Chapter 1

Introduction to the mathematical setting and main results

In the last decades, anomalous diffusion has been investigated as an appropriate substitute for normal diffusion in several branches of science, such as biology and in particular the foraging theory (see for instance [17, 31, 37, 40, 42]). In this context a special case of anomalous diffusion occurs when a forager in search of food, rather than diffusing according to the classical Brownian motion, performs long-jump patterns characterized by a space and time steps scale invariance, see, e.g., [24] and the references therein.

This type of searches fits the model of the Lévy flight, according to the probabilistic description given in [2, Section 4.3]. In contrast to what happens with the classical random walk, the forager performing these flights has less chances to revisit intensively the immediate surrounding areas and then being confined in a narrow region. Therefore, in the biological framework, Lévy flights seem to be a better search strategy when the source of food is scarce and sparsely distributed and there is a large area to be covered in order to succeed in the hunt.

These kinds of foraging search strategies have been empirically observed in many ecological systems, see, e.g., [3, 17, 20–22, 30, 32, 40]. Moreover, several studies have been made in order to validate the Lévy flight foraging hypothesis from a mathematical and statistical point of view [4, 39, 41, 42].

In these models a number of assumptions are usually made on the environment, on targets and foragers. For instance, a low prey density is often assumed and the targets are randomly distributed in a wide area; the forager does not keep memory of previous encounters; the forager has scarce information on the area to search and on the prey location. On the one hand, on some occasions, these structural assumptions are introduced in order to simplify the problem, which otherwise would be extremely challenging to be analyzed from a theoretical perspective; on the other hand, some of these conditions can actually be structurally necessary for the convenience of the Lévy flight strategy over more standard type of diffusive processes. In any case, the complexity of the raw problem is a consequence of its dependence on a great number of environmental, evolutionary and biological variables. Even though an oversimplification may lead to a less accurate model in some circumstances, we can evince from a simplified model some remarkable properties, advancing the knowledge on such a complex topic.

In this memoir we will investigate the Lévy flight foraging hypothesis relying on a fractional elliptic operator. This is motivated by the fact that in the limit of the time step going to zero, the distribution of a seeker performing Lévy flights converges to the solution of a fractional heat equation, see, e.g., [2, 6, 38].

In order to test the Lévy flights foraging hypothesis, we consider some efficiency functionals, accounting for the random encounter rate between the forager and the target. We maximize these efficiency functionals with respect to the fractional exponent, with the aim of understanding which flight was more advantageous for the forager. From a biological perspective, this optimization with respect to the fractional exponent corresponds to the possibility of a forager to modify its searching strategy by tuning, e.g., the average length of a hunting path and the waiting times between different paths.

We will assume that the forager is confined in some bounded region $\Omega \subset \mathbb{R}^n$, which plays the role of an ecological niche. Both Dirichlet and Neumann boundary conditions will be taken into account to describe absorbing and reflecting boundaries.

For us, the choice of a spectral fractional heat equation as a diffusion equation for the forager was motivated by its stochastic interpretation as a subordinate Brownian motion in Ω, see [10]. See also [8, 14, 15, 27, 35] and the references therein for several applications of fractional elliptic equations to biological problems.

In this memoir, we will test the Lévy flight foraging hypothesis by taking into account different biological configurations, such as

- the case in which the forager starting position and the target location coincide,
- the case in which the forager starting position is located in proximity of the target,
- the case in which the forager and the target, instead of being modeled as material points, are uniformly distributed in some regions of space.

The situation in which the biological population is not confined into a bounded region of space and can travel through the whole of $\mathbb{R}^n$ is technically different and has been treated in the papers [12, 13].

The memoir is structured as follows. In Section 1.1, we define the *efficiency functionals* for the spectral search in the bounded region $\Omega \subset \mathbb{R}^n$. They will be taken to be proportional to the encounter rate between the forager and the target. Moreover, different "penalizations quantities" will be considered, such as the average distance and the mean square displacement, in order to build physically reliable efficiency functionals.

Sections 1.2 and 1.3 are devoted to the study of the *maximizer* for the aforementioned functionals. These maximizers thus correspond to the most rewarding searching mode. In particular, in Section 1.2 we will assume that the forager starting position and the prey location coincide. This scenario, though physically less relevant, will let us detect some *monotonicity properties* of one of the functionals, when the domain satisfies suitable geometric properties, see Theorem 1.4 below. This result shows how *the search for a maximizer is related to the geometric structure of the play field*.

In Section 1.3, as well as in Section 1.4 for the case of distributed foragers and targets, we analyze the case in which the target is in some small neighborhood of the forager starting position. Here we establish that *if the target position converges to the initial location of the seeker, then the maximizer of the efficiency functionals is located in a neighborhood of* $s = 0$. This is the content of Theorems 1.7, 1.8, 1.15, and 1.16.

Furthermore, in Theorems 1.6 and 1.14 it will be proved that for some of these efficiency functionals the strategy

$$s = 0$$

is the *unique global minimizer*, thus corresponding to the unique pessimizer of the searching mode. This minimality result, together with the convergence of the best strategy, will entail that, roughly speaking, the most rewarding strategy may end up being not reliable, presenting arbitrarily close pessimizers, thus opening the dilemma of whether in practice one should follow the most performant option, or the safest one, or, say, a balanced combination of the two (see Remark 1.9 below).

In Chapter 2 we collect the main analytical tools that will be employed in order to prove our main results.

Finally, in Chapter 3 we prove the results stated in Sections 1.2 and 1.3.

1.1 Efficiency functionals

To measure the effectiveness of a foraging strategy, one can consider different functionals which account for the rate of hunting "success" for the predator versus the "effort" needed.

The possibility of accounting for different efficiency functionals plays, in our opinion, a crucial role in biology and ethology, since, while the notion of "foraging success" may be somewhat objective (as measured for instance by the amount of food eaten, or by the calories carried by such a food), the notion of "cost spent to achieve the success" is intrinsically more ambiguous and different biological theories may end up measuring this concept in different ways. As an example, we recall the debate about the way honey bees assess how far they have flown (whether based on the energy expended in flying or on the fatigue required by the action, as conjectured in former experiments, or on the image motion of the surrounding landscape through visual perception, as pointed out in the "optic flow hypothesis" and addressed in recent tests, see [36]). Related to this, we also recall that in some situations the measure of the distance traveled can be performed according to a number of possible strategies (e.g., in the case of ants, which can use optic flow, pheromone and chemical trails, as well as the "counting" of the number of steps, see [44]). See also [19] and the references therein for further reading on how animals measure distances.

The mathematical setting that we consider here goes as follows. We model a forager moving in some bounded region $\Omega \subset \mathbb{R}^n$ through a spectral fractional diffusion with either Dirichlet or Neumann homogeneous boundary conditions. The domain Ω where the diffusion occurs can be seen as an ecological niche where the forager is confined (the Dirichlet condition corresponding to the case in which the forager is killed at the boundary of the niche, and the Neumann datum corresponding, e.g., to fences that prevent the forager to exit the niche).

Specifically, the probability density $u = u(t, x)$ of the forager satisfies the diffusive equation

$$\partial_t u(t, x) = -(-\Delta)^s u(t, x) \quad \text{for all } (t, x) \in (0, +\infty) \times \Omega, \tag{1.1}$$

with either Dirichlet or Neumann homogeneous boundary conditions.

Here above s is a fractional parameter in $(0, 1)$ and the operator $(-\Delta)^s$ represents the spectral fractional Laplacian, see, e.g., [2, Sections 2.3 and 4.3] for the basics of this operator. See also [16] for different approaches to the problem of Lévy flights in (one-dimensional) bounded domains.

We also assume that the targets are scattered in Ω according to a distribution $p(t, x)$, where

$$(t, x) \in [0, +\infty) \times \Omega.$$

We consider, as an initial measure of the success of the hunting strategy of the predator, a *foraging success functional* which accounts for the random encounters between the forager following the anomalous diffusion equation in (1.1) and the targets.

Specifically, in the situation considered here, given $T \in (0, +\infty)$ and $y \in \Omega$, the foraging success functional takes the form

$$\int_0^T \int_\Omega r^s(t, x, y) p(t, x)\, dx\, dt, \tag{1.2}$$

where $r^s(t, x, y)$ represents either the Dirichlet or the Neumann spectral fractional heat kernel, for some fractional parameter $s \in (0, 1)$, see for instance [10, 11] and the beginning of the forthcoming Chapter 2 for definitions and basic properties of these kernels.

We notice indeed that the quantity in (1.2) is associated with the probability that a forager starting at the position $y \in \Omega$ and following the diffusion process modeled by the fractional heat equation with either Dirichlet or Neumann boundary condition hits a target distributed according to $p(t, x)$ in the time interval $(0, T)$.

To obtain an efficiency functional, we compare this quantity with some other quantities of physical and biological significance that instead provide a penalization for the seeker. Here, we will consider as penalization quantities the *time* T, the *average distance traveled by the forager* $l^y(s,T)$ after a time T and the *mean square displacement* $\mathcal{A}^y(s,T)$ after a time T.

More explicitly, the average distance traveled by the forager at time $T \in (0,+\infty)$ is given by

$$l^y(s,T) := \int_0^T \int_\Omega |\zeta - y| r^s(t,\zeta,y)\, d\zeta\, dt. \tag{1.3}$$

The probabilistic interpretation underpinning this definition consists in taking into account the random process Y_t starting at y corresponding to a subordinate Brownian motion which is either killed or reflected at the boundary (the generator of such a process corresponding to the spectral fractional Laplacian with either Dirichlet or Neumann datum).

In this framework, the quantity $|Y_t|$ represents the distance at time t for a single representation of the process, whence it is natural to consider its expected value

$$\mathbb{E}_y^s[|Y_t|] = \int_\Omega |x-y| r^s(t,x,y)\, dx$$

as the mean distance traveled at time t. The setting in (1.3) is thus the average over time $t \in (0,T)$ of this quantity.

Similarly, the mean square displacement is given by

$$\mathcal{A}^y(s,T) := \int_0^T \int_\Omega |\zeta - y|^2 r^s(t,\zeta,y)\, d\zeta\, dt \tag{1.4}$$

and represents the average over time $t \in (0,T)$ of the expected value of the squared distance

$$\mathbb{E}_y^s[|Y_t|^2] = \int_\Omega |x-y|^2 r^s(t,x,y)\, dx.$$

Interestingly, subordinators related to waiting times may have an intimate connection to biology, since spontaneous patterns of waiting times are known to occur in nature, and they can be species-specific, depend on body size, foraging modes, prey preference, etc., see [43].

While the notations in (1.3) and (1.4) are the same for the Dirichlet and the Neumann cases (the difference being only in the fractional heat kernel, which is sensitive to the boundary conditions), it is convenient to distinguish explicitly between the two types of boundary data and for this we add the subscript D or N to the notation, namely we write $l_D^y(s,T)$, $l_N^y(s,T)$, $\mathcal{A}_D^y(s,T)$, and $\mathcal{A}_N^y(s,T)$ to emphasize the dependence of the average distance traveled and of the mean square displacement with respect to the Dirichlet or the Neumann boundary condition.

As a special case of target distribution $p(t,\xi)$, we consider the situation in which there is only one target located at $x \in \Omega$. In this case, the distribution $p(t,\xi)$ reduces to the Dirac's delta $\delta_x(\xi)$ and the foraging success functional in (1.2) will be denoted (depending on the boundary condition) by

$$\Phi_D^{x,y}(s,T) = \int_0^T \int_\Omega r_D^s(t,\zeta,y)\delta_x(\zeta)\,d\zeta\,dt = \int_0^T r_D^s(t,x,y)\,dt \tag{1.5a}$$

or

$$\Phi_N^{x,y}(s,T) = \int_0^T \int_\Omega r_N^s(t,\zeta,y)\delta_x(\zeta)\,d\zeta\,dt = \int_0^T r_N^s(t,x,y)\,dt. \tag{1.5b}$$

In this memoir we focus on the optimal foraging strategy according to the following efficiency functionals:

$$\begin{aligned} \mathcal{E}_{1,D}^{x,y}(s,T) &:= \frac{\Phi_D^{x,y}(s,T)}{T}, & \mathcal{E}_{1,N}^{x,y}(s,T) &:= \frac{\Phi_N^{x,y}(s,T)}{T}, \\ \mathcal{E}_{2,D}^{x,y}(s,T) &:= \frac{\Phi_D^{x,y}(s,T)}{l_D^y(s,T)}, & \mathcal{E}_{2,N}^{x,y}(s,T) &:= \frac{\Phi_N^{x,y}(s,T)}{l_N^y(s,T)}, \\ \mathcal{E}_{3,D}^{x,y}(s,T) &:= \frac{\Phi_D^{x,y}(s,T)}{\mathcal{A}_D^y(s,T)}, & \mathcal{E}_{3,N}^{x,y}(s,T) &:= \frac{\Phi_N^{x,y}(s,T)}{\mathcal{A}_N^y(s,T)}. \end{aligned} \tag{1.6}$$

In addition to the functionals in (1.6), we consider the following set-dependent functionals. Here, the exact initial positions of target and forager are replaced by uniform densities in two subregions of Ω. Namely, we assume that the targets are distributed in Ω according to

$$p(t,x) := \frac{\chi_{\Omega_1}(x)}{|\Omega_1|},$$

for some measurable set $\Omega_1 \subset \Omega$, where χ_{Ω_1} is the characteristic function of Ω_1 and $|\Omega_1|$ denotes the Lebesgue measure of Ω_1.

The forager diffusing via the spectral fractional heat equation is initially uniformly distributed in some measurable set $\Omega_2 \subset \Omega$ and therefore, dropping for the moment the subscript D and N, its density in $(t,x) \in (0,+\infty) \times \Omega$ is given by

$$f^s(t,x) := \frac{1}{|\Omega_2|} \int_{\Omega_2} r^s(t,x,y)\,dy,$$

see, e.g., [10, Lemma 4] and [11, Lemma 5].

With this notation, the *set-dependent forager success functional* takes the form

$$\begin{aligned} \widetilde{\Phi}^{\Omega_1,\Omega_2}(s,T) &:= \int_0^T \int_\Omega f^s(t,x)p(t,x)\,dx\,dt \\ &= \frac{1}{|\Omega_1||\Omega_2|} \int_0^T \int_{\Omega_1\times\Omega_2} r^s(t,x,y)\,dx\,dy\,dt. \end{aligned} \tag{1.7}$$

Furthermore, in this framework, the average distance traveled by the forager and the mean square displacement are given by

$$\begin{aligned}\tilde{l}^{\Omega_2}(s,T) &:= \int_0^T \int_\Omega |\xi - y| f^s(t,\xi)\,d\xi\,dt \\ &= \frac{1}{|\Omega_2|}\int_0^T \int_{\Omega\times\Omega_2} |\xi - y| r^s(t,\xi,y)\,d\xi\,dy\,dt, \\ \tilde{\mathcal{A}}^{\Omega_2}(s,T) &:= \int_0^T \int_\Omega |\xi - y|^2 f^s(t,\xi)\,d\xi\,dt \\ &= \frac{1}{|\Omega_2|}\int_0^T \int_{\Omega\times\Omega_2} |\xi - y|^2 r^s(t,\xi,y)\,d\xi\,dy\,dt.\end{aligned} \tag{1.8}$$

Therefore, by using these set-dependent foraging success functionals and penalization quantities, we define the *set-dependent efficiency functionals* as

$$\begin{aligned}\tilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{T}, & \tilde{\mathcal{E}}_{1,N}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_N^{\Omega_1,\Omega_2}(s,T)}{T}, \\ \tilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{\tilde{l}_D^{\Omega_2}(s,T)}, & \tilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_N^{\Omega_1,\Omega_2}(s,T)}{\tilde{l}_N^{\Omega_2}(s,T)}, \\ \tilde{\mathcal{E}}_{3,D}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{\tilde{\mathcal{A}}_D^{\Omega_2}(s,T)}, & \tilde{\mathcal{E}}_{3,N}^{\Omega_1,\Omega_2}(s,T) &:= \frac{\tilde{\Phi}_N^{\Omega_1,\Omega_2}(s,T)}{\tilde{\mathcal{A}}_N^{\Omega_2}(s,T)}.\end{aligned} \tag{1.9}$$

1.2 Prey at forager starting position and change of monotonicity

In this section we will assume that the forager starts its search from the prey location. In this case, all the efficiency functionals in (1.6) diverge if $n \geqslant 2$ or $n = 1$ and $s \in (0, \frac{1}{2}]$, as better specified in the following proposition. For this reason, in this scenario where the forager starting position coincides with the target location, we will only work in one dimension.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected, $x \in \Omega$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$. Then, for each $T \in (0, +\infty)$, if either $n \geqslant 2$ or $n = 1$ and $s \in (0, \frac{1}{2}]$ it holds that $\mathcal{E}(s,T) = +\infty$.*

In the one-dimensional framework, the connectedness hypothesis on Ω forces the domain to be an interval. Thus, up to a translation, we can suppose that $\Omega = (0, a)$ for some $a \in (0, +\infty)$. In this case, several results can be obtained at the same time for all the efficiency functionals in (1.6).

In the following proposition we establish that the range of the fractional exponent in which these functionals achieve a finite value coincides with $(\frac{1}{2}, 1]$, and that in this interval they are continuous in s.

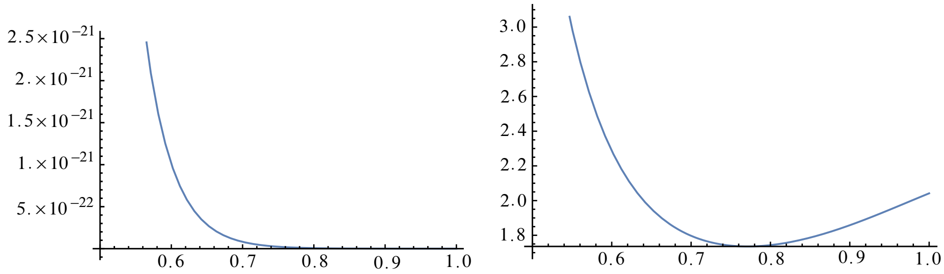

Figure 1.1. Plot of $(\frac{1}{2}, 1) \ni s \mapsto \Phi_D^{x,x}(s,T)$ for $\Omega = (0,a)$ with $x = 2.5$, $T = 100$ and $a \in \{3, 10\}$. We have approximated Φ_D, as explicitly given in (3.5), by summing to the 5×10^5th term.

Proposition 1.2. *Let $a \in (0,+\infty)$, $\Omega = (0,a)$, $x \in \Omega$, $T \in (0,+\infty)$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$. Then, $\mathcal{E}(s,T) \in (0,+\infty)$ for all $s \in (\frac{1}{2}, 1]$ and $\mathcal{E}(\cdot, T) \in C((12, 1])$.*

In terms of detecting the most rewarding foraging strategy with respect to the Lévy exponent s, we show that *if the initial position of the forager coincides with the location of the target, then $s = 1/2$ is the optimizer* for all the efficiency functionals in (1.6).

Theorem 1.3. *Let $a \in (0,+\infty)$, $\Omega = (0,a)$, $x \in \Omega$ and $\mathcal{E}$ be any of the efficiency functionals in* (1.6) *with $x = y$. Then, for all $T \in (0,+\infty)$, the supremum over $s \in (\frac{1}{2}, 1]$ of $\mathcal{E}$ is attained at $s = \frac{1}{2}$, with*

$$\lim_{s \searrow \frac{1}{2}} \mathcal{E}(s,T) = +\infty. \tag{1.10}$$

Even though the environmental scenario of a forager starting its search precisely from the target location is physically less relevant than the other cases, it can serve as an example of the complexity of the optimization problem and its dependence on external factors, such as the geometrical properties of the domain.

In what follows, we provide an example of *change of monotonicity* for the functionals in equation (1.5). Specifically, we show that *if the interval in which we consider the motion is small enough, then the functionals are strictly decreasing in s*. On the other hand, we prove that *if the interval is large enough, then there is a region of this interval such that if the search starts there, then the monotonicity property is violated* in a neighborhood of the Brownian strategy $s = 1$, see Figure 1.1.

Theorem 1.4. *Let $a \in (0,+\infty)$, $\Omega = (0,a)$, $T \in (0,+\infty)$ and $x \in \Omega$. Let Φ be any of the foraging success functional in* (1.5) *with*

$$x = y.$$

Then, if $a \in (0, \pi]$, *for every* $s_0 \in (\frac{1}{2}, 1]$ *and* $s_1 \in (s_0, 1]$, *we have that*

$$\Phi(s_0, T) > \Phi(s_1, T). \tag{1.11}$$

Also, for every $\nu \in (0, \frac{1}{2})$ *there exists* $a_\nu \in (\pi, +\infty)$ *such that if* $a \in (a_\nu, +\infty)$, *then, for every* $T \in [\nu a^{2s}, +\infty)$, $x \in (\nu a, (1-\nu)a)$, $s_0 \in (1 + \nu 2, 1)$ *and* $s_1 \in (s_0, 1]$, *it holds that*

$$\Phi_D^{x,x}(s_1, T) > \Phi_D^{x,x}(s_0, T). \tag{1.12}$$

Furthermore, for every $\nu \in (0, \frac{1}{2})$ *there exists* $a_\nu \in (\pi, +\infty)$ *such that if* $a \in (a_\nu, +\infty)$, *then, for every* $T \in [\nu a^{2s}, +\infty)$, $x \in (0, \frac{(1-\nu)a}{2}) \cup (\frac{(1+\nu)a}{2}, a)$, $s_0 \in (\frac{1+\nu}{2}, 1)$ *and* $s_1 \in (s_0, 1]$, *it holds that*

$$\Phi_N^{x,x}(s_1, T) > \Phi_N^{x,x}(s_0, T). \tag{1.13}$$

In [10, 11] we studied the monotonicity properties of the fractional heat kernel $r^s(t, x, x)$ with respect to the fractional parameter s and we showed that these properties depend on the geometry of the domain. This dependence is expressed via the eigenvalues of either the Dirichlet or the Neumann Laplacian, which are well known to depend on geometric features of the domain, like its measure or the Hausdorff measure of its boundary. For further details on this relation see the comments after [10, Theorem 8] and [11, Theorem 8] and the references therein.

More precisely, in [10, Theorem 7] we established that if the first eigenvalue of the Dirichlet Laplacian is greater than 1, then the fractional heat kernel $r_D^s(t, x, x)$ is strictly decreasing in s. Analogously, in [11, Theorem 7] we proved that if the first nonvanishing eigenvalue $\mu_{k(x)}$ of the Neumann Laplacian associated to a nonvanishing eigenfunction in x is greater than 1, then $r_N^s(t, x, x)$ is strictly decreasing in s. The monotonicity property given in (1.11) is thus a consequence of [10, Theorems 7] and [11] and the definitions in (1.5).

On the other hand, in Theorem 7 of both [10] and [11] we proved that under some circumstances there is a change of monotonicity for $r^s(t, x, x)$. Indeed, we showed that if the first eigenvalue of the Dirichlet Laplacian, or $\mu_{k(x)}$ as described above for the Neumann case, is smaller than 1, then for every $s_0, s_1 \in (0, 1)$ such that

$$s_0 < s_1$$

there exists some $T \in (0, +\infty)$ such that

$$r^{s_0}(t, x, x) < r^{s_1}(t, x, x)$$

for all $t \in (T, +\infty)$. This latter change of monotonicity in relation to the size of the eigenvalues inspired the search for a change of monotonicity also for the efficiency functionals $\Phi_D^{x,x}$ and $\Phi_N^{x,x}$, which is proved to be true, as expressed by equations (1.12) and (1.13) above.

1.3 Prey in proximity of the forager

We now turn our attention to the efficiency functionals in (1.6) when the initial position of the forager $y \in \Omega$ is different from the target location $x \in \Omega$. We begin by stating the following *continuity result* with respect to the fractional exponent s.

Proposition 1.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. For every*

$$(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$$

such that $x \neq y$, let us denote by $\mathcal{E}^{x,y}$ any of the efficiency functionals in (1.6). Then, $\mathcal{E}^{x,y}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\mathcal{E}^{x,y}(\cdot, T) \in C((0, 1])$.

In the following result we establish that for each $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$, satisfying $x \neq y$, the first Dirichlet functional $\mathcal{E}^{x,y}_{1,D}(s, T)$ *attains its infimum at* $s = 0$. Moreover, we show that the Dirichlet functionals in (1.6) admit a finite limit for $s \searrow 0$, as far as $x \neq y$.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, for every $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$ with $x \neq y$, it holds that*

$$\inf_{s \in (0,1)} \mathcal{E}^{x,y}_{1,D}(s, T) = \lim_{s \searrow 0} \mathcal{E}^{x,y}_{1,D}(s, T) = 0. \tag{1.14}$$

Moreover, we have that

$$\lim_{s \searrow 0} \mathcal{E}^{x,y}_{2,D}(s, T) \in (0, +\infty) \quad \textit{and} \quad \lim_{s \searrow 0} \mathcal{E}^{x,y}_{3,D}(s, T) \in (0, +\infty). \tag{1.15}$$

From Theorem 1.6 we evince that we can extend by continuity the Dirichlet functionals in (1.6) to the whole compact interval $[0, 1]$. Hence, from now on, we will adopt the notation

$$\mathcal{E}^{x,y}_{j,D}(0, T) := \lim_{s \searrow 0} \mathcal{E}^{x,y}_{j,D}(s, T),$$

for all $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$, with $x \neq y$ and $j \in \{1, 2, 3\}$.

The following two theorems are the most important results of this section. We state that *if the forager starting position $y \in \Omega$ is close enough to the prey location $x \in \Omega$, then the best search strategy for the efficiency functionals in (1.6) will be in some small neighborhood of $s = 0$.*

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $(y, T) \in \Omega \times (0, +\infty)$. Then, for each $\varepsilon \in (0, 1)$ there exists some $\delta = \delta_{\varepsilon,y,T,\Omega} \in (0, +\infty)$ such that for each $x \in B_\delta(y) \setminus \{y\}$ it holds that*

$$\sup_{s \in (0,1)} \mathcal{E}^{x,y}_{1,D}(s, T) = \mathcal{E}^{x,y}_{1,D}(s^{(1)}_{x,y,T}, T) \quad \textit{with } s^{(1)}_{x,y,T} \in (0, \varepsilon). \tag{1.16}$$

Moreover, for each $j \in \{2,3\}$ *it holds that*

$$\mathcal{E}^{x,y}_{j,D}(0,T) \geqslant \sup_{s\in(\varepsilon,1)} \mathcal{E}^{x,y}_{j,D}(s,T). \tag{1.17}$$

We stress that the situation $x \neq y$ treated in Theorem 1.7 is conceptually quite different from the case $x = y$ presented in Theorem 1.3: indeed, when the initial location of the predator is different from the position of the target, the efficiency functionals are finite for all $s \in (0,1]$ independently of the dimension, as stated in Proposition 1.5.

The result in Theorem 1.7 is general enough to include different Dirichlet efficiency functionals and detects a somewhat *"universal" qualitative behavior*.

Moreover, an analogous situation holds true also for the Neumann functionals in (1.6).

Theorem 1.8. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, smooth and connected and* $(y,T) \in \Omega \times (0,+\infty)$. *Then, for each* $\varepsilon \in (0,1)$ *there exists some* $\delta = \delta_{\varepsilon,y,T,\Omega} \in (0,+\infty)$ *such that for each* $x \in B_\delta(y) \setminus \{y\}$ *and for all* $j \in \{1,2,3\}$ *it holds that*

$$\sup_{s\in(0,1)} \mathcal{E}^{x,y}_{j,N}(s,T) = \mathcal{E}^{x,y}_{j,N}(s^{(j)}_{x,y,T}, T) \quad \textit{with } s^{(j)}_{x,y,T} \in (0,\varepsilon). \tag{1.18}$$

Therefore, from Theorems 1.7 and 1.8 we deduce that *if the initial position of the forager approaches the position of the target, the fractional parameter* $s \in (0,1)$ *maximizing the functionals in* (1.6) *converges to* 0. Thus, *in the regime of close proximity of seeker starting position and prey location, the above functionals are maximized by a search strategy with a very fat tail.*

It is interesting to notice that the maximizer $s^{(1)}_{x,y,T}$ of $\mathcal{E}_{1,D}$ given by Theorem 1.7 may turn out to be unreliable in practice, differently from the other two maximizers of the Dirichlet functionals, according to the following remark.

Remark 1.9. On the one hand, Theorem 1.6 establishes that $s = 0$ is a global minimizer for $\mathcal{E}_{1,D}$. On the other hand, if $s^{(1)}_{x,y,T}$ is a maximizer of $\mathcal{E}^{x,y}_{1,D}(\cdot,T)$, then from Theorem 1.7 we evince that

$$\lim_{x\to y} s^{(1)}_{x,y,T} = 0.$$

This means that as x approaches y, *the maximizer of the functional* $\mathcal{E}^{x,y}_{1,D}$ *converges to* $s = 0$, *which is a global minimizer*. Therefore, a small perturbation of $s^{(1)}_{x,y,T}$ can lead to very small values for $\mathcal{E}_{1,D}$, *making such choice of the most rewarding fractional exponent quite unreliable*. Therefore, in an environmental scenario where the forager starts its search in proximity of the target and the efficiency functional modelling the energy to maximize is given by $\mathcal{E}_{1,D}$, the "most rewarding" search strategy is to be considered "unreliable".

Things turn out to be different for $\mathcal{E}^{x,y}_{2,D}$ and $\mathcal{E}^{x,y}_{3,D}$. Indeed, if $s^{(j)}_{x,y,T}$ is a maximizer of the functional $\mathcal{E}^{x,y}_{j,D}(\cdot, T)$ with $j \in \{2, 3\}$, then, according to Theorem 1.7, one still has the limit

$$\lim_{x \to y} s^{(j)}_{x,y,T} = 0.$$

Nevertheless, in contrast with the case $j = 1$, now $s = 0$ is not necessarily a global minimum. Actually, see equation (1.17), for each $\varepsilon \in (0, 1)$, if x and y are close enough, then

$$\mathcal{E}^{x,y}_{j,D}(0, T) \geqslant \sup_{s \in (\varepsilon,1)} \mathcal{E}^{x,y}_{j,D}(s, T),$$

so that $s = 0$ in these two cases is "almost" a maximizer. Roughly speaking, we can say that the functionals $\mathcal{E}_{2,D}$ and $\mathcal{E}_{3,D}$ present more reliable optimal configurations than $\mathcal{E}_{1,D}$, since the maximizing fractional exponent is "separated" from the minimizers, whence the most rewarding strategy appears to be safer.

Remark 1.10. It has been observed in [43] that the case $s = 0$ occurs when some marine predators, such as anglers and blonde skates, specifically aim at a type of prey with a high energy content. It is therefore natural to relate the high-energy content of the prey and the high-risk/high-reward strategy related to $s = 0$; namely a high gain prospected by the energy content of the prey may serve as a mitigation of the chance of failure entailed by searching mode selected and as an indirect encouragement towards a potentially very beneficial, but intrinsically very risky, strategy.

Remark 1.11. One may wonder whether the unreliability of the most rewarding strategies and the corresponding high-risk/high-reward searching mode are specific of the situation considered in this memoir, i.e., of a forager confined in a bounded region and a nearby prey. This is not the case, in fact in the paper [13] we will show that the same pattern persists, for instance, for a predator diffusing in the whole space and also for a prey located arbitrarily far from the predator.

The case that will be addressed in [13] is technically different from the one here, since the spectral analysis cannot be performed in unbounded domains and we will have to rely on singular integral calculations instead.

In what follows we observe a phenomenon which arises in the one-dimensional framework as a consequence of Theorem 1.4. In particular, under the same geometric assumptions of Theorem 1.4 on the domain Ω, we show that if the target location $x \in \Omega$ is sufficiently close to the forager initial position $y \in \Omega$, then *there exists a local maximizer* $s^*_{x,y,T}$ *for* $\mathcal{E}^{x,y}_{1,D}$ *and* $\mathcal{E}^{x,y}_{1,N}$ *in a neighborhood of the Brownian strategy* $s = 1$.

Corollary 1.12. *Let* $a \in (0, +\infty)$, $\Omega = (0, a)$ *and* $T \in (0, +\infty)$. *Then, for every* $\nu \in (0, \frac{1}{2})$ *and* $\varepsilon \in (0, 1)$ *there exists* $a_\nu \in (\pi, +\infty)$ *such that if* $a \in (a_\nu, +\infty)$, *then, for* *every* $T \in [\nu a^{2s}, +\infty)$ *and* $y \in (\nu a, (1-\nu)a)$*, there exists some* $\delta = \delta_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ *such that if* $x \in B_\delta(y) \setminus \{y\}$*, then*

$$\sup_{s\in(\frac{1+\nu}{2},1)} \mathcal{E}_{1,D}^{x,y}(s,T) = \mathcal{E}_{1,D}^{x,y}(s^*_{x,y,T},T) \quad \textit{with } s^*_{x,y,T} \in (1-\varepsilon, 1]. \tag{1.19}$$

Also, for every $\nu \in (0, \frac{1}{2})$ *and* $\varepsilon \in (0,1)$ *there exists* $a_\nu \in (\pi, +\infty)$ *such that if* $a \in (a_\nu, +\infty)$*, then, for every* $T \in [\nu a^{2s}, +\infty)$ *and* $y \in (0, \frac{(1-\nu)a}{2}) \cup (\frac{(1+\nu)a}{2}, a)$*, there exists some* $\delta = \delta_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ *such that if* $x \in B_\delta(y) \setminus \{y\}$*, then*

$$\sup_{s\in(\frac{1+\nu}{2},1)} \mathcal{E}_{1,N}^{x,y}(s,T) = \mathcal{E}_{1,N}^{x,y}(\hat{s}_{x,y,T},T) \quad \textit{with } \hat{s}_{x,y,T} \in (1-\varepsilon, 1]. \tag{1.20}$$

It is interesting to compare this result with Remark 1.9 on the unreliability of the most rewarding search strategy for $\mathcal{E}_{1,D}^{x,y}$. Indeed, as a consequence of Theorem 1.7 and Corollary 1.12, we have that for each $\nu \in (0, \frac{1}{2})$ and $\varepsilon \in (0,1)$ there exists some $a_\nu \in (\pi, +\infty)$ such that for every a, T and y given as in the statement of Corollary 1.12, there exists some $\delta^* = \delta^*_{\nu,\varepsilon,y,T,\Omega} \in (0,+\infty)$ such that, for every $x \in B_{\delta^*}(y) \setminus \{y\}$,

$$\begin{aligned}
\sup_{s\in(0,1)} \mathcal{E}_{1,D}^{x,y}(s,T) &= \mathcal{E}_{1,D}^{x,y}(s^{(1)}_{x,y,T},T) \quad \text{with } s^{(1)}_{x,y,T} \in (0,\varepsilon),\\
\sup_{s\in(\frac{1+\nu}{2},1)} \mathcal{E}_{1,D}^{x,y}(s,T) &= \mathcal{E}_{1,D}^{x,y}(s^*_{x,y,T},T) \quad \text{with } s^*_{x,y,T} \in (1-\varepsilon,1].
\end{aligned}$$

From this, we deduce that in this framework *there exist a global and a local maximizer*. The global maximizer $s^{(1)}_{x,y,T}$ seems to be the most rewarding option for the forager performing the search. Nevertheless, thanks to Remark 1.9, we also know that it is extremely *unreliable* for practical purposes. Indeed, a small deviation from $s^{(1)}_{x,y,T}$ can lead to the unique global minimizer $s = 0$, that makes the functional vanish.

On the other hand, even though the local maximizer $s^*_{x,y,T}$ is not optimal, it could be a better choice due to its stability. As a matter of fact, as stated in Proposition 1.5, the functional $\mathcal{E}_{1,D}^{x,y}$ vanishes nowhere near the Brownian strategy $s = 1$. Therefore, by choosing $s^*_{x,y,T}$, even under the presence of a positive error in the choice of the strategy, the outcome would not be heavily affected, as it could be for the most rewarding, but unreliable, strategy $s^{(1)}_{x,y,T}$.

This observation highlights how the definition of “best search strategy” is arguable, and how in some contexts it could not coincide with the classical notion of maximizer of a given energy: after all, what does “best” mean, is it “most rewarding” or “safest”? Thus, it may be appropriate to define new efficiency functionals that, rather than depending on an “exact choice” of the fractional exponent $s \in (0,1)$, take into account a probability measure in $(0,1)$ that allows the existence of an error range for the forager. This new approach will be investigated by the authors in a forthcoming work.

1.4 Foragers and targets uniformly distributed in some regions

Now we focus our attention on the study of the functionals in equation (1.9). In this case, the forager starting position and the prey location are replaced by uniform densities in disjoint subsets $\Omega_1, \Omega_2 \subset \Omega$. We begin by analyzing the continuity of these functionals with respect to the fractional exponent $s \in (0, 1]$.

Proposition 1.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. For every $T \in (0, +\infty)$ and measurable sets $\Omega_1, \Omega_2 \subset \Omega$, let us denote by $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}$ any of the efficiency functionals in (1.9). Then, $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}(s, T) \in (0, +\infty)$ for all $s \in (0, 1]$ and $\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}(\cdot, T) \in C((0, 1])$.*

The following result can be considered as the set-dependent counterpart of Theorem 1.6.

Theorem 1.14. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, for all $T \in (0, +\infty)$ and smooth and disjoint sets $\Omega_1, \Omega_2 \subset \Omega$, it holds that*

$$\inf_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s, T) = \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s, T) = 0. \tag{1.21}$$

Moreover, we have that

$$\lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(s, T) \in (0, +\infty) \quad \textit{and} \quad \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{3,D}(s, T) \in (0, +\infty). \tag{1.22}$$

From Theorem 1.14 we deduce that we can extend by continuity also the Dirichlet functionals in (1.9) to the whole compact interval $[0, 1]$. From now on, for $j \in \{1, 2, 3\}$, we will adopt the notation

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(0, T) := \lim_{s\searrow 0} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(s, T),$$

for all $T \in (0, +\infty)$ and $\Omega_1, \Omega_2 \subset \Omega$ satisfying the hypothesis of Theorem 1.14.

In Theorems 1.7 and 1.8 we have established that the Neumann and Dirichlet functionals in (1.6) have a common feature. Indeed, if the prey location $x \in \Omega$ is in a sufficiently small neighborhood of the forager starting position $y \in \Omega$, then $\mathcal{E}^{x,y}_{j,D}$ and $\mathcal{E}^{x,y}_{j,N}$ attain their maximum for some value close to $s = 0$.

This characteristic is somewhat preserved if we consider the set-dependent functionals in (1.9). Indeed, we can show that if Ω_1, Ω_2 are close enough (in a sense that will be made precise later), then also for the functionals in (1.9) a strongly nonlocal search strategy will be preferred.

Before stating the precise results we fix some notation. For each $B \subset \mathbb{R}^n$, $y \in \mathbb{R}^n$ and $r \in (0, +\infty)$ we denote

$$r_y B := \{r(x - y) + y \text{ s.t. } x \in B\}. \tag{1.23}$$

Theorem 1.15. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, smooth and connected and* $(y, T) \in \Omega \times (0, +\infty)$. *Then, for each* $\varepsilon \in (0, 1)$ *there exists some* $r = r_{\varepsilon,y,T,\Omega} \in (0, +\infty)$ *such that for any smooth and disjoint sets* $\Omega_1, \Omega_2 \subset B_r(y)$ *it holds that*

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s, T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s^{(1)}_{\Omega_1,\Omega_2,T}, T) \quad \textit{with } s^{(1)}_{\Omega_1,\Omega_2,T} \in (0, \varepsilon).$$

Moreover, let $K \Subset \Omega$ *be star-shaped with respect to some* $y \in K$. *Then, for all* $j \in \{2, 3\}$ *and* $\varepsilon \in (0, 1)$, *there exists some* $r = r_{\varepsilon,K,T,\Omega}$ *such that if* $\Omega_1, \Omega_2 \subset r_y K$ *are smooth and disjoint it holds that*

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(0, T) \geqslant \sup_{s\in(\varepsilon,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,D}(s, T). \tag{1.24}$$

As a consequence of Theorems 1.14 and 1.15 we can deduce that the most rewarding strategy may not be the safest, similarly to what happens for the functional $\mathcal{E}_{1,D}$ (recall Remark 1.9). Also, a result analogous to Theorem 1.15 holds true when considering the Neumann functionals in (1.9).

Theorem 1.16. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, smooth and connected and* $(y, T) \in \Omega \times (0, +\infty)$. *Then, for each* $\varepsilon \in (0, 1)$ *there exists some* $r = r_{\varepsilon,y,T,\Omega} \in (0, +\infty)$ *such that for any smooth and disjoint sets* $\Omega_1, \Omega_2 \subset B_r(y)$ *and for each* $j \in \{1, 2, 3\}$ *it holds that*

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s, T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s^{(j)}_{\Omega_1,\Omega_2,T}, T) \quad \textit{with } s^{(j)}_{\Omega_1,\Omega_2,T} \in (0, \varepsilon). \tag{1.25}$$