb{C}}^\pm := \overline{\sum_{\pm n > 0} \widehat{\mathfrak{g}}_{\mathbb{C}}^n},$$

where

$$\widehat{\mathfrak{g}}_{\mathbb{C}}^n := \ker\left(\mathbf{d} + \frac{2\pi i n}{N}\mathbf{1}\right) \quad \text{for } n \in \mathbb{Z}$$

(see (A.1) in the appendix). For $\mathfrak{g} = \mathcal{L}_\varphi(\mathfrak{k})$, we have the analogous decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^+ \oplus \mathfrak{g}_{\mathbb{C}}^0 \oplus \mathfrak{g}_{\mathbb{C}}^-$$

with

$$\mathfrak{g}_{\mathbb{C}}^+ = \widehat{\mathfrak{g}}_{\mathbb{C}}^+ \quad \text{and} \quad \mathfrak{g}_{\mathbb{C}}^- = \widehat{\mathfrak{g}}_{\mathbb{C}}^-.$$

For a smooth unitary representation of $\widehat{\mathcal{L}}_\Phi(K)$, we define its *minimal energy subspace* with respect to $i\,\mathtt{d}\rho(\mathbf{d})$ by

$$\mathcal{E} := \overline{(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}} \quad \text{for } (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-} := \big\{\psi \in \mathcal{H}^\infty : (\forall x \in \mathfrak{g}_{\mathbb{C}}^-)\,\mathtt{d}\rho(x)\psi = 0\big\}. \tag{8.2}$$

Lemma 8.4. *For every smooth positive energy representation $(\rho, \mathcal{H})$ of $\widehat{\mathcal{L}}_\Phi(K)$, the subspace $\mathcal{E}$ is generating for $\mathcal{L}_\Phi^\sharp(K)$.*

Proof. Note that $\mathcal{E}$ is defined in terms of $\rho|_{\mathcal{L}_\Phi^\sharp(K)}$. In view of Corollary 3.9 and the fact that $\alpha_N = \mathrm{id}_{\mathcal{L}_\Phi(K)}$, we may therefore assume, without loss of generality, that ρ is periodic.

Let $\mathcal{H}' \subseteq \mathcal{H}$ denote the smallest closed $\mathcal{L}_\Phi^\sharp(K)$-invariant subspace containing $\mathcal{E}$. Then, $\mathcal{H}'$ is U-invariant, and the representation of $\widehat{\mathcal{L}}_\Phi(K)$ on $(\mathcal{H}')^\perp$ is also a positive energy representation. If $(\mathcal{H}')^\perp \neq \{0\}$, then its minimal energy subspace $\mathcal{F}$ is non-zero by Remark 8.2, and since it contains smooth vectors, we obtain a contradiction to $\mathcal{F} \perp \mathcal{E}$. Therefore, $(\mathcal{H}')^\perp = \{0\}$ and the subspace $\mathcal{E}$ is $\mathcal{L}_\Phi^\sharp(K)$-generating. ■

We now abbreviate

$$G := \mathcal{L}_\Phi(K), \quad \widehat{G} := \widehat{\mathcal{L}}_\Phi(K) \quad \text{and} \quad G^\sharp := \mathcal{L}_\Phi^\sharp(K) \tag{8.3}$$

and denote the corresponding groups of fixed points by

$$L = K^\Phi, \quad \widehat{L} := \mathrm{Fix}_\alpha(\widehat{G}) \cong \mathbb{T} \times K^\Phi \times \mathbb{R}, \quad L^\sharp := \widehat{L} \cap G^\sharp \cong \mathbb{T} \times L.$$

From the discussion in [79, Section 5.2 and Appendix C], it follows that the homogeneous space $G/L \cong \widehat{G}/\widehat{L} \cong G^\sharp/L^\sharp$ carries the structure of a complex Fréchet manifold on which $\widehat{G}$ acts analytically, and the tangent space in the base point is isomorphic to the quotient space $\widehat{\mathfrak{g}}_\mathbb{C}/(\widehat{\mathfrak{g}}_\mathbb{C}^0 + \mathfrak{g}_\mathbb{C}^+)$. For any bounded unitary representation (ρ^L, E) of $\widehat{L}$, we then obtain a holomorphic vector bundle $\mathbb{E} := \widehat{G} \times_{\widehat{L}} E$ over $\widehat{G}/\widehat{L}$. We write $\Gamma_{\mathrm{hol}}(G/L, \mathbb{E})$ for the space of holomorphic sections of $\mathbb{E}$.

Definition 8.5 (Holomorphically induced representations). A unitary representation $(\rho, \mathcal{H})$ of $\widehat{G}$ is said to be *holomorphically induced* from (ρ^L, E) if there exists a G-equivariant linear injection $\Psi\colon \mathcal{H} \to \Gamma_{\mathrm{hol}}(G/L, \mathbb{E})$ such that the adjoint of the evaluation map

$$\mathrm{ev}_{\mathbf{1}\widehat{L}}\colon \mathcal{H} \to E = \mathbb{E}_{\mathbf{1}\widehat{L}}$$

defines an isometry $\mathrm{ev}_{\mathbf{1}\widehat{L}}^*\colon E \hookrightarrow \mathcal{H}$. If there exists a unitary representation $(\rho, \mathcal{H})$ holomorphically induced from (ρ^L, E), then it is uniquely determined [77, Definition 3.10]. We then call the representation (ρ^L, E) of $\widehat{L}$ *(holomorphically) inducible*. The same statements apply to $G^\sharp$ and $L^\sharp$.

Let $\mathfrak{t}^\circ \subseteq \mathfrak{k}^\varphi$ be maximal abelian, so that

$$\mathfrak{t} = \mathbb{R}C \oplus \mathfrak{t}^\circ \oplus \mathbb{R}\mathbf{d}$$

is maximal abelian in $\widehat{\mathcal{L}}_\varphi(\mathfrak{k})$. We write $T^\sharp = \mathbb{T} \times T^\circ$ for the torus group with Lie algebra $\mathfrak{t}^\sharp = \mathbb{R}C \oplus \mathfrak{t}^\circ$. Let Δ^+ be a positive system for the affine Kac–Moody Lie algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_\mathbb{C})$ with respect to the Cartan subalgebra $\mathfrak{t}_\mathbb{C}$ such that, for all $\alpha \in \Delta$, the relation $\alpha(i\mathbf{d}) > 0$ implies $\alpha \in \Delta^+$ (cf. Appendix A and [38, Chapter X]).

Proposition 8.6. *A bounded representation* (ρ^L, E) *of*

$$L^\sharp = \exp(\mathbb{R}C) \times L \cong \mathbb{T} \times L$$

is holomorphically inducible if and only if

$$d\rho^L([z^*, z]) \geq 0 \quad \text{for all } z \in \mathfrak{g}^n_{\mathbb{C}},\ n > 0. \tag{8.4}$$

In particular, the irreducible, holomorphically inducible representations of $L^\sharp$ *are parametrized by the anti-dominant, integral weights* λ *of the form*

$$\lambda = (\lambda(C), \lambda_0, 0) \in i\mathfrak{t}^* \tag{8.5}$$

of the affine Kac–Moody Lie algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}})$ *with respect to the Cartan subalgebra* $\mathfrak{t}_{\mathbb{C}}$ *and the positive system* Δ^+*. Here the central charge* $c := -i\lambda(C)$ *is contained in* $\mathbb{N}_0$*, and for every central charge* c *there are only finitely many such representations with* $\lambda(C) = ic$

Proof. Since the representation ρ^L of the compact group $L^\sharp$ is a direct sum of irreducible representations, we may assume that it is a representation with lowest weight λ with respect to the positive system of roots Δ_0^+ of $(\mathfrak{k}^\varphi_{\mathbb{C}}, \mathfrak{t}^\circ_{\mathbb{C}})$.

The necessity of (8.4) follows from [79, Proposition 5.6]. To show that λ is anti-dominant for $(\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}}), \mathfrak{t}_{\mathbb{C}}, \Delta^+)$, we need that $\lambda((\alpha, n)^\vee) \leq 0$ for $(\alpha, n) \in \Delta_+$. We distinguish the cases $n > 0$ and $n = 0$. If $n > 0$, we use (A.2) in Appendix A, to see that (8.4) implies $\lambda((\alpha, n)^\vee) \leq 0$ for $0 \neq \alpha \in \Delta_0$, the root system of $(\mathfrak{k}^\varphi_{\mathbb{C}}, \mathfrak{t}^\circ_{\mathbb{C}})$. For $n = 0$, the assertion follows from $\lambda(\beta^\vee) \leq 0$ for $\beta \in \Delta_0^+$.

Next, we prove the integrality of λ. For $\alpha \neq 0$, the relation

$$\exp(2\pi i(\alpha, n)^\vee) = \mathbf{1} \tag{8.6}$$

in $T^\sharp$ follows from the fact that

$$\mathfrak{k}(\alpha, n) := \operatorname{span}_{\mathbb{R}}\left\{x \otimes e_n - x^* \otimes e_{-n}, i(x \otimes e_n + x^* \otimes e_{-n}), i(\alpha, n)^\vee\right\} \cong \mathfrak{su}(2, \mathbb{C}).$$

Since λ corresponds to a character of $T^\sharp$, the relation (8.6) implies that

$$\exp(2\pi i\lambda((\alpha, n)^\vee)) = 1,$$

so that $\lambda((\alpha, n)^\vee) \in \mathbb{Z}$. We conclude that λ is anti-dominant integral.

We now argue that every integral, anti-dominant weight λ as in (8.5) specifies a holomorphically inducible representation (ρ^L, E_λ) of $L^\sharp$. In fact, the unitarity of the corresponding lowest weight module $L(\lambda, -\Delta^+)$ of the affine Kac–Moody algebra $\widehat{\mathcal{L}}_\varphi(\mathfrak{k}_{\mathbb{C}})$ ([54, Theorem 11.7]) can be used as in the proof of [79, Theorem 5.10] to see with [79, Theorem C.6] that (ρ^L, E_λ) is holomorphically inducible. ∎

The following theorem is well-known for untwisted loop groups $\mathcal{L}(K)$, but we did not find an appropriate statement in the literature for the twisted case. It requires some refined methods based on holomorphic induction which we draw from [79].

Theorem 8.7. *If K is 1-connected and $(\rho, \mathcal{H})$ is a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$, then its restriction to $\mathcal{L}^\sharp_\Phi(K)$ is a finite direct sum of factor representations of type I, hence, in particular, a direct sum of irreducible representations.*

Proof. Since the assertion only refers to the restriction $\rho|_{\mathcal{L}^\sharp_\Phi(K)}$, we may assume, without loss of generality, that $\rho = \rho_0$ is minimal (Definition 3.8 and Theorem 3.7). Then, $\alpha_N = \mathrm{id}_{\mathcal{L}_\Phi(K)}$ implies that ρ is periodic and that every subrepresentation is generated by the fixed points of

$$U_t = \rho(\mathbf{1}, t) = e^{-itH}.$$

In view of Remark 8.2, the space $\mathcal{H}^\infty$ of smooth vectors for $\widehat{G} = \widehat{\mathcal{L}}_\Phi(K)$ (see (8.3)) is invariant under the projections $p_n \colon \mathcal{H} \to \mathcal{H}_n$ onto the eigenspaces of $H = i\,\mathtt{d}\rho(D)$. Since $\rho = \rho_0$ is minimal, we have $\mathcal{H}_n = \{0\}$ for $n < 0$ and $\mathcal{H}_0$ is generating. Now $\mathcal{H}_0 \cap \mathcal{H}^\infty$ is contained in $\mathcal{E}$, the closure of $(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ from (8.2). As the intersection $\mathcal{H}_0 \cap \mathcal{H}^\infty$ is dense in $\mathcal{H}_0$, we have $\mathcal{H}_0 \subseteq \mathcal{E}$.

Recall that

$$\widehat{L} = \mathrm{Fix}_\alpha(\widehat{G}) \cong L \times \mathbb{R} \cong \mathbb{T} \times K^\Phi \times \mathbb{R}.$$

As K^Φ is compact and $U_N = \mathbf{1}$ follows from $\varphi^N = \mathrm{id}_{\mathfrak{k}}$, $\rho(\widehat{L})$ is a compact subgroup of $\mathrm{U}(\mathcal{H})$. Hence, the $\widehat{L}$-invariant subspace $\mathcal{H}_0 \subseteq \mathcal{E}$ is a direct sum of finite-dimensional subrepresentations. In particular, it decomposes into isotypic components $\mathcal{E}_j := E_j \otimes \mathcal{M}_j$, $j \in J$, where $\mathcal{M}_j \cong B(E_j, \mathcal{H}_0)^{\widehat{L}}$ is the multiplicity space of the (finite-dimensional) irreducible representation (ρ_j^L, E_j). It also follows that the representation of $\widehat{L}$ on each $\mathcal{E}_j$ is semisimple in the algebraic sense and that the irreducible subrepresentations are of the form $E_j \otimes \psi$, $\psi \in \mathcal{M}_j$. As a consequence, every $\widehat{L}$-invariant subspace of $\mathcal{E}_j$ is of the form $E_j \otimes \mathcal{M}'_j$ for a linear subspace $\mathcal{M}'_j \subseteq \mathcal{M}_j$.

The dense subspace $(\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ of $\mathcal{E}$ is invariant under the projections onto the isotypic components because they are given by integration over a compact group[1]. This implies that $\mathcal{E}_j \cap (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-}$ is dense in $\mathcal{E}_j$. In view of the preceding discussion, we thus obtain

$$\mathcal{E}_j \cap (\mathcal{H}^\infty)^{\mathfrak{g}_{\mathbb{C}}^-} \cong E_j \otimes \mathcal{M}_j^\infty$$

for a dense linear subspace $\mathcal{M}_j^\infty \subseteq \mathcal{M}_j$. In view of Lemma 8.4, we now have to show that, for every $\psi \in \mathcal{M}_j^\infty$, the subspace $E_j \otimes \psi \subseteq \mathcal{E}$ generates an irreducible subrepresentation of

$$G^\sharp = \mathcal{L}^\sharp_\Phi(K).$$

[1]This follows by differentiation under the integral sign, see [30, Proposition 1.3.23].

For the untwisted case, i.e., $\Phi = \mathrm{id}_K$, this follows from [68, Proposition VII.1]. For the twisted case we have to invoke the machinery of holomorphic induction described in Definition 8.5. For the following argument, observe that $\mathcal{L}^\sharp_\Phi(K)$ is connected by Lemma 7.16. On the finite-dimensional subspace $E := E_j \otimes \psi \subseteq (\mathcal{H}^\infty)^{\mathfrak{g}^-_{\mathbb{C}}}$, the representation of $\widehat{L}$ is bounded. Hence, [79, Theorem C.3] implies that the $\widehat{G}$-subrepresentation $(\rho', \mathcal{H}')$ of $(\rho, \mathcal{H})$ generated by E is holomorphically induced from the $\widehat{L}$-representation (ρ^L, E). In view of [79, Theorem C.2], the irreducibility of (ρ^L, E) implies the irreducibility of $(\rho', \mathcal{H}')$.

We have seen in the proof of Proposition 8.6 that the holomorphically inducible irreducible representations ρ^L of $\widehat{L}$ are parametrized by a set of anti-dominant integral weights of an affine Kac–Moody algebra $\widehat{\mathcal{L}}_\psi(\mathfrak{k}_{\mathbb{C}})$ with a fixed central charge. This implies the finiteness of the possible types. ■

The following corollary can be used to deal with gauge groups if the structure group K is not 1-connected. It covers in particular the case $K = \mathrm{Aut}(\mathfrak{k})$ that arises from structure groups of Lie algebra bundles $\mathfrak{K} \to \mathbb{S}^1$.

Corollary 8.8 (Non-connected fibers). *If K is a compact Lie group with simple Lie algebra and $(\rho, \mathcal{H})$ a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$, then its restriction to $\mathcal{L}^\sharp_\Phi(K)$ is a finite direct sum of factor representations of type I, hence, in particular, a direct sum of irreducible representations.*

Proof. Since K is compact with simple Lie algebra, the groups $\pi_0(K)$ and $\pi_1(K)$ are finite. Therefore, the exact sequence

$$\mathbf{1} \to \pi_1(K)/\,\mathrm{im}(\pi_1(\Phi) - \mathrm{id}) \hookrightarrow \pi_0(\mathcal{L}_\Phi(K)) \twoheadrightarrow \pi_0(K)^\Phi \to \mathbf{1}$$

from [84, Remark 2.6 (a)] implies that $\pi_0(\mathcal{L}_\Phi(K))$ is finite. The identity component $\mathcal{L}_\Phi(K)_0$ is isomorphic to $\mathcal{L}_\Phi(\widetilde{K}_0)$, where $\widetilde{K}_0$ is the simply connected covering of the identity component K_0 of K. Now the assertion follows by combining Theorem 8.7 with Theorem C.1. ■

Remark 8.9 (Explicit aspects of the Borchers–Arveson theorem).

(a) Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{\mathcal{L}}_\Phi(K)$ for which the restriction $\rho^\sharp$ to $\mathcal{L}^\sharp_\Phi(K)$ is isotypic. Then, the proof of Theorem 8.7 shows that $\rho^\sharp$ is holomorphically induced from $(\rho^L, \mathcal{E})$, where $\mathcal{E} \cong E \otimes \mathcal{M}$ and (ρ^L, E) is an irreducible representation of $L^\sharp$, and hence of K^Φ.

That the representation is basic, $U_{\mathbb{R}} \subseteq \rho(G^\sharp)''$, is equivalent to $U_{\mathbb{R}}$ commuting with the commutant $\rho(G^\sharp)'$. Since the restriction to $\mathcal{E}$ yields an isomorphism $\rho(G^\sharp)' \to \rho^L(L^\sharp)' = \rho^L(L)'$ ([79, Theorem C.2]) and $\mathcal{E}$ is invariant under $U_{\mathbb{R}}$, the inclusion $U_{\mathbb{R}} \subseteq (\rho(G^\sharp)')'$ is equivalent to

$$U_{\mathbb{R}}|_{\mathcal{E}} \subseteq (\rho^L(L)')' = B(E) \otimes \mathbf{1}.$$

Since $\widehat{L} = \mathbb{T} \times K^{\Phi} \times \mathbb{R}$, where K^{Φ} is considered as a subgroup of constant sections, we have $U_{\mathbb{R}}|_{\mathcal{E}} \subseteq \rho(\widehat{L})'$. The representation is therefore basic if and only if $U_{\mathbb{R}}|_{\mathcal{E}}$ is contained in

$$\rho(\widehat{L})' \cap \rho^L(L)'' = \mathbb{C}\mathbf{1},$$

that is, if and only if U acts on $\mathcal{E}$ by a character.

(b) We construct an example which is not basic, but which is factorial on $G^\sharp$. Let $(\rho, \mathcal{H})$ be an irreducible positive energy representation of $\widehat{G} = \widehat{\mathcal{L}}_{\Phi}(K)$. For any non-trivial character $\chi \colon \mathbb{R} \to \mathbb{T}$, the representation $\rho \oplus (\widehat{\chi} \otimes \rho)$ with $\widehat{\chi}(g,t) := \chi(t)$ is factorial on $G^\sharp$, but not on $\widehat{G}$.

8.2 The classification theorem for compact base manifolds

Let M be a manifold on which the flow γ_M has no fixed points, and let K be a compact, connected, simple Lie group. We now obtain a full classification of the projective positive energy representations of $\Gamma_c(M, \mathcal{K})_0$ in the case where M is compact, by combining the localization Theorem 7.1 with the results on twisted loop groups from Section 8.1.

8.2.1 One-dimensional manifolds with compact components

By Theorem 7.1 and Corollary 7.5, every projective positive energy representation of $\Gamma_c(M, \mathcal{K})_0$ factors through the gauge group $\Gamma_c(S, \widetilde{\mathcal{K}})$ of a 1-dimensional, $\mathbb{R}$-equivariantly closed embedded submanifold $S \subseteq M$. If S is compact, then it is the disjoint union of finitely many circles S_j on which $\mathbb{R}$ acts with period T_j.

In this section we assume that S is a (not necessarily finite) union of circles. The restricted gauge group $G := \Gamma_c(S, \widetilde{\mathcal{K}})$ is then a restricted direct product of twisted loop groups $\mathcal{L}_{\Phi_j}(\widetilde{K}_j)$, where $\widetilde{K}_j$ is the 1-connected cover of the structure group K_j of $\mathcal{K}|_{S_j}$. On the Lie algebra level, we have a direct sum of Lie algebras

$$\mathfrak{g} \cong \bigoplus_{j \in J} \mathcal{L}_{\varphi_j}(\mathfrak{k}_j).$$

As in (8.1), the infinitesimal generator D of the $\mathbb{R}$-action acts on $\xi \in \mathfrak{g}$ by

$$D(\xi) = \bigoplus_{j \in J} \frac{1}{T_j}(\mathbf{d}_j \xi_j + [A_j, \xi_j]),$$

where $A_j \in \mathcal{L}_{\varphi_j}(\mathfrak{k})$ is determined by the $\mathbb{R}$-action according to (7.10).

Let $(\mathrm{d}\rho, \mathcal{H})$ be a positive energy representation of $\mathfrak{g}^\sharp = \mathbb{R}C \oplus_\omega \mathfrak{g}$ with cocycle

$$\omega(\xi, \eta) = \sum_{j \in J} \frac{c_j}{2\pi} \int_0^1 \kappa(\xi_j', \eta_j) dt \quad \text{with } c_j \in \mathbb{N}_0.$$

For each j, $\mathtt{d}\rho$ restricts to a positive energy representation of the centrally extended twisted loop algebra $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j. By Proposition 8.6 and Remark 8.3 (cf. [94, Chapter 9] for the untwisted case), the irreducible positive energy representations of $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j are precisely the irreducible unitary lowest weight representations $(\mathtt{d}\rho_\lambda, \mathcal{H}_\lambda)$ with integral anti-dominant weight λ satisfying $\lambda(C) = ic_j$. Since there are finitely many of these, the representation can be written as a finite sum

$$\mathcal{H} = \bigoplus_\lambda \mathcal{H}^j_\lambda \otimes \mathcal{M}^j_\lambda \tag{8.7}$$

where the sum runs over the integral anti-dominant weights of $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ with central charge c_j (cf. (8.5)) and $\mathcal{L}^\sharp_{\varphi_j}(\mathfrak{k}_j)$ acts trivially on the multiplicity space $\mathcal{M}^j_\lambda$ (Theorem 8.7).

Now suppose that $(\rho, \mathcal{H})$ is a positive energy factor representation of $G^\sharp$. Then, the restriction to a normal subgroup

$$G_j := \mathcal{L}_{\Phi_j}(\widetilde{K})$$

decomposes discretely with finitely many isotypes (Theorem 8.7). For a subset $F \subseteq J$, we denote the corresponding normal subgroup of G by

$$G_F := \bigoplus_{j \in F} \mathcal{L}_{\Phi_j}(\widetilde{K}_j).$$

Since G_j commutes with $G_{J\setminus\{j\}}$, the factoriality of ρ on $G^\sharp$ implies that the restriction of ρ to $G^\sharp_j$ is factorial as well. Hence, there is only one summand in (8.7), and we have

$$\mathcal{H} = \mathcal{H}^j_\lambda \otimes \mathcal{H}'$$

for some multiplicity space $\mathcal{H}'$. Although a priori we only have a single operator H for all components S_j, we now obtain an operator $\mathtt{d}\rho(\mathbf{d}_j)$ satisfying

$$[\mathtt{d}\rho(\mathbf{d}_j), \mathtt{d}\rho(\xi_i)] = \delta_{ij}\mathtt{d}\rho(\xi'_j)$$

from the minimal implementation[2] in Corollary 3.9.

Since

$$H' := H - \frac{i}{T_j}\mathtt{d}\rho(\mathbf{d}_j + A_j)$$

commutes with $\widehat{\mathcal{L}}_{\varphi_j}(\mathfrak{k})$, we obtain a positive energy representation on $\mathcal{H}'$ with Hamiltonian H', but now for the group $G^\sharp_{J\setminus\{j\}}$. Continuing this way, we obtain for each

[2]One could also use the Segal–Sugawara construction ([55, Section 3] and [32]), but this leads to a non-zero minimal eigenvalue; see Section 9.1.1 for more details.

$j \in J$ an integral anti-dominant weight λ_j of central charge c_j, and for each finite subset $F \subseteq J$ a tensor product decomposition

$$\rho = \rho_F \otimes \rho'_F, \quad \mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}'_F \quad \text{with } \mathcal{H}_F := \bigotimes_{j\in F} \mathcal{H}_{\lambda_j} \tag{8.8}$$

into positive energy representations for the gauge groups $G^\sharp_F$ and $G^\sharp_{J\setminus F}$.

8.2.2 Compact base manifolds

For gauge groups over a compact base manifold M, we thus obtain the following classification result. It contains in particular Torresani's classification for linear flows on a torus; see [105] and [3, Section 5.4].

Theorem 8.10. *Let M be a compact manifold with a fixed point free $\mathbb{R}$-action γ_M, and let $\mathcal{K} \to M$ be a bundle of Lie groups with compact, simple, connected fibers. Let $\bar{\rho}\colon \Gamma(M,\mathcal{K})_0 \rtimes \mathbb{R} \to \mathrm{PU}(\mathcal{H})$ be a minimal projective positive energy representation with respect to a lift γ of the $\mathbb{R}$-action to $\mathcal{K}$. Then, there exist finitely many $\mathbb{R}$-orbits $S_j \subseteq M$, $j \in J$, with central charge $c_j \in \mathbb{N}_0$ such that $\bar{\rho}$ arises by factorization from an isotypic positive energy representation ρ_S of*

$$\widehat{G} = G^\sharp \rtimes \mathbb{R},$$

where

$$G := \Gamma(S, \widetilde{\mathcal{K}}) \simeq \prod_{j\in J} \mathcal{L}_{\Phi_j}(\widetilde{K}_j).$$

If ρ_S is irreducible, then

$$\mathcal{H} = \bigotimes_{j\in J} \mathcal{H}_{\lambda_j}$$

is a tensor product of lowest weight representations $(\rho_{\lambda_j}, \mathcal{H}_{\lambda_j})$ of the corresponding affine Kac–Moody group $\widehat{\mathcal{L}}_{\Phi_j}(\widetilde{K}_j)$, where λ_j is an integral anti-dominant weight of central charge c_j. On the level of the Lie algebra

$$\widehat{\mathfrak{g}} = \mathbb{R}C \times_\omega \Big(\bigoplus_{j\in J} \mathcal{L}_{\varphi_j}(\mathfrak{k}) \rtimes \mathbb{R}D \Big),$$

the central element acts by $d\rho(C) = i\mathbf{1}$, and the generator D acts by

$$d\rho(D) = \sum_{j\in J} \frac{1}{T_j} d\rho_{\lambda_j}(\mathbf{d}_j + A_j),$$

where $A_j \in \mathcal{L}_{\varphi_j}(\mathfrak{k})$ is specified by the $\mathbb{R}$-action on G.

Proof. Since M is compact, the $\mathbb{R}$-invariant, embedded, one-dimensional submanifold S is a union of finitely many periodic orbits. By Corollary 7.5, the projective positive energy representation $\bar{\rho}$ of $\Gamma(S, \mathcal{K})_0$ thus arises by factorization from a projective positive energy representation of $G = \Gamma(S, \widetilde{\mathcal{K}})$, which is trivial on the image Z of the diagonal group $Z_{[M]}$ in $Z_{[S]}$ (cf. Remark 7.4). It then follows from (8.8) and the discussion in Section 8.2.1 that every factorial positive energy representation is a multiple of a product of lowest weight representations as described above. The only thing left to check is that this representation restricts to a character on the image Z of $Z_{[M]}$ in G. Since Z is a subgroup of the central group $Z_{[S]} = \prod_{j\in J} \pi_1(K_j)^{\Phi_j}$, it is in particular contained in the group $\prod_{j\in J}(\widetilde{K}_j)^{\Phi_j}$ of constant sections, which is connected by [39, Theorem 12.4.26]. Its Lie algebra $\bigoplus_{j\in J} \mathfrak{k}^{\varphi_j}$ is contained in the radical of the cocycle ω. Since $\mathcal{L}(\widetilde{K}_j)^{\Phi_j}$ is 1-connected (Lemma 7.16), this implies that Z is not only central in G, but also in $\widehat{G}$. In particular, every factor representation restricts to a character on Z. ∎

Remark 8.11 (Semisimple groups). In Theorem 8.10, the restriction to simple fibers is by no means essential. For Lie group bundles $\mathcal{K} \to M$ with compact semisimple, 1-connected fibers, the representation still localizes to an embedded 1-dimensional submanifold $S \subseteq M$ by Theorem 7.1. As M is compact, S consists of finitely many circles S_j. Since the fibers of $\mathcal{K} \to M$ are 1-connected, the passage from M to the finite cover $\widehat{M}$ (Theorem 4.9) yields not only a Lie algebra bundle $\widehat{\mathfrak{K}} \to \widehat{M}$, but also a Lie group bundle $\widehat{\mathcal{K}} \to \widehat{M}$ with simple, compact fibers. By the same argument as in Remark 4.10, the $\mathbb{R}$-action on $\mathcal{K} \to M$ lifts to $\widehat{\mathcal{K}} \to \widehat{M}$. Applying Theorem 8.10 to $\widehat{\mathcal{K}} \to \widehat{M}$, we find that the minimal factorial positive energy representations are again multiples of the irreducible ones. The latter are now parametrized by embedded circles $\widehat{S}_{j,r} \subseteq \widehat{M}$, together with an integral anti-dominant weight $\lambda_{j,r}$ with central charge $\lambda_{j,r}(C) = i c_{j,r}$. Here, the circle $\widehat{S}_{j,r} \subseteq \widehat{M}$ is a finite cover of the circle $S_j \subseteq M$. The weight $\lambda_{j,r}$ is associated to the Kac–Moody algebra $\widehat{\mathcal{L}}_{\Phi_{j,r}}(\mathfrak{k}_{j,r})$, where $\mathfrak{k}_{j,r}$ is a simple ideal in the semisimple Lie algebra $\mathfrak{k}_j$, and $\Phi_{j,r}$ is the smallest power of the holonomy around S_j that maps $\mathfrak{k}_{j,r}$ to itself.

8.3 Extensions to non-connected groups

In this section we discuss several phenomena related to non-connected variants of the group G. Dealing with non-connected groups is typically more complicated because they may not have a simply connected covering group, nor do central extensions or representations of the identity component always extend to the whole group.

This suggests the following classification scheme to deal with projective positive energy representations of $G \rtimes_\alpha \mathbb{R}$ if G is not connected.

- Determine which central extensions of G_0 extend to the non-connected groups G.

- Determine which of these do this in an $\mathbb{R}$-equivariant fashion. This leads us to central extensions of the non-connected group $G \rtimes_\alpha \mathbb{R}$.
- Determine the irreducible positive energy representations of the non-connected groups $\widehat{G}$ in terms of the representations of $\widehat{G}_0$ (this may be carried out with Mackey's method of unitary induction, as in [104]).

The following factorization theorem reduces the classification of the irreducible representations to the corresponding problem for the identity component G_0 and the group $\pi_0(G)$ of connected components. It shows in particular that no additional difficulties arise if K is a 1-connected simple group. We shall use the notation

$$G \to \pi_0(G), \quad g \mapsto [g]$$

for the quotient map.

Theorem 8.12 (Factorization theorem for non-connected gauge groups). *Suppose that K is a 1-connected simple compact Lie group, that M is compact and that $\mathbf{v}_M$ has no zeros. Then, every positive energy representation $(\rho, \mathcal{H})$ of $G^\sharp = \Gamma(M, \mathcal{K})^\sharp$ can be written as $\rho(g) = \rho'(g)\zeta([g])$, where ρ' factors through a 1-dimensional, closed, $\mathbb{R}$-equivariantly embedded submanifold $S \subseteq M$, and $\zeta\colon \pi_0(G) \to \mathrm{U}(\mathcal{H})$ is a representation that commutes with $\rho'(G_0^\sharp) = \rho(G_0^\sharp)$. In particular, every irreducible positive energy representation of $G^\sharp$ is of the form $\rho' \otimes \zeta$ where both ρ' and ζ are irreducible, and, conversely, any such tensor product is irreducible.*

Proof. Let $(\rho, \mathcal{H})$ be a positive energy representation of $G^\sharp$. From Theorem 8.10 we know that the restriction of ρ to $G_0^\sharp$ factors through an evaluation homomorphism

$$\mathrm{ev}\colon G \to G_S := \Gamma(S, \mathcal{K}) \cong \prod_{j \in J} \mathcal{L}_{\Phi_j}(K),$$

that is, there exists a positive energy representation ρ_1 of $G_S^\sharp$ such that

$$\rho|_{G_0^\sharp} = \rho_1 \circ \mathrm{ev}\,|_{G_0^\sharp}.$$

Since K is 1-connected, the groups $\mathcal{L}_{\Phi_j}(K)$ are connected and therefore G_S is connected. Then, $\rho' := \rho_1 \circ \mathrm{ev}$ is a positive energy representation of $G^\sharp$ that coincides with ρ on $G_0^\sharp$.

This construction shows in particular that $\pi_0(G)$ acts trivially on the set of equivalence classes of irreducible positive energy representation of $G_0^\sharp$. Indeed, for every irreducible representation ρ_1 of $G_S^\sharp$, the representation ρ' extends the representation $\rho_1 \circ \mathrm{ev}\,|_{G_0^\sharp}$ to a representation of $G^\sharp$ on the same space.

As every positive energy representation of $G^\sharp$ decomposes on $G_0^\sharp$ into irreducible ones (Theorem 8.10), it follows that $\rho(G^\sharp)$ preserves all the $G_0^\sharp$ isotypic subspaces

$\mathcal{H}_j \cong \mathcal{F}_j \otimes \mathcal{M}_j$, $j \in J$, and on these the representation of $G_0^\sharp$ has the form $\rho_j \otimes \mathbf{1}$. Extending ρ_j to a representation $\widetilde{\rho}_j$ of $G^\sharp$, the restriction of ρ from $\mathcal{H}$ to $\mathcal{H}_j$ takes the form $\widetilde{\rho}_j \otimes \zeta_j$, where $\zeta_j : \pi_0(G^\sharp) \cong \pi_0(G) \to \mathrm{U}(\mathcal{M}_j)$ is a unitary representation on the multiplicity space. Putting everything together, we obtain a factorization $\rho = \rho' \otimes \zeta$, where ζ is a representation of $\pi_0(G)$ that commutes with $\rho(G_0^\sharp)$.

In view of Schur's Lemma, our construction shows in particular that the representation ρ is irreducible if and only if it is isotypical on $G_0^\sharp$, that is, $\mathcal{H} = \mathcal{F} \otimes \mathcal{M}$, and the representation ζ of $\pi_0(G)$ on $\mathcal{M}$ is irreducible. ■

Remark 8.13. (a) If K is connected but not simply connected and $\mathfrak{k}$ is a compact simple Lie algebra, then the classification in [104] shows that not all central extensions of $\mathcal{L}(K)_0$ extend to the whole group $\mathcal{L}(K)$, so that the situation becomes more complicated. Likewise, irreducible projective positive energy representations of $\mathcal{L}(K)_0$ do not in general extend to the whole group $\mathcal{L}(K)$. In [104] one finds a classification of the irreducible projective positive energy representations of the groups $\mathcal{L}(K)$ for connected simple groups K. Here the new difficulty is that the group $\pi_0(\mathcal{L}(K)) \cong \pi_1(K)$ acts non-trivially on the alcove whose intersection with the weight lattice classifies the irreducible projective positive energy representations of the connected group $\mathcal{L}(K)_0 \cong \mathcal{L}(\widetilde{K})$ for a fixed central charge.

(b) If we start with a projective representation of the non-connected gauge group $\Gamma_c(M, \mathcal{K})$, we get a representation of the image of $\Gamma_c(M, \mathcal{K})$ in $\Gamma_c(S, \mathcal{K})$, which is a restricted direct product of twisted loop groups. It maps $\Gamma_c(M, \mathcal{K})_0$ onto the identity component, but additional information is contained in the images of the other connected components. We then get a projective representation of a Lie group whose Lie algebra is $\Gamma_c(S, \mathfrak{K})$ and whose group of connected components is an image of $\pi_0(\Gamma_c(M, \mathcal{K}))$. Its action on the Lie algebra does not permute the ideals of the type $\mathcal{L}_\varphi(\mathfrak{k})$, so it acts on each twisted loop algebra separately by the adjoint action of some element of $\mathcal{L}_\Phi(K)$. This suggests that one needs a description of those Lie algebra cocycles ω on $\Gamma_c(M, \mathfrak{K})$ that actually correspond to central Lie group extensions of the full group $\Gamma_c(M, \mathcal{K})$. Here the obstructions lie in $H^3(\pi_0(\Gamma_c(M, \mathcal{K})), \mathbb{T})$. We refer to [72] for further details on such obstructions and for methods of their computation.

(c) For a bundle of Lie groups $\mathcal{K} \to M$, passing to the simply connected covering of the structure group K may not always be possible. For this, an obstruction class in $H^3(M, \pi_1(K))$ has to vanish (see [83]). Since $\pi_1(K)$ is finite for semisimple compact groups K, this is a torsion class. So for a discrete central subgroup $D \subseteq K$, every bundle with structure group K factorizes to a bundle with structure group K/D, but in general, not all bundles with structure group K/D are of this form.