Appendix B

Twisted conjugacy classes in compact groups

In this appendix we collect some more details concerning twisted conjugacy classes in compact groups.

A *Cartan subgroup* of a compact Lie group K is an abelian subgroup S topologically generated by a single element s ($s^{\mathbb{Z}}$ is dense in S) which has finite index in its normalizer $N_K(S) = \{k \in K : kSk^{-1} = S\}$.

Remark B.1. (a) For any Cartan subgroup S, the identity component S_0 is an abelian compact Lie group, hence a torus, and since tori are divisible, the short exact sequence $S_0 \hookrightarrow S \twoheadrightarrow \pi_0(S)$ splits, so that $S \cong S_0 \times \pi_0(S)$. By construction, $\pi_0(S)$ is a finite cyclic group. If $s_0 \in S_0$ is a topological generator, then, for every $N \in \mathbb{Z}$, the closure of $s_0^{N\mathbb{Z}}$ is a closed subgroup of finite index in S_0, hence equal to S_0. This implies that the topological generators of S are the elements of the form $s = (s_0, s_1) \in S_0 \times \pi_0(S)$, where s_0 is a topological generator of S_0 and s_1 is a generator of the cyclic group $\pi_0(S)$.

(b) By [14, Proposition IV.4.2], every element $k \in K$ is contained in a Cartan subgroup S such that the connected component kS_0 generates $\pi_0(S)$. The preceding discussion now shows that there exists an element $s_0 \in S_0$ such that $z := ks_0$ is a topological generator of S. Now [14, Proposition IV.4.3] implies that every element $g \in kK_0 = zK_0$ is conjugate to an element of kS_0.

Theorem B.2. *Let K be a compact connected Lie group and $\Phi \in \mathrm{Aut}(K)$ be an automorphism of finite order $N \in \mathbb{N}$. We consider the twisted conjugation action of K on itself given by*

$$g * k := gk\Phi(g)^{-1} \quad \textit{for } g, k \in K.$$

Then, the orbit of every element in K under this action intersects a maximal torus T^{Φ} of the subgroup K^{Φ} of Φ-fixed points.

Proof. We consider the compact Lie group $K_1 := K \rtimes \Phi^{\mathbb{Z}}$, where $\Phi^{\mathbb{Z}} \subseteq \mathrm{Aut}(K)$ is the finite subgroup generated by Φ. For $g, k \in K$, we then have

$$(g, \mathbf{1})(k, \Phi)(g, \mathbf{1})^{-1} = (gk\Phi(g)^{-1}, \Phi),$$

so that the conjugacy classes in the coset $K \times \{\Phi\} \subseteq K_1$ correspond to the Φ-twisted conjugacy classes in K.

According to Remark B.1 (b), the element $(\mathbf{1}, \Phi) \in K_1$ is contained in a Cartan subgroup S which is generated by an element of the form $z = (s_0, \Phi)$. As S_0 is abelian and commutes with $(\mathbf{1}, \Phi)$, it is contained in K^Φ. Let $T^\Phi \subseteq K^\Phi$ be a maximal torus containing S_0. Then, T^Φ commutes with S, so that the finiteness of $N_K(S)/S$ shows that $T^\Phi \subseteq S_0$. We conclude that

$$S = T^\Phi \times \Phi^{\mathbb{Z}}$$

is a Cartan subgroup of K_1. Therefore, Remark B.1 (b) implies that every Φ-twisted conjugacy class in K intersects $S_0 = T^\Phi \subseteq K^\Phi$. ■

We refer to [65] for more details on twisted conjugacy classes in compact groups, representatives, and stabilizer groups.

Remark B.3. If Φ is not of finite order, then the situation is more complicated. If, however, K is a compact Lie group with semisimple Lie algebra, then $\mathrm{Aut}(K)$ is a compact group with the same Lie algebra and one can apply the theory of Cartan subgroups of compact Lie groups to $\mathrm{Aut}(K)$.