Appendix D
Vacuum representations

In this appendix, we show that vacuum representations of weak products of topological groups arise as products of vacuum representations.

D.1 Weak products and $\mathbb{R}$-actions

The *weak product* of a sequence $(G_n)_{n\in\mathbb{N}}$ of topological groups is defined as

$$G := \prod\nolimits'_{n\in\mathbb{N}} G_n = \bigcup_{N=1}^{\infty} G^N, \quad G^N = G_1 \times \cdots \times G_N,$$

where the group structure is inherited from the product group $\prod_{n\in\mathbb{N}} G_n$. However, we will need a topology that is finer than the product topology. We equip G with the *box topology*, for which a basis of e-neighborhoods consists of the sets $G \cap \prod_{n=1}^{\infty} U_n$, where $U_n \subseteq G_n$ is an e-neighborhood in G_n. By [27, Lemma 4.4], this turns G into a topological group, and G is the direct limit in the category of topological groups of the increasing sequence of subgroups G^N, endowed with the product topology.

To study vacuum representations of weak products, consider a sequence of topological groups $(G_n, \mathbb{R}, \alpha_n)_{n\in\mathbb{N}}$ with homomorphisms $\alpha_n \colon \mathbb{R} \to \mathrm{Aut}(G_n)$ that defines a continuous action of $\mathbb{R}$ on G_n. The homomorphisms $\alpha_n \colon \mathbb{R} \to \mathrm{Aut}(G_n)$ combine to a homomorphism $\alpha \colon \mathbb{R} \to \mathrm{Aut}(G)$ by

$$\alpha_t(g_1, \ldots, g_N, e, \ldots) := (\alpha_{1,t}(g_1), \ldots, \alpha_{N,t}(g_N), e, \ldots),$$

where $\mathrm{Aut}(G)$ denotes the group of topological automorphisms.

Proposition D.1. *The above map α is a continuous action of $\mathbb{R}$ on G.*

Proof. To see this, we first note that all orbit maps are continuous because the subgroups G^N carry the product topology. Since all automorphisms α_t are continuous by [27, Lemma 4.4], it suffices to verify continuity of the action in all pairs $(0, g) \in \mathbb{R} \times G^N$. So we have to find for every sequence $(U_n)_{n\in\mathbb{N}}$ of e-neighborhoods in G_n an $\varepsilon > 0$ and a sequence of e-neighborhoods $V_n \subseteq G_n$ such that

$$\alpha_{n,t}(g_n V_n) \subseteq g_n U_n \quad \text{for } |t| < \varepsilon, n \in \mathbb{N}.$$

As $[-1, 1] \subseteq \mathbb{R}$ is compact, we find for every $n \in \mathbb{N}$ an identity neighborhood $V_n \subseteq W_n \subseteq G_n$ such that $W_n W_n \subseteq U_n$ and $\alpha_{n,t}(V_n) \subseteq W_n$ for $|t| \leq 1$. For $n \leq N$ we now choose $\varepsilon > 0$ in such a way that $\alpha_{n,t}(g_n) \in g_n W_n$ holds for $|t| \leq \varepsilon$. Then

$$\alpha_{n,t}(g_n V_n) = \alpha_{n,t}(g_n)\alpha_{n,t}(V_n) \subseteq g_n W_n W_n \subseteq g_n U_n$$

holds for $|t| < \varepsilon$ and $n \leq N$. For $n > N$, we have $g_n = e$ and

$$\alpha_{n,t}(g_n V_n) = \alpha_{n,t}(V_n) \subseteq W_n \subseteq U_n \quad \text{for } |t| \leq \varepsilon.$$

Therefore, α defines a continuous action on the weak direct product G. ■

If, in addition, the groups G_n are Lie groups, then the box topology on G is compatible with a Lie group structure on G ([27, Remark 4.3]).

Lemma D.2. *If all groups G_n are locally exponential, then α defines a smooth action on G.*

Proof. By [27, Remark 4.3], the group G is locally exponential as well. Therefore, it suffices to show that the $\mathbb{R}$-action on the Lie algebra $\mathfrak{g} \cong \bigoplus_{n\in\mathbb{N}} \mathfrak{g}_n$ (the locally convex direct sum), is smooth. Let $D_n \in \mathrm{der}(\mathfrak{g}_n)$ denote the infinitesimal generator of the smooth actions α^n on $\mathfrak{g}_n$. Then

$$\alpha(t,x) = (e^{tD_n}x_n)_{n\in\mathbb{N}} = e^{tD}x \quad \text{for } D(x_n) = (D_n x_n)$$

and the tangent map of α is given by

$$\mathrm{d}\alpha(t,x)(s,y) = sD(\alpha(t,x)) + \alpha(t,y).$$

As $D\colon \mathfrak{g} \to \mathfrak{g}$ is a continuous linear operator, we inductively obtain from the continuity of α (Proposition D.1) that α is C^k for each $k \in \mathbb{N}$, and hence that α is smooth. ■

The weak products encountered in this memoir are mostly of the following form.

Lemma D.3. *Suppose that the smooth manifold S has countably many connected components and that $\mathcal{K} \to S$ is a Lie group bundle. Then, the Lie group $\Gamma_c(\mathcal{K})$ is isomorphic to the restricted Lie group product $\prod'_{n\in\mathbb{N}} \Gamma_c(\mathcal{K}|_{S_n})$.*

Proof. Since the groups $G = \Gamma_c(\mathcal{K})$ and $G_n = \Gamma_c(\mathcal{K}|_{S_n})$ are locally exponential, it suffices to verify that the Lie algebra $\mathfrak{g} = \Gamma_c(\mathfrak{K})$ is the locally convex direct sum of the ideals $\mathfrak{g}_n = \Gamma_c(\mathfrak{K}|_{S_n})$. That the summation map

$$\Phi\colon \quad \bigoplus_{n\in\mathbb{N}} \Gamma_c(\mathfrak{K}|_{S_n}) \to \mathfrak{g}$$

is continuous follows from the universal property of the locally convex direct sum. That its inverse Φ^{-1} is also continuous, follows from its continuity on the Fréchet subspaces $\Gamma_D(\mathcal{K})$, where $D \subseteq S$ is compact, because any compact subset intersects at most finitely many connected components. ■

D.2 Vacuum representations

Let G be a topological group, and let $\alpha\colon \mathbb{R} \to \mathrm{Aut}(G)$ be a homomorphism that defines a continuous action of $\mathbb{R}$ on G.

Definition D.4. A triple $(\rho, \mathcal{H}, \Omega)$ is called a *vacuum representation* of $(G, \mathbb{R}, \alpha)$, if $\rho\colon G \rtimes_\alpha \mathbb{R} \to \mathrm{U}(\mathcal{H})$ is a continuous unitary representation, $\Omega \in \mathcal{H}$ is a G-cyclic unit vector, and the selfadjoint operator H, defined by $U_t := \rho(e,t) = e^{-itH}$ for $t \in \mathbb{R}$, satisfies $\ker(H - E_0\mathbf{1}) = \mathbb{C}\Omega$ for $E_0 = \inf(\mathrm{spec}(H))$.

The following is an immediate consequence of [8, Proposition 5.4].

Proposition D.5. *For a vacuum representation* $(\rho, \mathcal{H}, \Omega)$ *of* $(G, \mathbb{R}, \alpha)$*, the following assertions hold:*

(a) $U_\mathbb{R} \subseteq \rho(G)''$,

(b) *the representation* $\rho|_G$ *of* G *on* $\mathcal{H}$ *is irreducible.*

Proof. (a) The one-parameter group $(U_t^0)_{t\in\mathbb{R}}$ defined by $U_t^0 := e^{itE_0}U_t$ is minimal for the von Neumann algebra $\rho(G)''$ (cf. Definition 3.8) by [8, Proposition 5.4], hence contained in $\rho(G)''$, and this implies (a).

(b) From (a) it follows that the closed subspace

$$\mathbb{C}\Omega = \ker(H - E_0\mathbf{1}) \subseteq \mathcal{H}$$

is invariant under the commutant $\mathcal{M}' := \rho(G)'$ of $\mathcal{M} := \rho(G)''$. As Ω is generating for $\mathcal{M}$, it is separating for $\mathcal{M}'$, so that $\dim\ker(H_0 - E_0\mathbf{1}) = 1$ leads to $\mathcal{M}' = \mathbb{C}\mathbf{1}$. Now the assertion follows from Schur's Lemma. ■

Let $(G_n, \mathbb{R}, \alpha_n)_{n\in\mathbb{N}}$ be a sequence of topological groups, with for each $n \in \mathbb{N}$ a homomorphism $\alpha_n\colon \mathbb{R} \to \mathrm{Aut}(G_n)$ that defines a continuous action of $\mathbb{R}$ on G_n. The following theorem identifies the vacuum representations of the weak product $(G, \mathbb{R}, \alpha)$ in terms of vacuum representations of the triples $(G_n, \mathbb{R}, \alpha_n)$.

Theorem D.6. *For any sequence* $(\rho_n, \mathcal{H}_n, \Omega_n)$ *of vacuum representations of* $(G_n, \mathbb{R}, \alpha_n)$ *with minimal energy* $E_0 = 0$*, the infinite tensor product*

$$(\mathcal{H}, \Omega) := \bigotimes_{n=1}^{\infty} (\mathcal{H}_n, \Omega_n) \tag{D.1}$$

carries a continuous vacuum representation of $(G, \mathbb{R}, \alpha)$*, defined by*

$$\rho(g_1, \ldots, g_n, e, \ldots) := \rho_1(g_1) \otimes \cdots \otimes \rho_n(g_n) \otimes \mathbf{1}_{n+1} \otimes \cdots . \tag{D.2}$$

Conversely, every vacuum representation of $(G, \mathbb{R}, \alpha)$ *with* $E_0 = 0$ *is equivalent to such a representation.*

Proof. First, we prove that if all $(\rho_n, \mathcal{H}_n, \Omega_n)$ are vacuum representations, then so is their infinite tensor product. Since the Ω_n are unit vectors, the infinite tensor product Hilbert space $\mathcal{H}$ is defined. It contains the subspaces

$$\mathcal{H}^N := \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \otimes \Omega_{N+1} \otimes \cdots \cong \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N,$$

whose union is dense in $\mathcal{H}$. On $\mathcal{H}^N$, the representation ρ^N of $G^N \rtimes \mathbb{R}$, defined by

$$\rho^N((g_1, \ldots, g_N), t) := \rho_1(g_1, t) \otimes \cdots \otimes \rho_N(g_N, t),$$

is continuous with cyclic vector $\Omega = \otimes_{n=1}^{\infty} \Omega_n$. The representation $(\rho, \mathcal{H})$ of G now is a direct limit of the representations $(\rho^N, \mathcal{H}^N)$ of the subgroups G^N, hence a continuous unitary representation. Further, the invariance of Ω_n under the one-parameter group $U_t^n := \rho_n(e, t)$ implies that

$$U_t(v_1 \otimes \cdots \otimes v_N \otimes \Omega_{N+1} \otimes \cdots) := U_t^1 v_1 \otimes \cdots \otimes U_t^N v_N \otimes \Omega_{N+1} \otimes \cdots \tag{D.3}$$

defines a continuous unitary one-parameter group on $\mathcal{H}$ satisfying

$$U_t \rho(g) U_t^* = \rho(\alpha_t(g)) \quad \text{for } g \in G, t \in \mathbb{R}.$$

By $\rho(g, t) := \rho(g) U_t$, we thus obtain a continuous unitary representation of G on $\mathcal{H}$ for which Ω is a G-cyclic unit vector fixed by the one-parameter group $(U_t)_{t \in \mathbb{R}}$. Writing

$$U_t = e^{-itH} \quad \text{and} \quad U_t^n = e^{-itH_n}$$

for selfadjoint operators $H_n \geq 0$, (D.3) implies that $H \geq 0$. To see that $\ker H = \mathbb{C}\Omega$, we decompose

$$\mathcal{H} = \mathcal{H}^N \otimes \mathcal{K}_N \quad \text{for } N \in \mathbb{N}.$$

Accordingly,

$$U_t = V_t \otimes W_t \quad \text{with } V_t = U_t^1 \otimes \cdots \otimes U_t^N,$$

and both one-parameter groups $(V_t)_{t \in \mathbb{R}}$ and $(W_t)_{t \in \mathbb{R}}$ have positive generators H_V and H_W. From [8, Lemma A.3] we thus infer that

$$H = (H_V \otimes \mathbf{1}_{\mathcal{K}^N}) + (\mathbf{1}_{\mathcal{H}^N} \otimes H_W)$$

in the sense of unbounded operators, hence, in particular, that

$$\mathcal{D}(H) = (\mathcal{D}(H_V) \otimes \mathcal{K}^N) \cap (\mathcal{H}^N \otimes \mathcal{D}(H_W)).$$

We conclude that, for every $N \in \mathbb{N}$,

$$\ker H \subseteq \ker H_V \otimes \mathcal{K}^N = \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \mathcal{K}^N,$$

and this shows that

$$\ker H \subseteq \bigcap_N \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \mathcal{K}^N = \mathbb{C}\Omega.$$

Therefore, $(\rho, \mathcal{H}, \Omega)$ is a vacuum representation of $(G, \mathbb{R}, \alpha)$.

Now we assume, conversely, that $(\rho, \mathcal{H}, \Omega)$ is a vacuum representation of the triple $(G, \mathbb{R}, \alpha)$. Then, the subspace

$$\mathcal{H}^N := \overline{\operatorname{span} \rho(G^N)\Omega}$$

carries a vacuum representation of $(G^N, \mathbb{R}, \alpha^N)$. In particular, this representation is irreducible by Proposition D.5. The group G is a topological product

$$G = G^N \times G^{>N}, \quad \text{where } G^{>N} := \prod\nolimits'_{n>N} G_n,$$

and the representation ρ is irreducible by Proposition D.5. Since its restriction to G^N carries an irreducible subrepresentation, the restriction to G^N is factorial of type I, hence of the form

$$\rho|_{G^N} = \rho^N \otimes \mathbf{1}$$

with respect to some factorization $\mathcal{H} = \mathcal{H}^N \otimes \mathcal{K}^N$. Starting with $N = 1$ and proceeding inductively, we see that

$$\rho^N \cong \rho_1 \otimes \cdots \otimes \rho_N$$

for vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ of $(G_n, \mathbb{R}, \alpha_n)$. In particular, we obtain factorizations

$$\Omega = \Omega^N \otimes \tilde{\Omega}_N = \Omega_1 \otimes \cdots \otimes \Omega_N \otimes \tilde{\Omega}_N,$$

so that we may identify $\mathcal{H}^N$ with the subspace

$$\mathcal{H}^N \otimes \tilde{\Omega}_N \subseteq \mathcal{H}.$$

As Ω is G-cyclic, the union of these G^N-invariant subspaces is dense in $\mathcal{H}$. This implies that the vacuum representation $(\rho, \mathcal{H}, \Omega)$ is equivalent to the infinite tensor product $\otimes_{n\in\mathbb{N}}(\rho_n, \mathcal{H}_n, \Omega_n)$ of the ground state representations $(\rho_n, \mathcal{H}_n, \Omega_n)$. This completes the proof. ■

The following allows us to reduce the classification of smooth vacuum representations to the local case, under the assumption that the ground state is smooth.

Proposition D.7. *Suppose that the G_n are Lie groups and that the $\mathbb{R}$-actions on G_n are smooth. Then, the vacuum representation $(\rho, \mathcal{H}, \Omega)$ is smooth with smooth vector Ω if and only if the vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ are smooth with smooth vector Ω_n.*

Proof. If $(\rho, \mathcal{H}, \Omega)$ is a smooth representation with $\Omega \in \mathcal{H}^\infty$, then Ω will be a smooth vector for every $(\mathcal{H}_n, \rho_n, \Omega)$ as well. Since Ω is cyclic in $\mathcal{H}_n$, the latter will be a smooth representation.

Suppose, conversely, that the vacuum representations $(\rho_n, \mathcal{H}_n, \Omega_n)$ are smooth, and that $\Omega_n \in \mathcal{H}_n^\infty$ for all $n \in \mathbb{N}$. From Theorem D.6, we know that the tensor product representation $(\rho, \mathcal{H}, \Omega)$ is continuous and cyclic. To show that the vacuum representation $(\rho, \mathcal{H}, \Omega) = \bigotimes_{n=1}^\infty (\rho_n, \mathcal{H}_n, \Omega_n)$ is smooth with smooth vector $\Omega \in \mathcal{H}^\infty$, it suffices by [74, Theorem 7.2] to show that $\varphi(g) := \langle \Omega, \rho(g)\Omega \rangle$ is a smooth function from G to $\mathbb{C}$.

Note that φ is the infinite product $\prod_{n=1}^\infty \varphi_n(g_n)$ of the smooth, positive definite functions $\varphi_n \colon G_n \to \mathbb{C}$ defined by $\varphi_n(g) := \langle \Omega_n, \rho_n(g)\Omega_n \rangle$. To see that $\varphi \colon G \to \mathbb{C}$ is smooth, note that it can be decomposed into the smooth maps

$$G = \prod\nolimits'_{n\in\mathbb{N}} G_n \xrightarrow{\Phi_1} \mathbf{1} + \prod\nolimits'_{n\in\mathbb{N}} \mathbb{C} \xrightarrow{\Phi_2} \mathbf{1} + \ell^1(\mathbb{N}) \xrightarrow{\Phi_3} \mathbb{C},$$

where $\mathbf{1} = (1)_{n\in\mathbb{N}}$ and

$$\Phi_1((g_n)) = (\varphi_n(g_n)), \quad \Phi_2((z_n)) = (z_n), \quad \Phi_3((z_n)) = \prod_{n\in\mathbb{N}} z_n.$$

Here, the smoothness of Φ_1 follows from the compatibility with the box manifold structure, Φ_2 is continuous affine, and Φ_3 is holomorphic. It follows that

$$\varphi = \Phi_3 \circ \Phi_2 \circ \Phi_1$$

is smooth, and hence that $(\rho, \mathcal{H}, \Omega)$ is a smooth vacuum representation with smooth vector Ω. ∎