Appendix E

Ergodic property of 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$

We give a simplified proof for the following characterization of the ergodic property for 1-parameter subgroups of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ due to Mautner and Moore. Define the 1-parameter groups $x(t)$, $y(t)$ and $h(t)$ in $\mathrm{SL}(2, \mathbb{R})$ by

$$x(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y(t) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \text{and} \quad h(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},$$

and let $\widetilde{x}(t)$, $\widetilde{y}(t)$ and $\widetilde{h}(t)$ be their lift to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$.

Lemma E.1. *Let* $(\pi, \mathcal{H})$ *be a continuous unitary representation of* $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, *and let* $\Omega \in \mathcal{H}$ *be a unit vector. Then, the following are equivalent:*

(a) $\pi(\widetilde{x}(t))\Omega = \Omega$ *for all* $t \in \mathbb{R}$,

(b) $\pi(\widetilde{h}(t))\Omega = \Omega$ *for all* $t \in \mathbb{R}$,

(c) $\pi(g)\Omega = \Omega$ *for all* $g \in \widetilde{\mathrm{SL}}(2, \mathbb{R})$.

This well-known result plays an important role in ergodic theory. It is due to Calvin Moore [66,67], and in the proof below we almost literally follow his argument for the implication (a)⇒(b) from [67, page 7]. The implication (b)⇒(c) is due to Mautner [63], and this is implicitly used by Moore in [67], and by Howe and Moore in their seminal paper [43]. In his proof, Mautner uses the classification of irreducible unitary representations of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. We bypass this with a simple argument.

Proof. For (a)⇒(b), let $w(t) = x(t)y(-t^{-1})x(t)$, and note that we have

$$h(t) = w(e^t)w(1)^{-1} \quad \text{for all } t \in \mathbb{R}.$$

If we define $\widetilde{w}(t) := \widetilde{x}(t)\widetilde{y}(-t^{-1})\widetilde{x}(t)$, then the curve $t \mapsto \widetilde{w}(e^t)\widetilde{w}(1)^{-1}$ covers $h(t)$. Since it is the identity for $t = 0$, we have $\widetilde{w}(e^t)\widetilde{w}(1)^{-1} = \widetilde{h}(t)$. Since $\|\pi(\widetilde{w}(t))\Omega\| = 1$ for all $t \neq 0$, it follows from

$$\lim_{|t|\to\infty} \langle \pi(\widetilde{w}(t))\Omega, \Omega\rangle = \lim_{|t|\to\infty} \langle \pi(\widetilde{y}(-t^{-1}))\Omega, \Omega\rangle = 1$$

that $\lim_{|t|\to\infty} \pi(\widetilde{w}(t))\Omega = \Omega$. So for $\psi = \pi(\widetilde{w}(1))\Omega$, we find $\lim_{t\to\infty} \pi(\widetilde{h}(t))\psi = \Omega$. For every $s \in \mathbb{R}$ we thus have

$$\Omega = \lim_{t\to\infty} \pi(\widetilde{h}(s+t))\psi = \pi(\widetilde{h}(s)) \lim_{t\to\infty} \pi(\widetilde{h}(t))\psi = \pi(\widetilde{h}(s))\Omega,$$

so Ω is fixed by $\widetilde{h}(s)$ for all $s \in \mathbb{R}$.

For (b)⇒(a), note that since

$$x(te^{-2s}) = h(-s)x(t)h(s) \quad \text{for all } s, t \in \mathbb{R},$$

the same equation $\widetilde{x}(te^{-2s}) = \widetilde{h}(-s)\widetilde{x}(t)\widetilde{h}(s)$ holds in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ (both sides are the identity for $s = t = 0$). The invariance of Ω under the 1-parameter group $\widetilde{h}$ then implies

$$\langle \pi(\widetilde{x}(te^{-2s}))\Omega, \Omega\rangle = \langle \pi(\widetilde{x}(t))\Omega, \Omega\rangle.$$

Since $\lim_{s\to\infty} \widetilde{x}(te^{-2s})$ is the identity, we have $\langle \pi(\widetilde{x}(t))\Omega, \Omega\rangle = 1$, and it follows that $\pi(\widetilde{x}(t))\Omega = \Omega$ for all $t \in \mathbb{R}$.

Since $h(s)y(t)h(-s) = y(te^{-2s})$, a similar argument shows that if Ω is fixed by $\widetilde{h}$, then it is fixed by $\widetilde{y}$. It follows that if either (a) or (b) hold, then Ω is fixed by $\widetilde{x}(t)$, $\widetilde{y}(t)$ and $\widetilde{h}(t)$ alike, and hence by the group $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ that they generate. ■