Preamble

On the one hand, many popular adages share the idea that *to achieve a prominent goal one has to take risks* (e.g., "no gain without pain", "nothing ventured, nothing gained", "no guts, no glory", just to name a few proverbs). On many occasions, the ambition to a high reward may lead individuals to face potential dangers, and in some situations there is a full master plan centered around a high-risk/high-gain plan: for instance, the blueprint of the European Research Council is to fund high-risk/high-reward research, in which severe conceptual challenges (which, by definition, are prone to scientific failure) are accepted downsides for a research project to be truly successful and impactful.

On the other hand, there is nowadays a great interest in the investigation of *optimal searching strategies*, e.g., in the study of animal behavior, and the research on this topic has necessarily to be somewhat controversial, given the complexity of the phenomenon into consideration.

Our view on this point is that the difficulty of addressing the topic of optimal searching is not only due to the *enormous amount of parameters* which should be accounted for (such as predators and prey distributions, previous knowledge of the territory, interactions with the environment, social factors, different reactions to adverse circumstances, competition phenomena, cooperative behaviors, etc.), and not only due to the difficulty of measuring many of these parameters via *objective empirical observations*.

In fact, in our opinion, a core difficulty in this topic stems from the difficulty of assessing unambiguously and indisputably a suitable notion of "gain" which should be maximized by a searching algorithm. This gain cannot be limited to the actual effectiveness of the procedure (i.e., whether or not the predator captures the prey), but it has to take into account the cost of the procedure itself (e.g., the time needed for the task, or the energy spent for it), and, at least on some occasions, the possible value of the outcome of the search.

One of the findings of our research is indeed that the high-risk/high-reward situation may appear naturally even in very simple situations, therefore the notion of "best strategy" requires a *very careful mathematical setting*, in which an *efficiency functional* is chosen and *maximized*, and *the location of the maxima is confronted with that of the minima*.

In doing so, one discovers immediately some interesting features. First of all, different efficiency functionals can produce different results. This already highlights a structural complication towards a full understanding of the notion of optimal searching strategy: for instance, in a biological study, different species, or different individuals of the same species, may, implicitly or explicitly, address a different type of efficiency functional.

In addition, in several concrete situations, the maximizers of some efficiency functional may end up to be dangerously close to the minimizers: this is a clear case of high-risk/high-reward pattern and, in this "unstable" situation, one should expect that the practical outcome of the optimal searching pattern be influenced by intermediate strategies aiming at a balance between top performances and conservative options (e.g., a risk assessment which compromises between the most rewarding and the safest result). Quite likely, in these conditions, different biological species, or different members of the same group, may end up adopting different search strategies.

Interestingly, in our setting, the situation in which the most rewarding strategy is arbitrarily close to a complete failure of the searching pattern is related to Lévy distributions with a very low exponent and a very fat tail. This pattern is known to be related to foraging modes of "ambush" type (see [13, 16]). The literature has also collected experimental evidence of some species, such as anglers and blonde skates, which do follow diffusive paths with very low Lévy exponent: remarkably, a correlation has been found between this type of diffusion and the high content of energy of the targets (see [16]).

In our setting, this correlation is possibly motivated precisely by the fact that the most rewarding Lévy exponent happens to be very close to the pessimizer. In a sense, it can be significant to imagine that such a high-risk/high-gain strategy becomes particularly suitable when the possible outcome is of exceptional value (in the case of a biological predator, a prey of exceptionally high energetic content).

That is, in an implicit risk assessment, the value of the target may mitigate the prospect of an unsuccessful search, thus favoring the emergence, in these specific situations, of high-risk/high-reward diffusive patterns.

In this work, this general vision will be embodied into a precise mathematical study of the Lévy flight foraging hypothesis, considering the possibility that processes with long jumps (instead of standard Gaussian random movements) can optimize search efficiency by diminishing the repetitions of visits to previously inspected sites. Different efficiency functionals will be taken into account, with a thorough analysis of their optimizers and pessimizers. This phenomenon in which optimizers and pessimizers cluster together will be also explicitly detected and discussed.

The Lévy flights will be modeled via a heat equation of fractional type in bounded domains. We consider the case of a hostile environment (such as a "fence", modeled by homogeneous Dirichlet conditions which "annihilate" a biological species outside a confinement domain) as well as the case of reflecting boundaries (modeled by homogeneous Neumann conditions which maintain a biological species within a niche without altering the number of individuals present in the region).

To implement these boundary conditions in the setting of the fractional heat equation, we will make use of the spectral version of the fractional Laplacian.

In some of the efficiency functionals that we consider, predators and targets are modeled as points in the space. In other cases instead we will model predators and targets as regions of space (assuming, e.g., that the biological individuals are uniformly distributed within these regions): this situation can also be considered as a technical and conceptual simplification of the notion of “direct vision” which was previously adopted in the literature, see, e.g., [42]. That is, here we do not introduce an additional parameter to truncate the Lévy distribution in the proximity of its singularity (which entails in itself some delicate issues, see [28]) and we do not alter the diffusive equation to account for foragers directly aiming at the prey when they lie at short mutual distance. Instead, the diffusive equation is supposed to hold at every spatial scale and the role of a different region of influence (e.g., induced by uncertainties in the data or by a different hunting pattern at a small scale) is encoded only in the efficiency functional.

Here, we do not restrict our analysis to the one-dimensional case; in fact, we deal with an arbitrary large number n of dimensions. We note that the case of higher dimension is, in many instances, not only a situation of utmost biological interest, but also a source of technical difficulties and scientific controversies, see, e.g., [7, 25, 26].