Chapter 9

The classification for M noncompact

Even in the noncompact case, the techniques developed so far open up a number of new perspectives. The localization Theorem 7.1 allows us to restrict attention to a 1-dimensional invariant submanifold $S \subseteq M$. If M is noncompact, then S can have infinitely many connected components S_j, each of which is diffeomorphic to either $\mathbb{R}$ or $\mathbb{S}^1$. We consider these two cases separately.

In Section 9.1 we consider the case where S consists of infinitely many lines. In order to arrive at a (partial) classification, we impose the additional condition that the positive energy representation $(\bar{\rho}, \mathcal{H})$ admits a cyclic ground state vector $\Omega \in \mathcal{H}$ that is unique up to scalar. In Theorem 9.11 we show that these *vacuum representations* are classified up to unitary equivalence by a central charge $c_j \in \mathbb{N}_0$ for every connected component $S_j \simeq \mathbb{R}$. The proof proceeds by reducing to the (important) special case $M = \mathbb{R}$, where the classification is essentially due to Tanimoto [102].

In Section 9.2 we consider the case where S consists of infinitely many circles. Here we impose the much less restrictive condition that $\mathcal{H}$ is a *ground state representation*. This means that $\mathcal{H}$ is generated by the space of ground states, but we do not require these ground states to be unique. We show that this condition is automatically satisfied if the periods (9.6) of the $\mathbb{R}$-action are uniformly bounded. In Theorem 9.16 we classify this type of representations in terms of C^*-algebraic data, using techniques similar to those used in [50] for norm-continuous representations. The possibility of an infinite-dimensional space of ground states gives rise to interesting phenomena, such as factor representations of type II and III.

Finally, in Section 9.3, we briefly explore a simple situation where the $\mathbb{R}$-action has a fixed point. The main thing we wish to point out is that the lift of the $\mathbb{R}$-action *at the fixed point* has a qualitative influence on the type of representation theory that one encounters. In Part II of this series we develop the necessary tools to resolve the positive energy representation theory in more detail.

9.1 Infinitely many lines

In contrast to the case of (twisted) loop groups, the classification of projective positive energy representations of $C^\infty_c(\mathbb{R}, K)$, for K a compact 1-connected simple Lie group, is an open problem—closely related to the classification problem for representations of loop group nets (cf. [103, 111] and Remark 9.4).

A large class of examples can be obtained by restricting projective positive energy representations of the loop group $G := \mathcal{L}(K)$ to $G_{\mathrm{cs}} := C^\infty_c(\mathbb{R}, K)$, where the latter is considered as a subgroup by identifying the circle with the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$. The restriction of an irreducible projective positive energy representation of $\mathcal{L}(K)$ remains irreducible, essentially by [103, Corollary IV.1.3.3]. In Section 9.1.1 we show that the restriction remains of positive energy as well. This is not a priori clear, since the positive energy is defined in terms of rotations of the circle for G and in terms of translations on the real line for G_{cs}.

It is not true that *all* projective unitary positive energy representations of G_{cs} arise by restriction in this way, and the classification remains an open problem. We can, however, classify the projective positive energy representations under the additional assumption that they admit a cyclic ground state vector which is unique up to scalar. These *vacuum representations* were classified by Tanimoto for the Lie algebra of $\mathfrak{k}$-valued Schwartz functions [102], and in Section 9.1.2 we use Theorem 6.30 to push these results to the compactly supported setting.

Finally, in Section 9.1.3, we classify the vacuum state representations for a noncompact manifold M with a free $\mathbb{R}$-action. The proof proceeds by identifying the restricted gauge group $\Gamma_c(S, \mathcal{K})$ with the weak product

$$\prod_j{}' C_c^\infty(S_j, K),$$

where j labels the connected components $S_j \simeq \mathbb{R}$. We then use the results from Appendix D, where we show that the classification of vacuum representations for a weak product of Lie groups reduces to the same problem for each of its factors.

9.1.1 Restriction from $\mathcal{L}(K)$ to $C_c^\infty(\mathbb{R}, K)$

By identifying the circle $\mathbb{S}^1$ with the real projective line $\mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{\infty\}$, we can consider $G_{\text{cs}} := C_c^\infty(\mathbb{R}, K)$ as a subgroup of the loop group $G := \mathcal{L}(K)$.

Note that the natural $\mathbb{R}$-action by translations on G_{cs} does not agree with the $\mathbb{R}$-action by rigid rotations on G. In terms of the real projective line, the rotation action of $\mathbb{R}/\mathbb{Z}$ is given by the fractional linear maps

$$R_t(x) = \frac{\cos \pi t \cdot x + \sin \pi t}{-\sin \pi t \cdot x + \cos \pi t}, \quad x \in \mathbb{R} \cup \{\infty\},\ [t] \in \mathbb{R}/\mathbb{Z},$$

whereas the translation action of is given by $T_t(x) = x + t$.

Proposition 9.1 (Restriction of positive energy representations). *Let $(\rho, \mathcal{H})$ be an irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ with respect to the $\mathbb{R}$-action by rotations. Then, the restriction of ρ to $C_c^\infty(\mathbb{R}, K)^\sharp$ is an irreducible positive energy with respect to the $\mathbb{R}$-action by translations.*

We first prove that the restriction remains irreducible, and then continue with the positive energy condition.

Proof of irreducibility. Let $G_* := \{\xi \in G : \xi(\infty) = \mathbf{1}\}$ be the subgroup of based loops. Since $\rho(G_*^\sharp)'' = \mathbb{C}\mathbf{1}$ by[1] [103, Corollary IV.1.3.3], it suffices to show that $\rho(G_{\mathrm{cs}}^\sharp)$ is dense in $\rho(G_*^\sharp)$ for the strong operator topology. By [79, Appendix A], the representation of $G^\sharp$ extends to a smooth representation of the Banach–Lie group $H^1(\mathbb{S}^1, K)$ of H^1-loops, whose Lie algebra is the space $H^1(\mathbb{S}^1, \mathfrak{k})$ of H^1-functions $\xi\colon \mathbb{S}^1 \to \mathfrak{k}$. Since these are the absolutely continuous functions whose derivatives are L^2, the derivative $\xi \mapsto \xi'$ maps the subspace $H^1_*(\mathbb{S}^1, \mathfrak{k})$ of H^1-functions that vanish in the base point homeomorphically to

$$L^2_*(\mathbb{S}^1, \mathfrak{k}) = \left\{\xi \in L^2(\mathbb{S}^1, \mathfrak{k}) : \int_{\mathbb{S}^1} \xi(t)dt = 0\right\}.$$

In this space the subspace $\{\eta' : \eta \in C_c^\infty(\mathbb{R}, \mathfrak{k})\}$ is easily seen to be dense. Since $G_*^\sharp$ is connected, this implies that $\rho(G_{\mathrm{cs}}^\sharp)$ is dense in $\rho(G_*^\sharp)$. ■

To prove the positive energy condition for the restriction, we need to compare the generator $\mathbf{d}_0$ of rigid rotations with the generator $\mathbf{d}_1$ of translations. In $\mathfrak{sl}(2, \mathbb{R})$, these are given by

$$\mathbf{d}_0 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{d}_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{9.1}$$

The fact that $\mathbf{d}_0$ and $\mathbf{d}_1$ generate the same Ad-invariant closed convex cone in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ leads to the following characterization (cf. [59, Section 1.3]).

Lemma 9.2. *For a unitary representation $(\rho, \mathcal{H})$ of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the generator $i\, d\rho(\mathbf{d}_0)$ is bounded from below if and only if $i\, d\rho(\mathbf{d}_1)$ is bounded from below. Moreover, if this is the case, then $i\, d\rho(\mathbf{d}_0) \geq 0$ and $i\, d\rho(\mathbf{d}_1) \geq 0$.*

In particular, an $\widetilde{\mathrm{SL}}(2, \mathbb{R})$-representation is of positive energy for $\mathbf{d}_0$ if and only if it is of positive energy for $\mathbf{d}_1$. To prove that the restriction from $\mathcal{L}(K)^\sharp$ to $C_c^\infty(\mathbb{R}, K)$ is of positive energy with respect to $\mathbf{d}_1$, it therefore suffices to extend the action by rigid rotations to an action of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. This is done using the Segal–Sugawara construction.

Proof of positive energy. Recall from [32, Section 7] and [33] that every irreducible projective positive energy representation $(\bar{\rho}_\lambda, \mathcal{H}_\lambda)$ of $\mathcal{L}(K)$ with lowest weight λ extends to a projective representation of the semidirect product

$$\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1),$$

where $\mathrm{Diff}_+(\mathbb{S}^1)$ acts on $\mathcal{L}(K)$ by $\alpha_\varphi(\xi) := \xi \circ \varphi^{-1}$. The cocycle

$$\omega(\xi, \eta) = \frac{c}{2\pi} \int_{\mathbb{S}^1} \kappa(\xi'(t), \eta(t))dt \tag{9.2}$$

[1] Alternatively, one can use [102, Theorem 6.4], which uses [7, Corollary 1.2.3].

is easily seen to be invariant under the action of $\mathrm{Diff}_+(\mathbb{S}^1)$, but it is much harder to verify the covariance of the representations $\bar{\rho}_\lambda$. In [32, Section 7.2], the representation of the Virasoro algebra obtained from the Segal–Sugawara construction is integrated to a group representation. Since this respects the semidirect product structure of $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$, it follows in particular that

$$\bar{\rho}_\lambda \circ \varphi \cong \bar{\rho}_\lambda \quad \text{for every } \varphi \in \mathrm{Diff}_+(\mathbb{S}_1). \tag{9.3}$$

By Schur's Lemma and the irreducibility of $\bar{\rho}_\lambda$, the projective representation $\bar{\rho}^P$ of $\mathrm{Diff}_+(\mathbb{S}^1)$ on $\mathcal{H}_\lambda$ is uniquely determined by the intertwining property

$$\bar{\rho}^P(\varphi)\bar{\rho}_\lambda(\xi)\bar{\rho}^P(\varphi)^{-1} = \bar{\rho}(\alpha_\varphi \xi) \quad \text{for } \xi \in \mathcal{L}(K), \varphi \in \mathrm{Diff}_+(\mathbb{S}^1).$$

Since $\mathrm{Diff}_+(\mathbb{S}^1)$ contains the group of rigid rotations with respect to which $\bar{\rho}_\lambda$ is a positive energy representation, the Hamiltonian $H = i\,\mathrm{d}\rho^P(\mathbf{d}_0)$ associated to $\mathbf{d}_0$ is bounded below. Since ρ^P is a positive energy representation of the Virasoro group, it restricts to a positive energy representation of its subgroup $\widetilde{\mathrm{SL}}(2,\mathbb{R})$, the simply connected cover of the group $\mathrm{PSL}(2,\mathbb{R}) \subseteq \mathrm{Diff}_+(\mathbb{S}^1)$ of fractional linear transformations of $\mathbb{S}^1 \cong \mathbb{P}_1(\mathbb{R})$. By Lemma 9.2, the generator $i\,\mathrm{d}\rho^P(\mathbf{d}_1)$ then has non-negative spectrum. ■

Remark 9.3. Since the cocycle (9.2) is invariant under the action of $\mathrm{Diff}_+(\mathbb{S}^1)$, twisting $\bar{\rho}_\lambda$with $\varphi \in \mathrm{Diff}_+(\mathbb{S}^1)$ leads to an irreducible projective unitary representation $\bar{\rho}_\lambda \circ \varphi$ with the same central charge c. By Proposition 8.6, there are only finitely many types of such representations satisfying the positive energy condition. If we knew a priori that this twist preserves the positive energy condition (which is presently not the case), then we could bypass the integration procedure in [32], and construct the projective representation of $\mathrm{Diff}_+(\mathbb{S}^1)$ as follows.

By the Epstein–Hermann–Thurston theorem, $\mathrm{Diff}(M)_0$ is a simple group for every compact connected smooth manifold M (see [18]). In particular, $\mathrm{Diff}_+(\mathbb{S}^1)$ is a simple group. Since it has no normal subgroup of finite index, it acts trivially on any finite set. This implies that $\bar{\rho}_\lambda \circ \varphi \cong \bar{\rho}_\lambda$ for every $\varphi \in \mathrm{Diff}_+(\mathbb{S}^1)$. The unitaries that implement this equivalence constitute a projective unitary representation of $\mathrm{Diff}_+(\mathbb{S}^1)$.

Remark 9.4. The class of positive energy representations is by no means exhausted by the representations of Proposition 9.1. We briefly sketch the construction of a class of type III_1 factor representations, following [19, 112].

Recall from [32, Section 7.2] that an irreducible positive energy vacuum representation ρ of $G^\sharp = \mathcal{L}^\sharp(K)$ gives rise to a vacuum representation ρ^P of $\mathrm{Diff}_+(\mathbb{S}^1)^\sharp$. If we lift the $\mathbb{R}$-action by translations along the 2-fold covering $q\colon \mathbb{S}^1 \to \mathbb{R} \cup \{\infty\} \cong \mathbb{S}^1$, we obtain a flow on $\mathbb{S}^1$ with exactly two fixed points. Its generator $\mathbf{v}$ is obtained from the vector field $\mathbf{d}_0$ generating rigid rotations by multiplication with a non-negative function. This implies that the operator $i\,\mathrm{d}\rho^P(\mathbf{v})$ is bounded from below.

Let $I \subseteq \mathbb{S}^1$ be one of the two connected components of $q^{-1}(\mathbb{R})$ and identify

$$G_{\mathrm{cs}} = C_c^\infty(\mathbb{R}, K)$$

with $C_c^\infty(I, K)$. Then, the restriction of ρ to $G_{\mathrm{cs}}^\sharp$ is a factor representation of type III_1. Combining this with the one-parameter group generated by the vector field $\mathbf{v}$, we obtain a projective positive energy representation of $G_{\mathrm{cs}}^\sharp \rtimes \mathbb{R}$ with respect to the translation action on $\mathbb{R}$ ([112, Proposition 3.2]). We refer to [17, 112] for further details (see also the Remark after [103, Theorem IV.2.2.1]).

More generally, we may consider smooth vector fields $\mathbf{v} \in \mathcal{V}(\mathbb{S}^1)$ which are non-negative multiples $f\mathbf{d}_0$, $f \geq 0$, of the generator $\mathbf{d}_0$ of rigid rotations. For vacuum representations of $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$, the corresponding selfadjoint operator $i\,\mathrm{d}\rho^P(\mathbf{v})$ is bounded from below (cf. [19]). If $I \subseteq \mathbb{S}^1$ is an open interval on which $\mathbf{v}$ has no zeros but for which $\mathbf{v}$ vanishes in the boundary ∂I, then we obtain an embedding

$$C_c^\infty(\mathbb{R}, \mathfrak{k}) \rtimes \mathbb{R} \cong C_c^\infty(I, \mathfrak{k}) \rtimes \mathbb{R} \hookrightarrow \mathcal{L}(\mathfrak{k}) \rtimes \mathbb{R}\mathbf{v}$$

that integrates to the group level, where we obtain a projective positive energy representation of $C_c^\infty(\mathbb{R}, K)$.

9.1.2 Vacuum representations of $C_c^\infty(\mathbb{R}, K)$

Although the classification of projective positive energy representations $(\bar{\rho}, \mathcal{H})$ of $C_c^\infty(\mathbb{R}, K)$ is an open problem in general, it can be resolved under the additional assumption that $\mathcal{H}$ admits a unique, cyclic ground state.

Definition 9.5. Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$.

(a) A *ground state vector* is a vector $\Omega \in \mathcal{D}(H) \subseteq \mathcal{H}$ such that $H\Omega = E_0\Omega$ for $E_0 := \inf(\mathrm{spec}(H))$. We denote the space of ground state vectors by $\mathcal{E}$.

(b) A *ground state representation* is a positive energy representation $(\rho, \mathcal{H})$ that is generated by its space of ground states, in the sense that the linear span of $\rho(\widehat{G})\mathcal{E}$ is dense in $\mathcal{H}$.

(c) A *vacuum representation* is a ground state representation where the ground state is unique up to scalar, $\mathcal{E} = \mathbb{C}\Omega$.

At the Lie algebra level, we obtain analogous definitions if we replace the requirement that $\rho(\widehat{G})\mathcal{E}$ is dense in $\mathcal{H}$ by the requirement that

$$\mathcal{U}(\mathfrak{g})\Omega = \mathcal{U}(\mathfrak{g}^\sharp)\Omega$$

is dense in $\mathcal{H}$. Although the translation between these two concepts requires some caution, the two notions turn out to be compatible for positive energy representations.

Proposition 9.6. *Let $(\rho, \mathcal{H})$ be a positive energy representation of $\widehat{G}$ with ground state vector Ω. Then, $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is dense in $\mathcal{H}$ if and only if Ω is cyclic under $\rho(G^\sharp)$.*

Proof. For a closed interval $I \subseteq \mathbb{R}$, let

$$G_I := \{\xi \in G = C_c^\infty(\mathbb{R}, K) : \xi(\mathbb{R} \setminus I) = \{e\}\}$$

denote the Fréchet–Lie subgroup of maps supported by I. We claim that the Lie group $G_I^\sharp$ is BCH, i.e., it is locally exponential and its Lie algebra $\mathfrak{g}_I^\sharp$ is BCH, which means that the Baker–Campbell–Hausdorff series defines an analytic local multiplication on a 0-neighborhood of $\mathfrak{g}$ ([30, Theorem 15.7.1]). For G_I this follows from [30, Example 7.1.4 (c)] because the BCH property is inherited from the target group K. Further [30, Theorem 15.4.19] implies that the centrally extended Lie algebra $\mathfrak{g}_I^\sharp$ is also locally exponential and the proof of this theorem shows that the analyticity of the local multiplication is inherited by the central extension.

Lemma 6.34 implies that Ω is an analytic vector for each element in $\mathfrak{g}_I^\sharp$, so that [76, Proposition 4.10] further entails that Ω is an analytic vector for $G_I^\sharp$. Hence, the closure of $\mathcal{U}(\mathfrak{g}_I^\sharp)\Omega$ is $G_I^\sharp$-invariant. As the interval I was arbitrary, the closure of $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is invariant under $G^\sharp$, hence also under[2] $\widehat{G}$, because Ω is an H-eigenvector. This shows that $\mathcal{U}(\mathfrak{g}^\sharp)\Omega$ is dense in $\mathcal{H}$ if and only if Ω is cyclic under $\rho(G^\sharp)$. ∎

The vacuum representations for the Lie algebra $\mathfrak{g}_{\mathcal{S}} = \mathcal{S}(\mathbb{R}, \mathfrak{k})$ of $\mathfrak{k}$-valued Schwartz functions have been classified by Yoh Tanimoto.

Theorem 9.7 (Tanimoto's classification theorem; [102, Corollary 5.8]). *Let $(\pi, \mathcal{H}^\infty)$ be a vacuum representation of $\widehat{\mathfrak{g}}_{\mathcal{S}}$ with respect to the $\mathbb{R}$-action by translations. Suppose that for all $\psi, \chi \in \mathcal{H}^\infty$, the functional $\xi \mapsto \langle \psi, \pi(\xi)\chi \rangle$ is a tempered distribution. Then, $(\pi, \mathcal{H}^\infty)$ is characterized up to unitary equivalence by its central charge $c \in \mathbb{N}_0$.*

Using the continuity results from Chapter 6, we show that Tanimoto's classification theorem remains true for the smaller Lie algebra $\mathfrak{g}_{\mathrm{cs}} := C_c^\infty(\mathbb{R}, \mathfrak{k})$ of compactly supported smooth $\mathfrak{k}$-valued functions. This is an important improvement because the relevant Lie algebra for the classification of ground states of loop group nets is not $\mathcal{S}(\mathbb{R}, \mathfrak{k})$, but $C_c^\infty(\mathbb{R}, \mathfrak{k})$ (cf. [102, Section 6]).

As usual, we denote

$$\widehat{\mathfrak{g}}_{\mathrm{cs}} := (\mathbb{R}C \oplus_\omega \mathfrak{g}_{\mathrm{cs}}) \rtimes \mathbb{R}D \quad \text{and} \quad \widehat{\mathfrak{g}}_{\mathcal{S}} := (\mathbb{R}C \oplus_\omega \mathfrak{g}_{\mathcal{S}}) \rtimes \mathbb{R}D,$$

[2] For the concept of an analytic map to make sense, we need the group to be analytic. Since the $\mathbb{R}$-action on G need not be analytic, the semidirect product $G \rtimes_\alpha \mathbb{R}$ is in general not an analytic Lie group. In particular, $\widehat{G}$ need not be an analytic Lie group.

where D acts by infinitesimal translations. Since the inclusion of $\mathcal{S}(\mathbb{R}, \mathfrak{k})$ in $H^1_\partial(\mathbb{R}, \mathfrak{k})$ is continuous, the following is an immediate consequence of Theorem 6.30.

Proposition 9.8. *Let $(\rho, \mathcal{H})$ be a positive energy representation of the group*

$$\widehat{G}_{\mathrm{cs}} := (C^\infty_c(\mathbb{R}, K) \rtimes \mathbb{R})^\sharp.$$

Then, the derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}_{\mathrm{cs}}$ extends uniquely to a positive energy representation r of $\widehat{\mathfrak{g}}_{\mathcal{S}}$ such that, for all $\psi, \chi \in \mathcal{H}^\infty$, the functional $\xi \mapsto \langle \psi, r(\xi)\chi\rangle$ is a tempered distribution.

Combined with Theorem 9.7, this immediately yields the classification of vacuum representations in the compactly supported setting.

Theorem 9.9 (Vacuum representations of $C^\infty_c(\mathbb{R}, K)$). *Let K be a 1-connected, compact, simple Lie group and $G_{\mathrm{cs}} = C^\infty_c(\mathbb{R}, K)$. Then, a vacuum representation $(\rho, \mathcal{H})$ of $\widehat{G}_{\mathrm{cs}}$ is characterized up to unitary equivalence by its central charge $c \in \mathbb{N}_0$.*

Proof. By Proposition 9.6, the derived representation $\mathtt{d}\rho$ of $\widehat{\mathfrak{g}}_{\mathrm{cs}}$ is a vacuum representation which by Proposition 9.8 extends to a continuous representation of the Lie algebra $\widehat{\mathfrak{g}}_{\mathcal{S}}$. By [102, Corollary 5.8], the latter is determined up to isomorphism by its central charge $c \in \mathbb{N}_0$. Since G_{cs} is connected (Lemma 7.16), the representation ρ of $\widehat{G}_{\mathrm{cs}}$ is uniquely determined by its derived Lie algebra representation (Theorem 2.13), and the result follows. ∎

In Section 9.1.1, we saw that the restriction of an irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ (with respect to rotations) yields an irreducible positive energy representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ (with respect to translations). We now show that the unique vacuum representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ with central charge c arises by restriction of the irreducible positive energy representation of $\mathcal{L}^\sharp(K)$ with lowest weight

$$\lambda = (ic, 0, 0).$$

Proposition 9.10. *The irreducible lowest weight representation of $\mathcal{L}^\sharp(K)$ with lowest weight λ restricts to a vacuum representation of $C^\infty_c(\mathbb{R}, K)^\sharp$ if and only if the restriction λ_0 of λ to $i\mathfrak{t}$ is zero.*

Proof. Recall from Section 9.1.1 that every irreducible projective positive energy representation of $\mathcal{L}(K) \rtimes \mathbb{R}\mathbf{d}_0$ with lowest weight λ extends to $\mathcal{L}(K) \rtimes \mathrm{Diff}_+(\mathbb{S}^1)$. By Lemma 9.2, this induces a unitary representation of

$$\widetilde{\mathrm{SL}}(2, \mathbb{R}) \subseteq \mathrm{Diff}_+(\mathbb{S}^1)^\sharp,$$

which is of positive energy not only with respect to $\mathbf{d}_0 \in \mathfrak{sl}(2, \mathbb{R})$, but also with respect to $\mathbf{d}_1 \in \mathfrak{sl}(2, \mathbb{R})$ (cf. (9.1)).

Recall from Remark 8.9 that the space $\mathcal{E}_0$ of ground states for $i\,\mathrm{d}\rho(\mathbf{d}_0)$ is an irreducible unitary K-representation. Its lowest weight λ_0 is the restriction of λ to $i\mathfrak{t}$. By the formula in [32, Theorem 3.5 (iii)], the minimal eigenvalue of $H_0 = i\,\mathrm{d}\rho(\mathbf{d}_0)$ is a positive multiple of the Casimir eigenvalue for K on $\mathcal{E}_0$. In particular, it vanishes if and only if $\lambda_0 = 0$, which is the case if and only if $\dim \mathcal{E}_0 = 1$,

$$\inf \operatorname{Spec}(H_0) = 0 \iff \lambda_0 = 0 \iff \dim \mathcal{E}_0 = 1. \tag{9.4}$$

By a result of Mautner and Moore [63, 66],

$$\ker(\mathrm{d}\rho(\mathbf{d}_0)) = \ker(\mathrm{d}\rho(\mathbf{d}_1)) \tag{9.5}$$

coincides with the subspace of vectors that are fixed under $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (See Appendix E for a simplified direct proof.). If $\lambda_0 = 0$, the ground state for $H_1 := i\,\mathrm{d}\rho(\mathbf{d}_1)$ is therefore unique up to a scalar.

Conversely, suppose that the space $\mathcal{E}_1$ of ground states for H_1 is non-trivial. Since the adjoint orbit through $\mathbf{d}_1$ contains $\mathbb{R}^+\mathbf{d}_1$, the spectrum of $H_1 = i\,\mathrm{d}\rho(\mathbf{d}_1)$ is scale invariant. Any ground state $H_1\Omega = E\Omega$ then has $E = 0$, and will satisfy $H_0\Omega = 0$ by (9.5). Since H_0 is non-negative, it has minimal eigenvalue zero, the space $\mathcal{E}_0$ of ground states for $i\,\mathrm{d}\rho(\mathbf{d}_0)$ is one-dimensional. We conclude that $\lambda_0 = 0$, and that $\mathcal{E}_1 \subseteq \mathcal{E}_0$ is one-dimensional as well. ∎

9.1.3 Vacuum representations for noncompact manifolds

Let $\mathcal{K} \to M$ be a bundle of 1-connected simple compact Lie groups over a 2nd countable manifold M, equipped with a smooth $\mathbb{R}$-action by automorphisms.

Theorem 9.11. *If the action of $\mathbb{R}$ on M is free, then up to unitary equivalence, there is a bijective correspondence between the following.*

(a) *Smooth projective unitary representations $\bar{\rho}\colon \Gamma_c(M,\mathcal{K})_0 \to \mathrm{PU}(\mathcal{H})$ extending to a vacuum representation of $\Gamma_c(M,\mathcal{K})_0^\sharp \rtimes_\alpha \mathbb{R}$ with smooth ground state vector Ω.*

(b) *Closed, embedded, 1-dimensional flow-invariant submanifolds S, together with a non-zero central charge $c_j \in \mathbb{N}$ for every connected component $S_j \simeq \mathbb{R}$ of S.*

Under this correspondence we have

$$(\mathcal{H},\Omega) = \bigotimes_{j\in J}(\mathcal{H}_j,\Omega_j) \quad \text{and} \quad \rho(g) = \bigotimes_{j\in J}\rho_j(g|_{S_j}),$$

where $(\rho_j,\mathcal{H}_j,\Omega_j)$ is the restriction to $C_c^\infty(\mathbb{R},K) \simeq \Gamma_c(S_j,\mathcal{K})$ of the lowest weight representation of $\mathcal{L}^\sharp(K)$ with lowest weight $\lambda = (c_j,0,0)$ and J is the countable set of connected components of S.

Proof. By the localization Theorem 7.1, every projective positive energy representation $\bar{\rho}$ factors through $\Gamma_c(S, \mathcal{K})$ for a closed, embedded, 1-dimensional submanifold $S \subseteq M$. It follows that ρ factors through $\Gamma_c(S, \mathcal{K})^\sharp$. Since M is 2^{nd} countable, S has at most countably many connected components S_j, $j \in J$, and the freeness of the action implies that each of these is $\mathbb{R}$-equivariantly isomorphic to $\mathbb{R}$. By Lemma D.3, the Lie group $\Gamma_c(S, \mathcal{K})$ is isomorphic to the weak product

$$G := \prod\nolimits'_{j \in J} G_j, \quad \text{with } G_j = \Gamma_c(S_j, \mathcal{K}).$$

The cocycle $\psi\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ on $\mathfrak{g} := \mathbf{L}(G) = \bigoplus_{j \in J} \mathfrak{g}_j$ vanishes on $\mathfrak{g}_i \times \mathfrak{g}_j$ for $i \neq j$. Since every G_j is connected, this implies that

$$G^\sharp \cong \left(\prod\nolimits'_{j \in J} G_j^\sharp\right) \Big/ N,$$

where $N \subseteq \prod'_{j \in J} \mathbb{T}_j$ is the kernel of the smooth character

$$\chi\colon \prod\nolimits'_{j \in J} \mathbb{T}_j \to \mathbb{T}, \quad (z_j)_{j \in J} \mapsto \prod_{j \in J} z_j.$$

The vacuum representations of $G^\sharp$ therefore correspond to vacuum representations of the weak product $\prod'_{j \in J} G_j^\sharp$ such that the central subgroup $\prod'_{j \in J} \mathbb{T}_j$ acts by χ. We may assume, without loss of generality, that the ground state energy is zero, $H\Omega = 0$.

By Theorem D.6, every vacuum representation $(\rho, \mathcal{H}, \Omega)$ of the weak product $\prod'_{j \in J} G_j^\sharp$ is a product of vacuum representations $(\rho_j, \mathcal{H}_j, \Omega_j)$ of $G_j^\sharp$, and by Proposition D.7, ρ is smooth with smooth ground state vector Ω if and only if all the ρ_i are smooth with smooth ground state Ω_j.

Since ρ_j is irreducible by Proposition D.5, its restriction to the central subgroup $\mathbb{T}_j \subseteq G_j^\sharp$ is a character $\chi_j\colon \mathbb{T}_j \to \mathbb{T}$. The product $\rho = \bigotimes_{j \in J} \rho_j$ acts by χ on the center $\prod'_{j \in J} \mathbb{T}_j$ if and only if $\chi_j(z) = z\mathbf{1}$ for all $j \in J$.

Using the free $\mathbb{R}$-action to identify $\mathcal{K}|_{S_j}$ with $\mathbb{R} \times K$, we obtain an $\mathbb{R}$-equivariant isomorphism between $G_j = \Gamma_c(S_j, \mathcal{K})$ and $C_c^\infty(\mathbb{R}, K)$ (cf. Section 7.3). By Theorem 9.9, the vacuum representations of $G_j^\sharp$ are characterized up to unitary equivalence by their central charge $c_j \in \mathbb{N}_0$, and by Proposition 9.10, $(\rho_j, \mathcal{H}_j, \Omega_j)$ is unitarily equivalent to the restriction to $C_c^\infty(\mathbb{R}, K)$ of the lowest weight representation of $\mathcal{L}^\sharp(K)$ with $\lambda = (c_j, 0, 0)$. If $c_j = 0$, then the corresponding representation is trivial, so we can omit both S_j and c_j from the description. ∎

9.2 Infinitely many circles

We continue with the case where all connected components S_j of S are circles. In marked contrast with the case of infinitely many lines, the projective positive energy representations associated to a single connected component S_j are well understood, allowing us to classify the projective positive energy representations of $\Gamma_c(S, \mathcal{K})$ under the much weaker condition that the Hilbert space $\mathcal{H}$ is generated by the space $\mathcal{E}$ of ground states. These are the *ground state representations* of Definition 9.5.

As before, we assume that K is a 1-connected compact Lie group, which is not a serious restriction as long as K is connected (cf. Remark 7.4). In Section 9.2.1 we describe the *spectral gap condition*, an essentially geometric sufficient condition for all positive energy representations to be generated by the space of ground states. The main result of this section is Theorem 9.16 in Section 9.2.2, where we describe the ground state representations in terms of the representation theory of UHF C^*-algebras.

9.2.1 The spectral gap condition

Following the line of reasoning in Chapter 8, we associate to every compact connected component S_j a "local" Hamiltonian H_j. If these local Hamiltonians have a uniform spectral gap, we say that $(\rho, \mathcal{H})$ satisfies the *spectral gap condition*. We show that this (essentially geometric) condition guarantees that the positive energy representations are generated by their space of ground states.

We continue with the notation

$$G = \Gamma_c(S, \mathcal{K}) \cong \prod\nolimits'_{j \in J} \mathcal{L}_{\Phi_j}(K_j),$$

where $\prod'_{j \in J}$ denotes the weak direct product as in Section D.1. As in Section 7.3, we identify S_j with $\mathbb{R}/\mathbb{Z}$, where the time translation $\gamma_{S,t}$ acts on $[x_j] \in S_j$ by

$$\gamma_{S,t}([x_j]) = \left[x_j + \frac{t}{T_j}\right].$$

The derivation acts on $\xi_j \in \mathcal{L}_{\Phi_j}(\mathfrak{k}_j)$ by

$$D\xi_j = \frac{1}{T_j}(\mathbf{d}_j \xi_j + [A_j, \xi_j]).$$

By choosing a suitable parametrization of $\mathcal{K}|_{S_j}$, we may assume that A_j is constant (see [79, Proposition 2.14] or [65, Section 5.2]) and lies in the maximal abelian subalgebra $\mathfrak{t}^\circ$ of $\mathfrak{k}^{\varphi_j}$ (Theorem B.2). By acting with the φ_j-twisted Weyl group $\mathcal{W}$, i.e., the Weyl group of the underlying Kac–Moody Lie algebra, we may also assume that $\mathbf{d}_j + A_j$ lies in the positive Weyl chamber, i.e., $(\alpha, n)(i(\mathbf{d}_j + A_j)) \geq 0$ for all positive roots $(\alpha, n) \in \Delta^+$ ([65, Section 3] and Appendix A).

In the following $(\rho_{\lambda_j}, \mathcal{H}_{\lambda_j})$ denotes the irreducible positive energy representation of

$$G_j^\sharp \cong \mathcal{L}_{\Phi_j}^\sharp(K_j) \cong \Gamma(S_j, \mathcal{K})^\sharp$$

with lowest weight λ_j (cf. Section 8.2). Then, the minimal eigenspace V_j^0 of $\mathbf{d}_j$ in $\mathcal{H}_{\lambda_j}$ is an irreducible K^Φ-representation. Since A_j is anti-dominant, the minimal eigenspace W_j^0 of H_j (which is also finite-dimensional by Kac–Moody theory) contains all weight vectors v_μ in V_j^0 with $\mu(A_j) = 0$. Note that W_j^0 is 1-dimensional for generic A_j and increases in dimension as $\mathbf{d}_j + A_j$ is contained in a smaller face of the Weyl chamber (or, equivalently, as A_j is contained in a smaller face of the Weyl alcove), and that $W_j^0 = V_j^0 = V_{\lambda_j^0}$ if $A_j = 0$. We denote the orthogonal projection $\mathcal{H}_{\lambda_j} \to V_j^0$ by P_j, and for a finite subset $F \subseteq J$, we set

$$P_F := \prod_{j \in F} P_j.$$

Let $(\rho, \mathcal{H})$ be a factorial projective positive energy representation of $G^\sharp$. Recall from Section 8.2.1 that, for every finite subset $F \subseteq J$, we have a tensor product decomposition $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}'_F$. Here

$$\mathcal{H}_F = \bigotimes_{j \in F} \mathcal{H}_{\lambda_j}$$

is a positive energy representation of $G^\sharp_F$ with Hamiltonian

$$H_F = \sum_{j \in F} H_{\lambda_j},$$

where

$$H_{\lambda_j} = i\,\mathrm{d}\rho_{\lambda_j}\left(\frac{1}{T_j}(\mathbf{d}_j + A_j)\right) - \frac{i}{T_j}\lambda_j(\mathbf{d}_j + A_j)\mathbf{1}$$

is the minimal non-negative Hamiltonian on $\mathcal{H}_{\lambda_j}$ from Section 3.3. The other factor $\mathcal{H}'_F$ is a minimal positive energy representation of $G^\sharp_{J\setminus F}$ with Hamiltonian H', and we have

$$H = H_F \otimes \mathbf{1} + \mathbf{1} \otimes H'.$$

The ground states for a $\widehat{G}$-representation (in the sense of Definition 9.5) can be characterized in terms of the "local" Hamiltonians as follows.

Lemma 9.12. *For a factorial minimal positive energy representation $(\rho, \mathcal{H})$ of $\widehat{G}$, a vector $\Omega \in \mathcal{D}(H) \subseteq \mathcal{H}$ is a ground state vector if and only if $H_{\lambda_j}\Omega = 0$ for every $j \in J$.*

Proof. "⇒": Suppose first that Ω is a ground state vector. Then, $0 \leq H_{\lambda_j} \leq H$ implies that $H_{\lambda_j}\Omega = 0$.

"⇐": Conversely, suppose that $H_{\lambda_j}\Omega = 0$ holds for all $j \in J$. By minimality, the cyclic subspace generated by Ω under $G^\sharp$ is $\widehat{G}$-invariant and the corresponding representation on this subspace is minimal. We may therefore assume that Ω is cyclic.

For every finite subset $F \subseteq J$, Ω is fixed by the operators $V_t^F := e^{-itH_F}$, $t \in \mathbb{R}$. These operators satisfy

$$V_t^F \rho(g) V_{-t}^F = \rho(\alpha_t(g)) \quad \text{for } g \in G_F^\sharp,\ t \in \mathbb{R}.$$

For any finite superset $F' \supseteq F$ we then have

$$V_t^{F'} \rho(g)\Omega = \rho(\alpha_t(g))\Omega = V_t^F \rho(g)\Omega \quad \text{for } g \in G_F^\sharp.$$

This means that V_t^F and $V_t^{F'}$ coincide on the closed subspace $\mathcal{H}^F$ generated by $\rho(G_F^\sharp)\Omega$. Since the union of these subspaces is dense in $\mathcal{H}$, we obtain a unitary one-parameter group $(V_t)_{t\in\mathbb{R}}$ on $\mathcal{H}$ whose restriction to $\mathcal{H}^F$ coincides with $(V_t^F)_{t\in\mathbb{R}}$. This implies that

$$V_t \rho(g) V_{-t} = \rho(\alpha_t(g)) \quad \text{for } g \in G^\sharp,\ t \in \mathbb{R}.$$

Write $V_t = e^{-it\widetilde{H}}$ for a positive selfadjoint operator $\widetilde{H}$. Then, our construction shows that $\widetilde{H}$ coincides with $H_F = \sum_{j\in F} H_{\lambda_j}$ on $\mathcal{H}^F$, and thus $\widetilde{H} \geq 0$. By minimality of H, we have $0 \leq H \leq \widetilde{H}$, so that $\widetilde{H}\Omega = 0$ leads to $H\Omega = 0$. ■

Definition 9.13 (Spectral gap). We say that the family $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the *spectral gap condition* if there exists a positive real number ΔE such that, for every $j \in J$,

$$\mathrm{Spec}(H_{\lambda_j}) \subseteq \{0\} \cup [\Delta E, \infty).$$

The spectral gap condition is essentially geometric in nature. Recall that for $m \in M$, the $\mathbb{R}$-action γ_t yields a group automomorphism $\gamma_t(m)\colon \mathcal{K}_m \to \mathcal{K}_{\gamma_M(m)}$. The spectral gap condition is automatically satisfied if the period

$$T(m) := \inf\bigl\{t > 0; \gamma_{M,t}(m) = m, \gamma_t = \mathrm{Id} \in \mathrm{Aut}(\mathcal{K}_m)\bigr\} \tag{9.6}$$

is uniformly bounded on M. Indeed, the $\mathbb{R}$-action on $\Gamma(S_j, \mathcal{K})$ then has period $T_j \leq \sup_{m\in M} T(m)$, so the spectrum of $\mathbf{d}_j + A_j$ in every minimal unitary positive energy representation will be contained in $(2\pi i/T_j)\mathbb{Z}$.

Proposition 9.14 (Spectral gaps yield ground state vectors). *Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ such that the corresponding family $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the spectral gap condition with some $\Delta E > 0$. Then, $\mathcal{H}$ is generated under $G^\sharp$ by the subspace* $\ker H$ *of ground state vectors.*

Proof. The minimality implies that 0 is the infimum of the spectrum of H, so that the spectral projection

$$P := P([0, \Delta E/2])$$

is non-zero. First we show that $P\mathcal{H}$ is contained in the kernel of every H_{λ_j}. In fact, the operator $H - H_{\lambda_j}$ is non-negative. Since the minimal non-zero spectral value of

H_{λ_j} is $\geq \Delta E$, it follows that $P\mathcal{H} \subseteq \ker H_{\lambda_j}$. Lemma 9.12 now shows that $H\Omega = 0$. Therefore, $\mathcal{F} := P\mathcal{H}$ coincides with the subspace $\ker H$ of ground state vectors.

Next we show that $\mathcal{F}$ is generating under $G^\sharp$. Let $\mathcal{H}^1 \subseteq \mathcal{H}$ be the closed subspace generated by $\mathcal{F}$ under $G^\sharp$. Then, we obtain a $G^\sharp$-invariant decomposition $\mathcal{H} = \mathcal{H}^1 \oplus \mathcal{H}^2$. Minimality of ρ now implies that it is also $\widehat{G}$-invariant, so that the Hamiltonian H decomposes accordingly as $H = H^1 \oplus H^2$. Since $\mathcal{F} \cap \mathcal{H}^2 = \{0\}$, we obtain $\mathcal{H}^2 = \{0\}$ by minimality of H^2 and the first part of the proof. This shows that $\mathcal{H} = \mathcal{H}^1$ is generated by $\mathcal{F}$ under $G^\sharp$. ■

9.2.2 Classification in terms of UHF C^*-algebras

As in [50], where we dealt with norm continuous representations of gauge groups, we aim at a description of the factor representations of positive energy in terms of C^*-algebras. As semiboundedness is crucial to obtain corresponding C^*-algebras ([81]), we first observe that positive energy representations are semibounded (cf. Definition 6.31).

Applying Corollary 6.33 with $M = S$, we immediately obtain the following result.

Theorem 9.15. *If all connected components of S are compact, then every projective positive energy representation $(\rho, \mathcal{H})$ of*

$$\Gamma_c(S, \mathcal{K}) \cong \prod\nolimits'_{j\in J} \mathcal{L}_{\Phi_j}(K_j)$$

is semibounded with the affine hyperplane $\Gamma_c(S, \mathfrak{K})^\sharp - D$ contained in the open cone W_ρ, so that W_ρ is an open half space. In particular, it is a positive energy representation for all derivations

$$D_A := D - \operatorname{ad} A, \quad A \in \Gamma_c(S, \mathfrak{K}).$$

Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ and let $(\lambda_j, A_j, T_j)_{j\in J}$ be as above. Since the projection $P_j : \mathcal{H}_{\lambda_j} \to V_j^0$ onto the minimal energy space for H_{λ_j} in $\mathcal{H}_{\lambda_j}$ is finite-dimensional, P_j is a compact operator. We may therefore consider the direct limit

$$\mathcal{B} := \bigotimes_{j\in J} \big(K(\mathcal{H}_{\lambda_j}), P_j\big) \tag{9.7}$$

of the C^*-algebras

$$\mathcal{B}_F := \bigotimes_{j\in F} \big(K(\mathcal{H}_{\lambda_j}), P_j\big), \quad F \subseteq J \text{ finite},$$

where the tensor product of the non-unital algebras $K(\mathcal{H}_{\lambda_j})$ is constructed as in [34] with the inclusions

$$\begin{aligned} \mathcal{B}_{F_1} &\hookrightarrow \mathcal{B}_{F_2}, \\ A &\mapsto A \otimes \bigotimes_{j \in F_2 \setminus F_1} P_j \end{aligned}$$

for finite subsets $F_1 \subseteq F_2$ of J. We write $B \otimes \bigotimes_{j \in J \setminus F} P_j$ for the image of $B \in \mathcal{B}_F$ in $\mathcal{B}$ and

$$P_\infty := \bigotimes_{j \in J} P_j .$$

If J is finite, then $\mathcal{B} \cong \mathcal{B}_J$ and the above tensor product is finite. The C^*-algebra $\mathcal{B}$ carries a natural one-parameter group of automorphisms $(\alpha_t^{\mathcal{B}})_{t \in \mathbb{R}}$ specified by

$$\alpha_t^{\mathcal{B}}(B) = e^{-itH_F} B e^{itH_F} \quad \text{for } t \in \mathbb{R},\ B \in \mathcal{B}_F,$$

which fixes the projection P_∞.

Since every ground state representation can be written as a direct sum of cyclic ones, we may assume, without loss of generality, that $\mathcal{H}$ has a cyclic ground state $\Omega \in \mathcal{H}$. This defines a state of $\mathcal{B}$ by

$$\omega(B) := \langle \Omega, B\Omega \rangle \quad \text{for } B \in \mathcal{B}_F$$

because P_j projects onto the kernel of H_{λ_j} which contains Ω. Conversely, if $(\pi, \mathcal{H})$ is a representation of the C^*-algebra $\mathcal{B}$ that is generated by a vector Ω with

$$\pi(P_\infty)\Omega = \Omega,$$

then we obtain commuting representations of the multiplier algebras $B(\mathcal{H}_{\lambda_j})$ of $K(\mathcal{H}_{\lambda_j})$. In particular, we recover a unitary representation of the restricted product

$$\prod\nolimits'_{j \in J} \mathrm{U}(\mathcal{H}_{\lambda_j}),$$

and hence, a unitary representation of $G^\sharp$. This representation extends canonically to a minimal positive energy representation of $\widehat{G}$, where the Hamiltonian H is determined uniquely by

$$e^{-itH}\pi(B)\Omega = \pi(\alpha_t^{\mathcal{B}}(B))\Omega \quad \text{for } B \in \mathcal{B}.$$

The representations constructed above are now positive energy representations for the C^*-dynamical system $(\mathcal{B}, \mathbb{R}, \alpha^{\mathcal{B}})$ generated by ground states (cf. [13]).

From this correspondence, we derive the following noncompact analog of Theorem 8.10.

Theorem 9.16. *Let $\mathcal{B}$ be the C^*-algebra constructed for $(\lambda_j, A_j, T_j)_{j\in J}$ with a possibly infinite index set J as above. Then, the above construction yields a one-to-one correspondence between the following.*

(a) *Isomorphism classes of minimal factorial positive energy representations of $\widehat{G}$ corresponding to the family $(\lambda_j, A_j, T_j)_{j\in J}$.*

(b) *Isomorphism classes of factorial representations of $\mathcal{B}$ that are generated by fixed points of the projection P_∞.*

Proof. "(a)⇒(b)": Let $(\rho, \mathcal{H})$ be a factorial minimal positive energy representation of $\widehat{G}$ corresponding to the family $(\lambda_j, A_j, T_j)_{j\in J}$. As J is at most countably infinite, we may assume, without loss of generality, that $J = \mathbb{N}$ (the case of finite J is proved along the same lines) and put $\mathcal{B}_n := \mathcal{B}_{F_n}$ for $F_n = \{1, \ldots, n\}$. Then, we inductively choose factorizations of $(\rho, \mathcal{H})$ as $(\rho_{F_n} \otimes \rho'_{F_n}, \mathcal{H}_{F_n} \otimes \mathcal{H}'_{F_n})$ with

$$\mathcal{H}_{F_n} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n}, \quad \rho_{F_n} \cong \rho_{\lambda_1} \otimes \cdots \otimes \rho_{\lambda_n}.$$

We then obtain a consistent sequence of representations of the C^*-algebras $\mathcal{B}_n$ on the subspaces $\mathcal{H}_n := \mathcal{H}_{F_n} \otimes \mathcal{E}'_n$, where $\mathcal{E}'_n \subseteq \mathcal{H}'_{F_n}$ is the minimal eigenspace of H'_F on $\mathcal{H}'_{F_n}$, by

$$\pi_n \colon \mathcal{B}_n \to B(\mathcal{H}_{F_n} \otimes \mathcal{E}'_n), \quad \pi_n(B) := B \otimes \mathbf{1} \quad \text{for } B \in \mathcal{B}_n.$$

As the union of the subspaces $(\mathcal{H}_n)_{n\in\mathbb{N}}$ is dense in $\mathcal{H}$, we thus obtain a non-degenerate representation $(\pi, \mathcal{H})$ of $\mathcal{B}$ satisfying

$$\rho_{F_n}(g)\pi_n(B) = \pi_n(\rho_{F_n}(g)B) \quad \text{for } g \in G^\sharp_{F_n}, B \in \mathcal{B}_n, \tag{9.8}$$

and $\pi(P_\infty)$ is the projection onto the minimal eigenspace of H. Note that (9.8) determines the representation ρ uniquely in terms of the representation $(\pi, \mathcal{H})$ of $\mathcal{B}$.

"(b)⇒(a)": Suppose, conversely, that $(\pi, \mathcal{H})$ is a factorial representation of $\mathcal{B}$ generated by the subspace $\mathcal{E} := P_\infty \mathcal{H}$. Then, the union of the closed subspaces

$$\mathcal{H}_n := \pi(\mathcal{B}_n)\pi(P_\infty)\mathcal{H}$$

is dense in $\mathcal{H}$. Since the representation of $\mathcal{B}_n \cong \mathcal{K}(\mathcal{H}_{F_n})$ on $\mathcal{H}_n$ is non-degenerate, we obtain consistent factorizations

$$\mathcal{H}_n \cong \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_n} \otimes \mathcal{E}'_n \cong \mathcal{H}_{F_n} \otimes \mathcal{E}'_n \quad \text{with } \pi(B) = B \otimes \mathbf{1}_{\mathcal{E}'_n}, \ B \in \mathcal{B}_n.$$

This implies the existence of smooth unitary representations ρ_n of the groups $G^\sharp_{F_n}$ on $\mathcal{H}_n$ which are uniquely determined by

$$\rho_n(g)\pi(B)\pi(P_\infty) = \pi(\rho_{F_n}(g)B)\pi(P_\infty) \quad \text{for } g \in G^\sharp_{F_n}, B \in \mathcal{B}_n.$$

The uniqueness implies that $\rho_{n+1}(g)|_{\mathcal{H}_n} = \rho_n(g)$ for $g \in G^\sharp_{F_n}$, so that we obtain a unitary representation of $G^\sharp = \bigcup_F G^\sharp_F$ on $\mathcal{H}$ which naturally extends to $\widehat{G} = G^\sharp \rtimes \mathbb{R}$. Its continuity follows from [27, Lemma 4.4].

For the smoothness we use [114, Theorem 2.9]: The Lie algebra $\widehat{\mathfrak{g}}$ is the union of the subalgebras $\widehat{\mathfrak{g}}_{F_n}$, and the representation is smooth on the corresponding subgroup $\widehat{G}_{F_n}$. Further, the element $D \in \bigcap_n \widehat{\mathfrak{g}}_{F_n}$ lies in the interior of the open cones

$$W_{\rho|_{\widehat{G}_{F_n}}} \supseteq \mathfrak{g}^\sharp_{F_n} + (0,\infty)D,$$

which are open half spaces (Theorem 9.15). To apply Zellner's theorem, we have to show that the groups $\widehat{G}_{F_n}$ have the Trotter property, i.e., for any two elements x, y in the Lie algebra, we have

$$\exp(t(x+y)) = \lim_{n\to\infty} \left(\exp\left(\frac{t}{n}x\right) \exp\left(\frac{t}{n}y\right)\right)^n$$

in the sense of uniform convergence on compact subsets of $\mathbb{R}$. We first use [80, Theorem 4.11] to see that $G_{F_n} \rtimes \mathbb{R}$ has the Trotter property; as these groups are C^0-regular ([29, Theorem J]) and [80, Theorem 4.15] implies that the central extension $\widehat{G}_{F_n}$ also has the Trotter property. As any two elements $x, y \in \widehat{\mathfrak{g}}$ are contained in some $\widehat{\mathfrak{g}}_{F_n}$, the group $\widehat{G}$ also has the Trotter property. Therefore, [114, Theorem 2.9(a)] implies that the dense subspace $\mathcal{D}^\infty(\mathtt{d}\rho(D))$ of smooth vectors of the Hamiltonian coincides with $\mathcal{D}^\infty_c(\widehat{\mathfrak{g}})$, the set of all vectors ξ in the common domain of all finite products of elements in $\widehat{\mathfrak{g}}$, for which all maps

$$\widehat{\mathfrak{g}}^n \to \mathcal{H}, \quad (x_1,\ldots,x_n) \mapsto \mathtt{d}\rho(x_1)\cdots \mathtt{d}\rho(x_n)\xi$$

are continuous and n-linear. As the subgroup $G^\sharp$ is locally exponential (see [74, Lemma 4.3]) now implies that ξ is a smooth vector for $G^\sharp$, and since it is also smooth for $H = i\overline{\mathtt{d}\rho(D)}$, [74, Theorem 7.2] further entails that it is smooth for $\widehat{G}$. This proves the smoothness of ρ.

Clearly, the two constructions are mutually inverse, up to unitary equivalence. ∎

Remark 9.17. (a) By Lemma 9.12, the preceding theorem covers all minimal factorial representations for which $(\lambda_j, A_j, T_j)_{j\in J}$ satisfies the spectral gap condition.

(b) The projection $P_\infty \in \mathcal{B}$ defines the hereditary subalgebra $\mathcal{A} := P_\infty \mathcal{B} P_\infty$ onto which

$$\varepsilon\colon \mathcal{B} \to \mathcal{A}, \quad B \mapsto P_\infty B P_\infty$$

defines a conditional expectation, so, in particular, a completely positive map. From this perspective, the representations specified in Theorem 9.16 are precisely those obtained by Stinespring dilation from the completely positive maps that have the form $\omega = \pi \circ \varepsilon$, where $(\pi, \mathcal{F})$ is a non-degenerate representation of $\mathcal{A}$.

For $n_j := \operatorname{tr} P_j$, we have

$$\mathcal{A} \cong \bigotimes_{j \in J} M_{n_j}(\mathbb{C}),$$

showing that $\mathcal{A}$ is a UHF algebra [93]. The representation theory of these algebras also appears naturally in the context of norm continuous representations of gauge groups (cf. [50]). If infinitely many of the n_j are > 1, this leads to factor representations of type II and III. So the situation depends on the size of the minimal energy spaces in $\mathcal{H}_{\lambda_j}$. In particular, we obtain factorial representations as infinite tensor products corresponding to factorial product states on $\mathcal{A}$ because they correspond to product states on $\mathcal{B}$. We refer to [50] for details on the connection between norm-continuous representations of the restricted product $\prod'_{j \in J} K_j$ of the compact groups K_j and representations of infinite tensor products of matrix algebras.

9.3 A simple example with fixed points

In Part II of this series, we will focus on the type of phenomena one encounters when the $\mathbb{R}$-action on M is *not* fixed point free. To give a preview of the problems one encounters there, we briefly revisit the simple example of the circle action on $\mathbb{S}^2$, lifted to an $\mathbb{R}$-action on the trivial bundle $\mathcal{K} = \mathbb{S}^2 \times K$ (cf. Example 7.9). The fixed points are then the "north pole" $n = (0, 0, 1)$ and the "south pole" $(0, 0, -1)$.

Since every projective positive energy representation of $G = C^\infty(\mathbb{S}^2, K)$ restricts to a projective positive energy representation of the normal subgroup

$$G^\times = C_c^\infty(\mathbb{S}^2 \setminus \{n, s\}, K),$$

we can apply the techniques developed so far to $G^\times$. The two problems that remain are then to determine if a representation extends from $G^\times$ to G, and, if so, to classify the possible extensions. We will pursue these problems elsewhere, and for the moment content ourselves with describing the representation theory of $G^\times$. Although the Lie algebra bundle $\mathfrak{K} \to \mathbb{S}^2$ is trivial, the $\mathbb{R}$-action (7.5) on $\mathfrak{K}$ that covers the circle action on $\mathbb{S}^2$ will in general not be trivializable. It turns out that the lift of the circle action at the fixed points $n, s \in \mathbb{S}^2$ has a qualitative effect on the positive energy representation theory of $G^\times$.

By Theorem 7.1, every projective positive energy representation of $G^\times$ factors through a projective positive energy representation of $C_c^\infty(S, K)$, where

$$S = \{(x, y, z) \in \mathbb{S}^2 : z \in J\}$$

is a union of circles labeled by a discrete subset $J \subset (-1, 1)$ that has at most two accumulation points ± 1, corresponding to the fixed points n and s. Recall from

Example 7.9 that the fundamental vector field for the $\mathbb{R}$-action is of the form

$$\mathbf{v}(x, y, z) = (y\partial_x - x\partial_y) + A(x, y, z),$$

with $A(x, y, z) \in \mathfrak{k}$. For simplicity, consider first the case where $A \in \mathfrak{k}$ is *independent* of (x, y, z). If we identify the loop algebras of the various circles in the obvious manner, then the infinitesimal $\mathbb{R}$-action is represented by the *same* element $\mathbf{d}_j + A_j = \mathbf{d} + A$ for every circle S_j. It follows that $(\lambda_j, A_j, T_j) = (\lambda_j, A, 2\pi)$, so the C^*-algebra $\mathcal{A} = P_\infty \mathcal{B} P_\infty$ that governs the ground state representations is essentially determined by a sequence λ_j of anti-dominant integral weights for the affine Kac–Moody algebra $\widehat{\mathcal{L}}(\mathfrak{k})$.

The operators $H_j = i\pi_{\lambda_j}(\mathbf{d} + A)$ are readily seen to satisfy the spectral gap property 9.13. Indeed, the operators $i\pi_{\lambda_j}(\mathbf{d})$ on $\mathcal{H}_{\lambda_j}$ have a uniform spectral gap because the $\mathbb{R}$-action on $\mathbb{S}^2$ is periodic. Since $A \in \mathfrak{k}$ has a uniform spectral gap in all finite-dimensional lowest weight representations, it also has a uniform spectral gap in the minimal eigenspaces W_j^0 of the operators $i\pi_{\lambda_j}(\mathbf{d})$. The spectral gap for H_j then follows from the fact that $\mathbf{d}$ commutes with A in $\widehat{\mathcal{L}}(\mathfrak{k})$.

By Proposition 9.14, every factorial positive energy representation of $\widehat{G}^\times$ is a ground state representation, so the factorial projective positive energy representations are completely classified by Theorem 9.16.

(a) If A is an inner point of the Weyl chamber, then the minimal eigenspace of H_j in an irreducible $\mathfrak{k}$-representation is always 1-dimensional, $W_j^0 = \mathbb{C}\Omega_j$. In this case every projective irreducible positive energy representation is a *vacuum representation*, and it is of the form

$$(\mathcal{H}, \Omega) = \bigotimes_{j \in j} (\mathcal{H}_{\lambda_j}, \Omega_j) \tag{9.9}$$

by the results in Section D.2. Moreover, every factorial positive energy representation of $\widehat{G}^\times$ is of type I, i.e., a direct sum of irreducible representations. This follows from the fact that, if in the construction of Section 9.2 all projections P_j are of rank 1, then the projection $P_\infty \in \mathcal{B}$ has the property that the subalgebra $P_\infty \mathcal{B} P_\infty$ is one-dimensional. In particular, $P_\infty a P_\infty = \varphi(a) P_\infty$ defines a state of $\mathcal{B}$ and every representation $(\pi, \mathcal{H})$ of $\mathcal{B}$ generated by the range of $\pi(P_\infty)$ is a multiple of the GNS representation $(\pi_\varphi, \mathcal{H}_\varphi)$. For unit vectors $\Omega_j \in \mathrm{im}(P_j)$, (9.7) implies that

$$(\mathcal{H}_\varphi, \Omega_\varphi) \cong \bigotimes_{j \in J} (\mathcal{H}_{\lambda_j}, \Omega_j),$$

that the representation π_φ is faithful, and that $\pi_\varphi(\mathcal{B}) \cong K(\mathcal{H}_\varphi)$.

(b) If A lies in at least one face of the Weyl alcove, then the space W_j^0 of ground states need not be 1-dimensional. The projective positive energy factor representations of $\mathfrak{g}^\times$ are then classified by the lowest weights λ_j, together with a representation of the UHF C^*-algebra

$$\mathcal{A} = \bigotimes_{j \in J} B(W_j^0).$$

If W_j^0 is of dimension > 1 for infinitely many j, then this is an infinite tensor product of matrix algebras. By [93] it follows that $G^\times$ admits factor representations of type II and III.

If $A(x, y, z)$ is not constant and $A(n)$ and $A(s)$ are inner points of the Weyl alcove, then the situation remains qualitatively the same as in (a). Indeed, the holonomy with respect to A on S_j will approach $\exp(A(n))$ or $\exp(A(s))$ as $z_j \to \pm 1$, so the spectral gap condition holds for all but finitely many circles. In this case one finds a tensor product decomposition analogous to (9.9), where all but finitely terms are vacuum representations. In particular, the space of ground states is finite-dimensional.

However, if either $A(n)$ or $A(s)$ is not an inner point of the Weyl alcove, then the spectral gap condition need no longer be satisfied. The ground state representations can still be classified in the manner outlined above, but these can no longer be expected to exhaust the positive energy representations.