Appendix A

Twisted loop algebras and groups

Let K be a simple compact Lie group, $\Phi \in \mathrm{Aut}(K)$ and $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$. We assume that $\varphi^N = \mathrm{id}_K$ and let

$$\mathcal{L}_\Phi^T(K) := \big\{\xi \in C^\infty(\mathbb{R}, K) : (\forall t \in \mathbb{R})\xi(t+T) = \Phi^{-1}(\xi(t))\big\}$$

be the corresponding *twisted loop group*. The rotation action $(\alpha_t f)(s) := f(s+t)$ satisfies

$$\alpha_{NT} = \mathrm{id}_{\mathcal{L}_\Phi(K)}.$$

The Lie algebra of $\mathcal{L}_\varphi^T(K)$ is the twisted loop algebra

$$\mathcal{L}_\varphi^T(\mathfrak{k}) := \big\{\xi \in C^\infty(\mathbb{R}, \mathfrak{k}) : (\forall t \in \mathbb{R})\xi(t+T) = \varphi^{-1}(\xi(t))\big\}.$$

Accordingly, we obtain

$$\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k}) := \big(\mathbb{R} \oplus_\omega \mathcal{L}_\varphi^T(\mathfrak{k})\big) \rtimes_D \mathbb{R}, \quad D\xi = \xi',$$

where

$$\omega(\xi, \eta) := \frac{c}{2\pi T}\int_0^T \kappa(\xi'(t), \eta(t))dt$$

for some $c \in \mathbb{Z}$ (the *central charge*). Here κ is the Killing form of $\mathfrak{k}$, normalized as in (4.2) by $\kappa(i\alpha^\vee, i\alpha^\vee) = 2$ for the coroots corresponding to long roots.

We write

$$\mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}) = \mathbb{R} \oplus_\omega \mathcal{L}_\varphi^T(\mathfrak{k}).$$

Let $\mathfrak{t}^\circ \subseteq \mathfrak{k}^\varphi$ be a maximal abelian subalgebra, so that $\mathfrak{z}_\mathfrak{k}(\mathfrak{t}^\circ)$ is maximal abelian in $\mathfrak{k}$ by [79, Lemma D.2] (see also [54]). Then, $\mathfrak{t} = \mathbb{R} \oplus \mathfrak{t}^\circ \oplus \mathbb{R}$ is maximal abelian in $\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})$ and the corresponding set of roots Δ can be identified with the set of pairs (α, n), where

$$(\alpha, n)(z, h, s) := (0, \alpha, n)(z, h, s) = \alpha(h) + is\frac{2\pi n}{NT}, \quad n \in \mathbb{Z}, \alpha \in \Delta_n. \qquad \text{(A.1)}$$

Here, $\Delta_n \subseteq i(\mathfrak{t}^\circ)^*$ is the set of $\mathfrak{t}^\circ$-weights in

$$\mathfrak{k}_\mathbb{C}^n = \big\{x \in \mathfrak{k}_\mathbb{C} : \varphi^{-1}(x) = e^{2\pi i n/N}x\big\}.$$

For $(\alpha, n) \neq (0, 0)$, the corresponding root space is

$$\mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_\mathbb{C})^{(\alpha,n)} = \mathfrak{k}_\mathbb{C}^{(\alpha,n)} \otimes e_n = (\mathfrak{k}_\mathbb{C}^\alpha \cap \mathfrak{k}_\mathbb{C}^n) \otimes e_n, \quad \text{where } e_n(t) = e^{\frac{2\pi i n t}{NT}}.$$

The set

$$\Delta^\times = \{(\alpha, n) : 0 \neq \alpha \in \Delta_n, n \in \mathbb{Z}\}$$

has an N-fold layer structure

$$\Delta^\times = \bigcup_{n=0}^{N-1} \Delta_n^\times \times (n + N\mathbb{Z}), \quad \text{where } \Delta_n^\times := \Delta_n \setminus \{0\}.$$

For $n \in \mathbb{Z}$ and $x \in \mathfrak{k}_{\mathbb{C}}^{(\alpha,n)}$ with $[x, x^*] = \alpha^\vee$, the element $e_n \otimes x \in \mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_{\mathbb{C}})^{(\alpha,n)}$ satisfies $(e_n \otimes x)^* = e_{-n} \otimes x^*$, which leads to the coroot

$$[e_n \otimes x, (e_n \otimes x)^*] = (\alpha, n)^\vee = \Big(-i \frac{cn}{NT} \frac{\|\alpha^\vee\|^2}{2}, \alpha^\vee, 0 \Big) = \alpha^\vee - \frac{icn}{NT} \frac{\|\alpha^\vee\|^2}{2} C, \tag{A.2}$$

where $C = (1, 0, 0)$. Here, we have used that

$$\omega(e_n \otimes x, e_{-n} \otimes x^*) = \frac{icn}{NT} \kappa(x, x^*)$$

and

$$\kappa(x, x^*) = \frac{1}{2}\kappa([\alpha^\vee, x], x^*) = \frac{1}{2}\kappa(\alpha^\vee, [x, x^*]) = \frac{1}{2}\kappa(\alpha^\vee, \alpha^\vee) = -\frac{1}{2}\|\alpha^\vee\|^2.$$

Since $\mathfrak{k}$ is simple, $\Delta^\times$ does not decompose into two mutually orthogonal proper subsets ([79, Lemma D.3]), so that

$$\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})_{\mathbb{C}}^{\mathrm{alg}} := \mathfrak{t}_{\mathbb{C}} + \sum_{(\alpha,n)\in\Delta} \mathcal{L}_\varphi^{T,\sharp}(\mathfrak{k}_{\mathbb{C}})^{(\alpha,n)}$$

is an affine Kac–Moody–Lie algebra (see [54, Theorem 8.5] and [38, Chapter X]). In this context the root (α, n) is real if and only if $\alpha \neq 0$. Choosing a positive system $\Delta^+ \subseteq \Delta$ such that the roots (α, n), $n > 0$, are positive, the lowest weights of unitary lowest weight representations of $\widehat{\mathcal{L}}_\varphi^T(\mathfrak{k})$ are the anti-dominant integral weights

$$\mathcal{P}(\mathfrak{t}, \Delta^+) := \{\lambda \in i\mathfrak{t}^* : (\forall (\alpha, n))\ 0 \neq \alpha, (\alpha, n) \in \Delta^+ \Rightarrow \lambda((\alpha, n)^\vee) \in \mathbb{N}_0\}.$$

Note that, for $n > 0$, we have

$$\lambda((\alpha, n)^\vee) = \lambda(\alpha^\vee) + \frac{cn}{NT} \frac{\|\alpha^\vee\|^2}{2},$$

so that we obtain $c > 0$ as a necessary condition for the existence of non-trivial unitary lowest weight modules.