Appendix C

Restricting representations to normal subgroups

Theorem C.1. *Let G be a group, and let $N \trianglelefteq G$ be a normal subgroup of finite index. Suppose that $(\pi, \mathcal{H})$ is a unitary representation of G whose restriction $\pi|_N$ decomposes discretely with finitely many isotypic components. Then, the same holds for π.*

Proof. We consider the two von Neumann algebras

$$\mathcal{N} := \pi(N)'' \subseteq \mathcal{M} := \pi(G)''.$$

Let

$$\mathcal{H} = \bigoplus_{j=1}^{m} \mathcal{H}_j, \quad \text{with } \mathcal{H}_j = \mathcal{F}_j \otimes \mathcal{C}_j$$

be the isotypic decomposition for N, where the representations $(\rho_j, \mathcal{F}_j)$ of N are irreducible and N acts on $\mathcal{H}_j$ by $\pi_j := \rho_j \otimes \mathbf{1}$. Then

$$\mathcal{N}' = \pi(N)' \cong \bigoplus_{j=1}^{m} B(\mathcal{C}_j).$$

The conjugation action of G on $\mathcal{N}'$ factors through an action of the finite group G/N. We have to show that $\mathcal{M}' = (\mathcal{N}')^{G/N}$ also is a finite direct sum of full operator algebras.

Let $F := \{[\rho_j] : j = 1, \ldots, m\} \subseteq \widehat{N}$ be the support of the restriction $\rho|_N$. This set decomposes under the natural action of G/N on the unitary dual $\widehat{N}$ into finitely many orbits $F_1, \ldots, F_k$. The group G permutes the isotypic subspaces $\mathcal{H}_j$ of N and, accordingly,

$$\pi_k \cong \pi_j \circ c_g^{-1}|_N \ \text{ if and only if } \ \pi(g)\mathcal{H}_j = \mathcal{H}_k.$$

This follows from the relation $\pi_k(n)\pi(g) = \pi(g)\pi_j(g^{-1}ng)$ for $g \in G, n \in N$. We conclude that

$$P_j := \{g \in G : \pi(g)\mathcal{H}_j = \mathcal{H}_j\} = \{g \in G : \rho_j \circ c_g \cong \rho_j\}.$$

For every $g \in P_j$, we thus obtain a unitary operator $U_g \colon \mathcal{F}_j \to \mathcal{F}_j$ such that

$$\rho \circ c_g = U_g \rho U_g^{-1}.$$

Since $\mathcal{F}_j$ is irreducible, U_g is well defined modulo $\mathbb{T}$. The projective unitary representation $\bar{\rho}_j(g) = [U_g]$ of P_j yields a central extension $q_j \colon P_j^\sharp \to P_j$, a homomorphic lift $N \hookrightarrow P_j^\sharp$ and an extension $\rho_j^\sharp\colon P_j^\sharp \to \mathrm{U}(\mathcal{F}_j)$ of the unitary representation ρ_j of N to $P_j^\sharp$. Accordingly, the representation of $P_j^\sharp$ on $\mathcal{H}_j$ takes the form

$$\pi_j(q_j(p)) = \rho_j^\sharp(p) \otimes \beta_j(p),$$

where $\beta_j\colon P_j^\sharp \to \mathrm{U}(\mathcal{C}_j)$ a unitary representation with $\ker \beta_j \supseteq N$.

Let $\mathcal{H} = \mathcal{H}^1 \oplus \cdots \oplus \mathcal{H}^k$ denote the decomposition of $\mathcal{H}$ under G, corresponding to the decomposition of F under G/N, so that each subspace $\mathcal{H}^j$ is a sum of certain subspaces $\mathcal{H}_\ell$. We may assume, without loss of generality, that G/N acts transitively on F, i.e., that

$$\mathcal{H} = \mathrm{span}(\pi(G)\mathcal{H}_1) = \bigoplus_{[g]\in G/P_1} \pi(g)\mathcal{H}_1.$$

This means that $(\pi, \mathcal{H})$ is induced from the representation $\rho_1^\sharp \otimes \beta_1$ of P_1 on $\mathcal{H}_1$.

The subspace $\mathcal{H}_1$ is generating for G, and hence separating for the commutant $\mathcal{M}'$. As $\mathcal{H}_1$ is isotypic for N, the commutant $\mathcal{M}' \subseteq \mathcal{N}'$ leaves $\mathcal{H}_1$ invariant; likewise all subspaces $\mathcal{H}_j$ are $\mathcal{M}'$-invariant. Since an operator $A \in B(\mathcal{H}_1)$ extends to an element of $\mathcal{M}'$ if and only if it commutes with P_1, we have

$$\mathcal{M}' \cong (\rho_1^\sharp \otimes \beta_1)(P_1)' = \mathbf{1} \otimes \beta(P_1^\sharp)'.$$

Since $\beta(P_1^\sharp)$ is a finite group, the assertion follows from the fact that every unitary representation of a finite group decomposes discretely with finitely many isotypes. ∎