Chapter 2

Mathematical framework for the efficiency functionals

2.1 Preliminary results and asymptotics

In this section we establish some technical results regarding the efficiency functionals in (1.5), (1.6), and (1.9). These are the main analytical tools that we will use to prove the results stated in the introduction.

In Section 2.2 we provide some estimates for the functionals in (1.5) and (1.7). This is the content of Lemma 2.7, Theorem 2.9, and Corollary 2.11. These results will be employed in Section 3.2 in order to discuss the environmental scenario where the prey is in proximity of the forager starting location, and thus to prove Theorems 1.7, 1.8, 1.15, and 1.16. Moreover, we establish the limits of the Dirichlet functionals in (1.6) and (1.9) as $s \searrow 0$ as stated in Lemma 2.13. These asymptotics will be used to prove Theorems 1.6, 1.14, 1.7, and 1.15.

To conclude, in Lemmas 2.15 and 2.16 we show that the Neumann functionals in (1.3), (1.4), (1.5), (1.7), and (1.8) do not vanish for $s \searrow 0$, and we provide upper and lower bounds for their lim inf and lim sup. These results will be used in the proofs of Theorem 1.8 and 1.16.

To prove these results, it is useful to recall some properties regarding the fractional heat kernels r_D^s and r_N^s. It is well known that for each $s \in (0,1)$ these two kernels can be written for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$ as

$$\begin{aligned} r_D^s(t,x,y) &= \int_0^{+\infty} p_D^\Omega(l,x,y)\mu_t^s(l)\,dl, \\ r_N^s(t,x,y) &= \int_0^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl, \end{aligned} \tag{2.1}$$

where p_D^Ω and p_N^Ω are the classical Dirichlet and Neumann heat kernels in Ω, while μ_t^s is the density of an s-stable subordinator in $(0,+\infty)$ (see, e.g., [10, Definition 4]). For a proof of this latter fact see for instance [10, Proposition 2] and [11, Proposition 2].

If $s = 1$, the kernels r_N^1 and r_D^1 coincide respectively with the classical kernels p_N^Ω and p_D^Ω. Furthermore, we also know that the density μ_t^s admits the explicit representation formula

$$\mu_t^s(l) = \frac{1}{\pi}\int_0^{+\infty} e^{-lu - tu^s\cos(\pi s)}\sin(tu^s\sin(\pi s))\,du \quad \text{for all } (l,s) \in (0,+\infty)\times(0,1), \tag{2.2}$$

see [23, Proposition 3.1].

Moreover, we also recall the following fact on the spectral representation of r^s_D and r^s_N. In what follows we denote by $\{\zeta_{D,k}\}_k$ and $\{\zeta_{N,k}\}_k$ two orthonormal basis of $L^2(\Omega)$ satisfying

$$\begin{cases} -\Delta\zeta_{D,k} = \beta_{D,k}\zeta_{D,k} & \text{in } \Omega, \\ \zeta_{D,k} \in H^1_0(\Omega) \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\zeta_{N,k} = \beta_{N,k}\zeta_{N,k} & \text{in } \Omega, \\ \dfrac{\partial\zeta_{N,k}}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $0 < \beta_{D,1} < \beta_{D,2} \cdots$ and $0 = \beta_{N,0} < \beta_{N,1} < \cdots$ are respectively the eigenvalues of the Laplace operator with homogeneous Dirichlet and homogeneous Neumann boundary conditions.

Thus, thanks to [10, Theorem 5] and [11, Theorem 5], we can rewrite the Dirichlet and Neumann kernels r^s_D and r^s_N as

$$\begin{aligned} r^s_D(t,x,y) &= \sum_{k=1}^{+\infty} \zeta_{D,k}(x)\zeta_{D,k}(y)\exp(-t\beta^s_{D,k}), \\ r^s_N(t,x,y) &= \sum_{k=0}^{+\infty} \zeta_{N,k}(x)\zeta_{N,k}(y)\exp(-t\beta^s_{N,k}), \end{aligned} \tag{2.4}$$

for all $s \in (0,1]$ and $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$.

Now, we establish some results on μ^s_t. In what follows we recall a scaling property for the density μ^s_t of the s-stable subordinator. For the convenience of the reader the statement is proved.

Lemma 2.1. *Let $l \in (0,+\infty)$, $t \in (0,+\infty)$ and $s \in (0,1)$. Then, we have that*

$$\mu^s_t(l) = \frac{1}{t^{\frac{1}{s}}}\mu^s_1\left(\frac{l}{t^{\frac{1}{s}}}\right). \tag{2.5}$$

Proof. Let $\alpha := u^s\cos(\pi s)$, $\beta := u^s\sin(\pi s)$ and

$$g(\alpha,\beta) := e^{-t\alpha}\sin(t\beta).$$

With this notation, we integrate by parts the expression on the right-hand side of (2.2) and obtain that

$$\begin{aligned} \mu^s_t(l) &= -\frac{1}{l\pi}e^{-lu}g(\alpha,\beta)\Big|_0^{+\infty} + \frac{1}{\pi l}\int_0^{+\infty} e^{-lu}\frac{d}{du}g(\alpha,\beta)\,du \\ &= 0 + \frac{1}{\pi l}\int_0^{+\infty} e^{-lu}e^{-t\alpha}stu^{s-1}(-\cos(\pi s)\sin(t\beta) + \sin(\pi s)\cos(t\beta))\,du \\ &= \frac{st}{\pi l}\int_0^{+\infty} e^{-lu}e^{-tu^s\cos(\pi s)}u^{s-1}\sin(\pi s - tu^s\sin(\pi s))\,du. \end{aligned} \tag{2.6}$$

We employ the change of variable $v = ut^{\frac{1}{s}}$ and infer from the last identity that

$$\begin{aligned}\mu_t^s(l) &= \frac{s}{\pi l}\int_0^{+\infty} e^{-\frac{l}{t^{\frac{1}{s}}}v} e^{-v^s\cos(\pi s)} v^{s-1}\sin(\pi s - v^s\sin(\pi s))\,dv\\ &= \frac{1}{t^{\frac{1}{s}}}\frac{st^{\frac{1}{s}}}{\pi l}\int_0^{+\infty} e^{-\frac{l}{t^{\frac{1}{s}}}v} e^{-v^s\cos(\pi s)} v^{s-1}\sin(\pi s - v^s\sin(\pi s))\,dv\\ &= \frac{1}{t^{\frac{1}{s}}}\mu_1^s\Big(\frac{l}{t^{\frac{1}{s}}}\Big).\end{aligned}$$

■

Now, we discuss some asymptotic estimates for the density $\mu_t^s(l)$ in l. As it is recalled in [5] by R. Song and proved by Skorohod in [33], one has that

$$\mu_1^s(l) \sim 2\pi\Gamma(1+s)\sin\Big(\frac{\pi s}{2}\Big)\frac{1}{l^{1+s}} \quad \text{for } l \to +\infty. \tag{2.7}$$

Using this estimate and Lemma 2.1 on the time-scaling property of μ_t^s one obtains an interesting asymptotic expansion in the forthcoming Lemma 2.2. As a side comment, we point out that the asymptotic properties of this type of distributions are relevant to understand how the tail of μ_t^s changes by varying the fractional parameter s, which in turn provides some important information about the optimization problem that we analyze in this memoir.

Lemma 2.2. *Let $s \in (0,1)$ and $t \in (0,+\infty)$. Then, we have that*

$$\mu_t^s(t) \sim 2\pi\Gamma(1+s)\sin\Big(\frac{\pi s}{2}\Big)\frac{t}{l^{1+s}} \quad \textit{for } l \to +\infty. \tag{2.8}$$

Proof. Thanks to Lemma 2.1, we know that for each $s \in (0,1)$, $l \in (0,+\infty)$ and $t \in (0,+\infty)$ one has that

$$\mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}}\mu_1^s\Big(\frac{l}{t^{\frac{1}{s}}}\Big).$$

Thus, using this identity and the estimate in (2.7) one readily obtains that

$$\begin{aligned}\mu_t^s(l) &\sim \frac{1}{t^{\frac{1}{s}}}2\pi\Gamma(1+s)\sin\Big(\frac{\pi s}{2}\Big)\frac{t^{\frac{1+s}{s}}}{l^{1+s}}\\ &= 2\pi\Gamma(1+s)\sin\Big(\frac{\pi s}{2}\Big)\frac{t}{l^{1+s}},\end{aligned}$$

for $l \to +\infty$. ■

The following theorem provides similar estimates to the one given in (2.8) in the range $s \in (0, \frac{1}{2})$. Here, the constants involved are less accurate than the one appearing in (2.8), but on the other hand we gain some important information. In particular, while the estimate in (2.8) holds true for $l \to +\infty$, the ones that we prove below are true for each $l \in (t^{\frac{1}{s}}, +\infty)$. This additional information will be used several times.

Theorem 2.3. *Let $s \in (0, \frac{1}{2})$ and $t \in (0, +\infty)$. Then, there exists some constant $C_1 \in (0, +\infty)$, independent of s and l, such that*

$$\begin{aligned} \frac{stC_1}{\pi l^{1+s}} &\leqslant \mu_t^s(l) \qquad \text{for all } l \in (t^{\frac{1}{s}}, +\infty), \\ \mu_t^s(l) &\leqslant \frac{st\Gamma(1+s)}{l^{1+s}} \quad \text{for all } l \in (0, +\infty). \end{aligned} \tag{2.9}$$

Proof. Thanks to the scaling property proved in Lemma 2.1, it is enough to show the result for $t = 1$. Indeed, if for $t = 1$ the inequalities in (2.9) hold true, then if $t > 1$ and $l \geqslant t^{\frac{1}{s}}$, we have in view of (2.5) that

$$\mu_t^s(l) = \frac{1}{t^{\frac{1}{s}}} \mu_1^s\left(\frac{l}{t^{\frac{1}{s}}}\right) \geqslant \frac{sC_1 t}{\pi l^{1+s}}.$$

The second inequality in (2.9) is proved similarly. For this reason, we focus our attention on the case $t = 1$.

We will first prove the second inequality in (2.9). If $s \in (0, \frac{1}{2})$, from (2.2) we notice that

$$\mu_1^s(l) \leqslant \frac{\sin(\pi s)}{\pi} \int_0^{+\infty} e^{-lu} tu^s \, du \leqslant \frac{s}{\pi l^{1+s}} \Gamma(1+s),$$

which concludes the proof of the second inequality in (2.9).

Now we focus on the proof of the first inequality. To do so, we observe that thanks to equation (2.6) one has that

$$\mu_1^s(l) = \frac{s}{\pi l} \int_0^{+\infty} e^{-lu} e^{-u^s \cos(\pi s)} u^{s-1} \sin(\pi s - u^s \sin(\pi s)) \, du.$$

We perform the change of variable $lu = \theta$ and obtain that

$$\begin{aligned} \mu_s^1(l) &= \frac{s}{\pi l^{1+s}} \int_0^{+\infty} e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) d\theta \\ &=: \frac{s}{\pi l^{1+s}} f(s, l), \end{aligned} \tag{2.10}$$

where by construction $f(s, l) > 0$ for each $l \in (0, +\infty)$ and $s \in (0, 1)$.

Now we observe that, for each $\theta \geqslant 1$ and $s \in (0, \frac{1}{2}]$,

$$\left| e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) \right| \leqslant e^{-\theta}.$$

Thus, by the dominated convergence theorem we obtain that

$$\lim_{s \searrow 0} \int_1^{+\infty} e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) d\theta = 0.$$

Also, by using the change of variable $\theta^s = l^s z$ we deduce that

$$\int_0^1 e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) d\theta$$

$$= \frac{l^s}{s} \int_0^{\frac{1}{l^s}} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)} \sin(\pi s - z\sin(\pi s))\, dz$$

$$= l^s \int_0^{+\infty} \chi_{[0,l^{-s}]} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)} \frac{\sin(\pi s - z\sin(\pi s))}{s}\, dz.$$

If $s \in (0, \frac{1}{3})$ we also notice that

$$\left| \chi_{[0,l^{-s}]} e^{-lz^{\frac{1}{s}}} e^{-z\cos(\pi s)} \frac{\sin(\pi s - z\sin(\pi s))}{s} \right| \leqslant \pi e^{-\frac{z}{2}}(1+z),$$

and therefore, since $l \geqslant 1$, by the dominated convergence theorem we obtain that

$$\lim_{s\searrow 0} \int_0^1 e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) d\theta = \pi \int_0^1 e^{-z}(1-z)\, dz.$$

Consequently, for each $l \geqslant 1$

$$\lim_{s\searrow 0} f(s,l) = \pi \int_0^1 e^{-z}(1-z)\, dz = \frac{\pi}{e}. \tag{2.11}$$

We also observe that, if $s \in (0, \frac{1}{2}]$,

$$\left| e^{-\theta} e^{-\frac{\theta^s}{l^s}\cos(\pi s)} \theta^{s-1} \sin\left(\pi s - \frac{\theta^s}{l^s}\sin(\pi s)\right) \right| \leqslant e^{-\theta}\theta^{s-1},$$

for all $\theta \in (0, +\infty)$.

As a consequence, by the dominated convergence theorem we evince that

$$\lim_{l\to+\infty} f(s,l) = \sin(\pi s)\Gamma(s) > 0, \tag{2.12}$$

for all $s \in (0, \frac{1}{2}]$.

Besides, by the definition of $f(s,l)$, we have that $f \in C((0, \frac{1}{2}) \times (1, +\infty))$ and

$$f(s,l) > 0 \quad \text{for all } (s,l) \in \left(0, \frac{1}{2}\right] \times [1, +\infty). \tag{2.13}$$

Therefore, using (2.11), (2.12), and (2.13) we deduce that there exists some $C_1 \in (0, +\infty)$ such that

$$C_1 \leqslant f(s,l) \quad \text{for all } (s,l) \in \left(0, \frac{1}{2}\right) \times [1, +\infty).$$

In light of this observation and equation (2.10) we deduce that

$$\frac{C_1 s}{\pi l^{1+s}} \leqslant \mu_1^s(l).$$

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2.2 Structural results for the efficiency functionals

Now we develop the main technical tools that will be employed in the proofs of the results contained in Sections 1.2 and 1.3.

In what follows we adopt the subscript $*$ to refer to the fact that the functional considered can be the one associated with both the Dirichlet and the Neumann case.

We begin by recalling here the following estimates for the classical Dirichlet heat kernel in relation to the classical heat kernel. Using the weak maximum principle for the heat equation one can show that

$$p_D^\Omega(t,x,y) \leqslant \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) \quad \text{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega. \tag{2.14}$$

On compact subsets of Ω and for finite time spans, one can prove the following lower bound for $p_D^\Omega(t,x,y)$.

Lemma 2.4 (See [45, Lemma 2.1]). *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, there exists a constant $T_\Omega \in (0,+\infty)$ such that for each $K \Subset \Omega$, if we define*

$$T_{K,\Omega} := \min\left\{T_\Omega,\ \min_{x\in K} \frac{d^2(x,\partial\Omega)}{2}\right\}, \tag{2.15}$$

then there exist two constants $c_1, c_2 \in (0,+\infty)$, depending on K and Ω, such that

$$p_D^\Omega(t,x,y) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \quad \textit{for all } (t,x,y) \in (0,T_{K,\Omega}] \times K \times K. \tag{2.16}$$

Using the weak maximum principle, it is also possible to compare the Neumann heat kernel with the Dirichlet one, as better specified in the following result.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $K' \Subset \Omega$. Then, for each $s \in (0,1]$ we have that*

$$r_D^s(t,x,y) \leqslant r_N^s(t,x,y) \quad \textit{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega. \tag{2.17}$$

Furthermore, if $K \subseteq K' \Subset \Omega$ is star-shaped with respect to some $x_0 \in K$, there exist some constants $C_{K',\Omega}, c_{K',\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, depending on K' and Ω, such that

$$r_N^s(t,x,y) \leqslant C_{K',\Omega} r_D^s(t,x_\varepsilon,y_\varepsilon) + c_{K',\Omega} \quad \textit{for all } (t,x,y) \in (0,+\infty) \times K \times K, \tag{2.18}$$

for each $\varepsilon \in (0,\varepsilon_0)$, where $(x_\varepsilon, y_\varepsilon) := (\varepsilon x + (1-\varepsilon)x_0, \varepsilon y + (1-\varepsilon)x_0)$.

Proof. We begin by proving the lower bound in (2.17). To do so, we observe that thanks to the maximum principle for the heat equation, one has that

$$p_D^\Omega(t,x,y) \leqslant p_N^\Omega(t,x,y) \quad \text{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega.$$

Therefore, using (2.1) and the latter inequality, we obtain that

$$\begin{aligned} r_D^s(t,x,y) &= \int_0^{+\infty} p_D^{\Omega}(l,x,y)\mu_t^s(l)\,dl \\ &\leqslant \int_0^{+\infty} p_N^{\Omega}(l,x,y)\mu_t^s(l)\,dl = r_N^s(t,x,y), \end{aligned}$$

for each $(t,x,y) \in (0,+\infty) \times \Omega \times \Omega$. This concludes the proof of (2.17).

Now we show (2.18). Thanks to [9, Theorem 3.2.9], we have that there exists some constant c_Ω such that

$$p_N^{\Omega}(t,x,y) \leqslant c_\Omega \max\left\{1, \frac{1}{t^{\frac{n}{2}}}\right\} \exp\left(-\frac{|x-y|^2}{6t}\right) \quad \text{for all } (t,x,y) \in (0,+\infty)\times\Omega\times\Omega. \tag{2.19}$$

Furthermore, if $K \subseteq K' \Subset \Omega$, thanks to Lemma 2.4 we obtain that

$$p_D^{\Omega}(t,x,y) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-y|^2}{t}\right) \quad \text{for all } (t,x,y) \in (0,T_{K',\Omega}] \times K' \times K', \tag{2.20}$$

where $T_{K',\Omega}$ is introduced in (2.15) and c_1, c_2 depends on K' and Ω.

Up to a translation we can assume that K is star-shaped with respect to

$$x_0 = 0.$$

Now we observe that there exists two constants $C_{K',\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, such that

$$C_{K',\Omega} c_1 \geqslant c_\Omega \quad \text{and} \quad c_2\varepsilon^2 \leqslant \frac{1}{6} \quad \text{for all } \varepsilon \in (0,\varepsilon_0).$$

As a consequence, if for each $\varepsilon \in (0,\varepsilon_0)$ we call

$$(x_\varepsilon, y_\varepsilon) = \varepsilon(x,y),$$

then from (2.19) and (2.20) we obtain that

$$\begin{aligned} & C_{K',\Omega}\, p_D^{\Omega}(t,x_\varepsilon,y_\varepsilon) - p_N^{\Omega}(t,x,y) \\ &\quad \geqslant C_{K',\Omega} \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x_\varepsilon - y_\varepsilon|^2}{t}\right) - \frac{c_\Omega}{t^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6t}\right) \\ &\quad = \frac{c_\Omega}{t^{\frac{n}{2}}} \left(C_{K',\Omega} \frac{c_1}{c_\Omega} \exp\left(-\varepsilon^2 c_2 \frac{|x-y|^2}{t}\right) - \exp\left(-\frac{|x-y|^2}{6t}\right)\right) \\ &\quad \geqslant \frac{c_\Omega}{t^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{6t}\right) \left(\exp\left(-\left(\varepsilon^2 c_2 - \frac{1}{6}\right)\frac{|x-y|^2}{t}\right) - 1\right) \\ &\quad \geqslant 0, \end{aligned} \tag{2.21}$$

for each $(t,x,y) \in (0,T_{K',\Omega}] \times K \times K$.

Thus, using equation (2.1) and the relation in (2.21) we obtain that

$$\begin{aligned}
r_N^s(t,x,y) &= \int_0^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl \\
&= \int_0^{T_{K',\Omega}} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl + \int_{T_{K',\Omega}}^{+\infty} p_N^\Omega(l,x,y)\mu_t^s(l)\,dl \\
&\leqslant C_{K',\Omega}\int_0^{T_{K',\Omega}} p_D^\Omega(l,x_\varepsilon,y_\varepsilon)\mu_t^s(l)\,dl \\
&\quad + c_\Omega \int_{T_{K',\Omega}}^{+\infty} \max\left\{1, \frac{1}{l^{\frac{n}{2}}}\right\}\exp\left(-\frac{|x-y|^2}{6l}\right)\mu_t^s(l)\,dl \\
&\leqslant C_{K',\Omega}\int_0^{+\infty} p_D^\Omega(l,x_\varepsilon,y_\varepsilon)\mu_t^s(l)\,dl + c_{K',\Omega} \\
&= C_{K',\Omega} r_D^s(t,x_\varepsilon,y_\varepsilon) + c_{K',\Omega},
\end{aligned}$$

for each $(t,x,y) \in (0,+\infty)\times K \times K$, where we defined

$$c_{K',\Omega} := \max_{x,y\in K'}\max_{l\in[T_{K,\Omega},+\infty)} c_\Omega \max\left\{1, \frac{1}{l^{\frac{n}{2}}}\right\}\exp\left(-\frac{|x-y|^2}{6l}\right). \qquad \blacksquare$$

As a useful consequence of Theorem 2.5, we obtain that it is possible to compare the Neumann functional Φ_N with the Dirichlet one Φ_D. The result goes as follows.

Corollary 2.6. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $K' \Subset \Omega$. Then, for each $s \in (0,1]$ and $T \in (0,+\infty)$ it holds that*

$$\Phi_D^{x,y}(s,T) \leqslant \Phi_N^{x,y}(s,T) \quad \text{for all } (x,y) \in \mathcal{C}, \tag{2.22}$$

where

$$\mathcal{C} := (\Omega\times\Omega)\setminus\{(p,p) \text{ s.t. } p\in\Omega\} = \big\{(p,q)\in\Omega\times\Omega \text{ s.t. } p\neq q\big\}. \tag{2.23}$$

Furthermore, for each $K \subseteq K' \Subset \Omega$ star-shaped with respect to some $x_0 \in K$, $s \in (0,1)$ and $T \in (0,+\infty)$, there exists some $\varepsilon_0 \in (0,1)$ such that

$$\Phi_N^{x,y}(s,T) \leqslant C_{K',\Omega}\Phi_D^{x_\varepsilon,y_\varepsilon}(s,T) + c_{K',\Omega}T \quad \text{for all } (x,y)\in\mathcal{C}\cap(K\times K), \tag{2.24}$$

for each $\varepsilon \in (0,\varepsilon_0)$, where

$$(x_\varepsilon,y_\varepsilon) := (\varepsilon x + (1-\varepsilon)x_0, \varepsilon y + (1-\varepsilon)x_0)$$

and $C_{K',\Omega}, c_{K',\Omega} \in (0,+\infty)$ are given in Theorem 2.5.

Proof. Inequalities (2.22) and (2.24) are respectively obtained by integrating over the time t in $(0,T)$ both sides of (2.17) and (2.18). $\blacksquare$

In Lemma 2.7 below we establish a lower bound for $\Phi_*^{x,y}(s,T)$, for $x \in \Omega$ in a sufficiently small neighborhood of $y \in \Omega$.

This estimate is pivotal to determine the asymptotic behavior of the functionals in (1.6) when x approaches y, providing some information on the best search strategy in the environmental scenario addressed in Section 1.3, namely where the forager starts its search in proximity of the target.

Lemma 2.7. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. If $(y,T) \in \Omega \times (0,+\infty)$ and $s \in (0,1)$, then there exists some $\widehat{\delta} = \widehat{\delta}_{s,y,T,\Omega} \in (0,+\infty)$ such that, for each $x, z \in B_{\widehat{\delta}}(y)$ satisfying $x \neq z$,*

$$\Phi_*^{x,z}(s,T) \geqslant \frac{C_{s,y,\Omega}}{|x-z|^{n-2s}}, \tag{2.25}$$

for some constant $C_{s,y,\Omega} \in (0,+\infty)$.

Proof. In virtue of inequality (2.22) it is enough to show the result for Φ_D.

Let $y \in \Omega$ and let us denote $d_y := \frac{d(y,\partial\Omega)}{2}$, where

$$d(y,\partial\Omega) := \inf_{x \in \partial\Omega} |x-y|.$$

With this notation we set

$$B_y := B_{d_y}(y).$$

Now, by (2.15) and (2.16) (used here with $K := B_y$),

$$p_D^\Omega(t,x,z) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{t}\right) \quad \text{for all } (t,x,z) \in (0, T_{B_y,\Omega}] \times B_y \times B_y. \tag{2.26}$$

We also observe that for each $x, z \in \mathbb{R}^n$ such that $x \neq z$, the function

$$g(t) := \frac{c_1}{t^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{t}\right)$$

has a maximum in $\varepsilon_{x,z} := \frac{2c_2}{n}|x-z|^2$ and it is increasing in $(0,\varepsilon_{x,z})$ and decreasing in $(\varepsilon_{x,z},+\infty)$.

We set

$$l_{s,y,T} := \min\{T_{B_y,\Omega},\, T^{\frac{1}{s}}\}$$

and we choose $\widehat{\delta} = \widehat{\delta}_{s,y,T,\Omega}$ such that

$$\widehat{\delta}_{s,y,T,\Omega} := \min\left\{\left(\frac{n\, l_{s,y,T}}{2c_2}\right)^{\frac{1}{2}}, d_y\right\}. \tag{2.27}$$

It follows that if $x, z \in B_{\widehat{\delta}}(y)$ with $x \neq z$, then $\varepsilon_{x,z} \leqslant l_{s,y,T}$ and $x, z \in B_y$.

To simplify the notation, we simply write $\varepsilon = \varepsilon_{x,z}$. In this way, by (2.1) and (2.26), if $x, z \in B_{\widehat{\delta}}(y)$ and $x \neq z$ we have that

$$\begin{aligned}\Phi_D^{x,z}(s,T) &= \int_0^T \int_0^{+\infty} p_D^{\Omega}(l,x,z)\mu_t^s(l)\,dl\,dt \\ &\geqslant \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} p_D^{\Omega}(l,x,z)\mu_t^s(l)\,dl\,dt \\ &\geqslant \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \frac{c_1}{l^{\frac{n}{2}}} \exp\left(-\frac{c_2|x-z|^2}{l}\right)\mu_t^s(l)\,dl\,dt \\ &\geqslant \frac{C}{\varepsilon^{\frac{n}{2}}} \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \mu_t^s(l)\,dl\,dt, \end{aligned} \tag{2.28}$$

where we set $C := c_1 2^{\frac{n}{2}} e^{-n}$.

Now we substitute μ_t^s in (2.28) with the expression in (2.2) and obtain that

$$\begin{aligned}\Phi_D^{x,z}(s,T) &\geqslant \frac{C}{\pi\varepsilon^{\frac{n}{2}}} \int_0^{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{+\infty} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s\sin(\pi s))\,du\,dl\,dt \\ &= \frac{C}{\pi\varepsilon^{\frac{n}{2}}} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{+\infty} \int_0^{\varepsilon^s} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s\sin(\pi s))\,dt\,du\,dl \\ &=: \mathcal{L}. \end{aligned} \tag{2.29}$$

Setting $F(t) := e^{-t\alpha}\sin(t\beta)$, with $\alpha := u^s\cos(\pi s)$ and $\beta := u^s\sin(\pi s)$, for each $T \in (0,+\infty)$ we integrate by parts and see that

$$\begin{aligned}\int_0^T F(t)\,dt &= -\frac{1}{\alpha}e^{-t\alpha}\sin(t\beta)\Big|_0^T + \frac{\beta}{\alpha}\int_0^T e^{-t\alpha}\cos(t\beta)\,dt \\ &= -\frac{1}{\alpha}e^{-T\alpha}\sin(T\beta) - \frac{\beta}{\alpha^2}e^{-t\alpha}\cos(t\beta)\Big|_0^T - \frac{\beta^2}{\alpha^2}\int_0^T e^{-t\alpha}\sin(t\beta)\,dt \\ &= -\frac{1}{\alpha}e^{-T\alpha}\sin(T\beta) - \frac{\beta}{\alpha^2}e^{-T\alpha}\cos(T\beta) + \frac{\beta}{\alpha^2} - \frac{\beta^2}{\alpha^2}\int_0^T F(t)\,dt. \end{aligned}$$

Therefore, by replacing α and β with their corresponding values, one obtains that

$$\begin{aligned}\int_0^T F(t)\,dt &= -\frac{\cos(\pi s)}{u^s}e^{-Tu^s\cos(\pi s)}\sin(Tu^s\sin(\pi s)) \\ &\quad - \frac{\sin(\pi s)}{u^s}e^{-Tu^s\cos(\pi s)}\cos(Tu^s\sin(\pi s)) + \frac{\sin(\pi s)}{u^s} \\ &= \frac{1}{u^s}(\sin(\pi s) - e^{-Tu^s\cos(\pi s)}\sin(Tu^s\sin(\pi s) + \pi s)). \end{aligned} \tag{2.30}$$

By (2.29), (2.30) and the change of variables $(U, L) = (u\varepsilon, \frac{l}{\varepsilon})$ one obtains that

$$\begin{aligned}\mathcal{L} &= \frac{C}{\pi \varepsilon^{\frac{n}{2}}} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{+\infty} \frac{e^{-lu}}{u^s} (\sin(\pi s) - e^{-\varepsilon^s u^s \cos(\pi s)} \sin(\pi s + \varepsilon^s u^s \sin(\pi s)))\, du\, dl \\ &= \frac{C}{\pi \varepsilon^{\frac{n}{2}-s}} \int_{\frac{1}{2}}^{1} \int_0^{+\infty} \frac{e^{-LU}}{U^s} (\sin(\pi s) - e^{-U^s \cos(\pi s)} \sin(\pi s + U^s \sin(\pi s)))\, dU\, dL \\ &=: \frac{C}{\pi \varepsilon^{\frac{n}{2}-s}} \mathcal{J}_s, \end{aligned} \tag{2.31}$$

where $\mathcal{J}_s$ does not depend on ε and is defined by

$$\mathcal{J}_s := \int_{\frac{1}{2}}^{1} \int_0^{+\infty} \frac{e^{-LU}}{U^s} (\sin(\pi s) - e^{-U^s \cos(\pi s)} \sin(\pi s + U^s \sin(\pi s)))\, dU\, dL.$$

Note also that by construction

$$\mathcal{J}_s = \frac{1}{\varepsilon^s} \int_{\frac{\varepsilon}{2}}^{\varepsilon} \int_0^{\varepsilon^s} \mu_t^s(l)\, dt\, dl,$$

which means that $\mathcal{J}_s \in (0, +\infty)$, since $\mu_t^s(l) \in (0, +\infty)$ for each s, t and l.

Accordingly, from (2.29) and (2.31),

$$\begin{aligned}\Phi_D^{x,z}(s, T) &\geqslant \frac{C\pi^{-1}}{\varepsilon^{n-2s}} \mathcal{J}_s \\ &= \frac{2^s e^{-n} c_1 n^{\frac{n}{2}-s}}{\pi c_2^{\frac{n}{2}-s} |x - z|^{n-2s}} \mathcal{J}_s \\ &\geqslant \frac{C_{y,\Omega}}{|x - z|^{n-2s}} \mathcal{J}_s \\ &= \frac{C_{s,y,\Omega}}{|x - z|^{n-2s}}, \end{aligned}$$

where we have defined

$$C_{y,\Omega} := \min_{s\in(0,1)} \frac{2^s e^{-n} c_1 n^{\frac{n}{2}-s}}{\pi c_2^{\frac{n}{2}-s}} \quad \text{and} \quad C_{s,y,\Omega} := C_{y,\Omega} \mathcal{J}_s. \tag{2.32}$$

This gives the desired result. ■

As a consequence of Lemma 2.7, we have that if x approaches y, then the functional $\Phi_*^{x,y}(s, T)$ diverges to infinity as far as $n > 2s$.

In the following result we make this statement precise. In particular, we show that divergence holds true as far as $n \geqslant 2s$.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected and $T \in (0,+\infty)$. If $n \geqslant 2$ or $n = 1$ and $s \in (0, \frac{1}{2}]$ we have that*

$$\lim_{(x,y)\to(z,z)} \Phi_*^{x,y}(s,T) = +\infty, \tag{2.33}$$

$$\Phi_*^{z,z}(s,T) = +\infty, \tag{2.34}$$

for each $z \in \Omega$.

Proof. We will prove only the Dirichlet case, since the Neumann one follows easily from the Dirichlet one and (2.17).

We first focus on the proof of (2.34). Using the identity (2.1), equations (2.15) and (2.16) together with the formula in (2.2) we deduce that if

$$\delta_{s,x,T} := \min\{T_{x,\Omega}, T^{\frac{1}{s}}\},$$

where $T_{x,\Omega}$ is given in (2.15), then for each $\delta \in (0, \delta_{s,x,T})$ it holds that

$$\begin{aligned} \Phi_D^{x,x}(s,T) &= \int_0^T \int_0^{+\infty} p_D^{\Omega}(l,x,x)\mu_t^s(l)\,dl\,dt \\ &= \int_0^{+\infty} p_D^{\Omega}(l,x,x) \int_0^T \mu_t^s(l)\,dt\,dl \\ &\geqslant \int_0^{\delta} \frac{c_1}{l^{\frac{n}{2}}} \int_0^{\delta^s} \mu_t^s(l)\,dt\,dl \\ &= \frac{1}{\pi} \int_0^{\delta} \frac{c_1}{l^{\frac{n}{2}}} \int_0^{\delta^s} \int_0^{+\infty} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s \sin(\pi s))\,du\,dt\,dl \\ &\geqslant \frac{c_1}{\pi\delta^{\frac{n}{2}}} \int_0^{\delta} \int_0^{+\infty} \int_0^{\delta^s} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s\sin(\pi s))\,dt\,du\,dl, \end{aligned} \tag{2.35}$$

where c_1 is introduced in (2.16).

Now, in light of (2.30) and (2.35), and using the change of variables $(L,U) = (\frac{l}{\delta}, u\delta)$, we find that

$$\begin{aligned} &\Phi_D^{x,x}(s,T) \\ &\geqslant \frac{c_1}{\pi\delta^{\frac{n}{2}}} \int_0^{\delta} \int_0^{+\infty} \int_0^{\delta^s} e^{-lu-tu^s\cos(\pi s)} \sin(tu^s\sin(\pi s))\,dt\,du\,dl \\ &= \frac{c_1}{\pi\delta^{\frac{n}{2}}} \int_0^{\delta} \int_0^{+\infty} e^{-lu} \frac{1}{u^s} (\sin(\pi s) - e^{-\delta^s u^s \cos(\pi s)} \sin(\delta^s u^s \sin(\pi s) + \pi s))\,du\,dl \\ &= \frac{c_1}{\pi\delta^{\frac{n}{2}-s}} \int_0^1 \int_0^{+\infty} e^{-LU} \frac{1}{U^s} (\sin(\pi s) - e^{-U^s\cos(\pi s)} \sin(U^s \sin(\pi s) + \pi s))\,dU\,dL \\ &=: \frac{c_1}{\pi\delta^{\frac{n}{2}-s}} \mathcal{G}_s. \end{aligned} \tag{2.36}$$

We also observe that $\mathcal{G}_s$ does not depend on δ and by construction

$$\mathcal{G}_s = \frac{1}{\delta^s}\int_0^{\delta}\int_0^{\delta^s} \mu_t^s(l)\,dt\,dl,$$

which means that $\mathcal{G}_s \in (0, +\infty)$, since $\mu_t^s(l) \in (0, +\infty)$ for each $s \in (0, 1)$, $t \in (0, +\infty)$ and $l \in (0, +\infty)$.

Therefore, recalling (2.36) we deduce that

$$\Phi_D^{x,x}(s, T) \geqslant \lim_{\delta \searrow 0} \frac{c_1}{\pi\delta^{\frac{n}{2}-s}}\mathcal{G}_s = +\infty,$$

if either $n \geqslant 2$ or $n = 1$ and $s \in (0, \frac{1}{2})$.

Hence, to complete the proof of (2.34), it is left to consider the case $n = 1$ and $s = \frac{1}{2}$. When $s = \frac{1}{2}$ equation (2.2) boils down to

$$\mu_t^{\frac{1}{2}}(l) = \frac{1}{\pi}\int_0^{+\infty} e^{-lu}\sin(tu^{\frac{1}{2}})\,du. \tag{2.37}$$

Therefore, using the latter identity, (2.15) and (2.16) we obtain that there exists $T_{x,\Omega} \in (0, +\infty)$ such that if $\delta \in (0, T_{x,\Omega})$, then

$$\begin{aligned}
\Phi_D^{x,x}\left(\frac{1}{2}, T\right) &= \int_0^T\int_0^{+\infty} p_D^{\Omega}(l, x, x)\mu_t^{\frac{1}{2}}(l)\,dl\,dt \\
&\geqslant \int_0^{\delta}\int_0^T p_D^{\Omega}(l, x, x)\mu_t^{\frac{1}{2}}(l)\,dt\,dl \\
&\geqslant c_1\int_0^{\delta}\int_0^T \frac{1}{l^{\frac{1}{2}}}\mu_t^{\frac{1}{2}}(l)\,dt\,dl \\
&= \frac{c_1}{\pi}\int_0^{\delta}\int_0^T\int_0^{+\infty} \frac{1}{l^{\frac{1}{2}}}e^{-lu}\sin(tu^{\frac{1}{2}})\,du\,dt\,dl \\
&= \frac{c_1}{\pi}\int_0^{\delta}\int_0^{+\infty}\int_0^T \frac{1}{l^{\frac{1}{2}}}e^{-lu}\sin(tu^{\frac{1}{2}})\,dt\,du\,dl \\
&= \frac{c_1}{\pi}\int_0^{\delta}\int_0^{+\infty} \frac{1}{l^{\frac{1}{2}}}e^{-lu}(1-\cos(Tu^{\frac{1}{2}}))\frac{1}{u^{\frac{1}{2}}}\,du\,dl,
\end{aligned}$$

where $c_1 \in (0, +\infty)$ has been introduced in (2.16).

Furthermore, by making the change of variable $lu = a$ in the l variable we deduce that

$$\begin{aligned}
\Phi_D^{x,x}\left(\frac{1}{2}, T\right) &\geqslant \frac{c_1}{\pi}\int_0^{+\infty}\int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}}(1-\cos(Tu^{\frac{1}{2}}))\frac{1}{u}\,da\,du \\
&\geqslant \frac{c_1}{\pi}\int_{\frac{1}{\delta}}^{+\infty}\int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}}(1-\cos(Tu^{\frac{1}{2}}))\frac{1}{u}\,da\,du \\
&=: \mathcal{I}.
\end{aligned} \tag{2.38}$$

We also observe that for each $u \geqslant \frac{1}{\delta}$ one has that

$$0 < c := \int_0^1 \frac{e^{-a}}{a^{\frac{1}{2}}}\, da \leqslant \int_0^{\delta u} \frac{e^{-a}}{a^{\frac{1}{2}}}\, da \leqslant \Gamma\left(\frac{1}{2}\right).$$

Moreover, defining

$$\tilde{k} := \min\left\{k \in \mathbb{N} \text{ s.t. } k \geqslant -\frac{1}{4} + \frac{T}{2\pi\delta^{\frac{1}{2}}}\right\},$$

we find that $(\frac{\pi}{2T} + \frac{2\pi\tilde{k}}{T})^2 > \frac{1}{\delta}$, and thus we deduce from (2.38) that

$$\begin{aligned}
\mathcal{I} &\geqslant \frac{cc_1}{\pi}\int_{\frac{1}{\delta}}^{+\infty} \frac{1-\cos(Tu^{\frac{1}{2}})}{u}\, du \\
&\geqslant \frac{cc_1}{\pi}\sum_{k=\tilde{k}}^{+\infty}\int_{(\frac{\pi}{2T}+\frac{2\pi k}{T})^2}^{(\frac{3\pi}{2T}+\frac{2\pi k}{T})^2} \frac{du}{u} \\
&= \frac{2cc_1}{\pi}\sum_{k=\tilde{k}}^{+\infty} \ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) \\
&=: \mathcal{I}\mathcal{I}.
\end{aligned}$$

Therefore, using Taylor's expansion we infer that there exists some $\tilde{K} \in \mathbb{N}$ with $\tilde{K} \geqslant \tilde{k}$ such that

$$\mathcal{I}\mathcal{I} \geqslant \frac{2cc_1}{\pi}\sum_{k=\tilde{K}}^{+\infty} \frac{1}{2k} + o\left(\frac{1}{k^2}\right) = +\infty. \tag{2.39}$$

This concludes the proof of (2.34).

Now we prove (2.33). If $n > 2s$, equation (2.33) is a direct consequence of inequality (2.25). Therefore, to conclude the proof of (2.33) it is left to show the case $n = 1$ and $s = \frac{1}{2}$. In order to achieve this, we observe that if $V_x \Subset \Omega$ is some neighborhood of x in Ω, then there exists some $k_0 \in \mathbb{N}$ such that for each $k \geqslant k_0$ it holds that $(x_k, y_k) \in V_x \times V_x$.

Thus, if we define $\varepsilon_k := |x_k - y_k|^2$, in view of (2.15) and (2.16), and recalling (2.37), we obtain that, if $k \geqslant k_0$,

$$\begin{aligned}
&\Phi_D^{x_k,y_k}\left(\frac{1}{2}, T\right) \\
&= \int_0^T \int_0^{+\infty} p_D^{\Omega}(l, x_k, y_k)\mu_t^{\frac{1}{2}}(l)\, dl\, dt \\
&= \int_0^{+\infty} \int_0^T p_D^{\Omega}(l, x_k, y_k)\mu_t^{\frac{1}{2}}(l)\, dt\, dl \\
&\geqslant \frac{1}{\pi}\int_0^{T_{V_x,\Omega}} \int_0^T \int_0^{+\infty} \frac{c_1}{l^{\frac{1}{2}}}\exp\left(-\frac{c_2\varepsilon_k}{l}\right)e^{-lu}\sin(tu^{\frac{1}{2}})du\, dt\, dl
\end{aligned}$$

$$= \frac{1}{\pi} \int_0^{T_{V_x,\Omega}} \int_0^{+\infty} \frac{c_1}{l^{\frac{1}{2}}} \exp\Big(-\frac{c_2 \varepsilon_k}{l}\Big) e^{-lu} \frac{(1-\cos(Tu^{\frac{1}{2}}))}{u^{\frac{1}{2}}}\, du\, dl$$
$$\geqslant \frac{c_1}{\pi} \int_{\frac{2}{T_{V_x,\Omega}}}^{+\infty} \int_{\frac{1}{u}}^{\frac{2}{u}} \frac{1}{l^{\frac{1}{2}}} \exp\Big(-\frac{c_2 \varepsilon_k}{l}\Big) e^{-lu} \frac{(1-\cos(Tu^{\frac{1}{2}}))}{u^{\frac{1}{2}}}\, dl\, du, \qquad (2.40)$$

where $T_{V_x,\Omega} \in (0,+\infty)$ and $c_1, c_2 \in (0,+\infty)$ are given respectively in (2.15) and (2.16).

Now we choose $\tilde{j}, j(\varepsilon_k) \in \mathbb{N}$ such that

$$\tilde{j} := \min\Big\{ j \in \mathbb{N} \text{ s.t. } j \geqslant \frac{T}{2\pi}\Big(\frac{2}{T_{V_x,\Omega}}\Big)^{\frac{1}{2}} - \frac{1}{4}\Big\},$$
$$j(\varepsilon_k) := \max\Big\{ j \in \mathbb{N} \text{ s.t. } j \leqslant \frac{T}{2\pi \varepsilon_k^{\frac{1}{2}}} - \frac{3}{4}\Big\}.$$

Note that if ε_k is chosen small enough, then $\tilde{j} < j(\varepsilon_k)$.

With this choices one has that

$$\Big(\frac{\pi}{2T} + \frac{2\pi\tilde{j}}{T}\Big)^2 \geqslant \frac{2}{T_{V_x,\Omega}} \quad \text{and} \quad \Big(\frac{3\pi}{2T} + \frac{2\pi j(\varepsilon_k)}{T}\Big)^2 \leqslant \frac{1}{\varepsilon_k}. \qquad (2.41)$$

Therefore, with this latter notation we obtain from (2.40) that

$$\Phi_D^{x_k,y_k}\Big(\frac{1}{2},T\Big) \geqslant \frac{c_1}{\pi} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \int_{(\frac{\pi}{2T}+\frac{2\pi j}{T})^2}^{(\frac{3\pi}{2T}+\frac{2\pi j}{T})^2} \int_{\frac{1}{u}}^{\frac{2}{u}} \frac{1}{l^{\frac{1}{2}}} \exp\Big(-\frac{c_2\varepsilon_k}{l}\Big) e^{-lu} \frac{1}{u^{\frac{1}{2}}}\, dl\, du$$
$$\geqslant \frac{c_1}{\pi} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \int_{(\frac{\pi}{2T}+\frac{2\pi j}{T})^2}^{(\frac{3\pi}{2T}+\frac{2\pi j}{T})^2} \int_{\frac{1}{u}}^{\frac{2}{u}} \exp(-c_2 u \varepsilon_k) e^{-2}\, dl\, du$$
$$= \frac{c_1 e^{-2}}{\pi} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \int_{(\frac{\pi}{2T}+\frac{2\pi j}{T})^2}^{(\frac{3\pi}{2T}+\frac{2\pi j}{T})^2} \frac{1}{u} \exp(-c_2 u \varepsilon_k)\, du.$$

Now, we deduce from (2.41) that since $u \leqslant (\frac{3\pi}{2T} + \frac{2\pi j(\varepsilon_k)}{T})^2$, then $u\varepsilon_k \leqslant 1$, and thus from the latter computations we obtain that

$$\Phi_D^{x_k,y_k}\Big(\frac{1}{2},T\Big) \geqslant \frac{c_1 e^{-c_2}}{\pi e^2} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \int_{(\frac{\pi}{2T}+\frac{2\pi j}{T})^2}^{(\frac{3\pi}{2T}+\frac{2\pi j}{T})^2} \frac{1}{u}\, du$$
$$= \frac{2c_1 e^{-c_2}}{\pi e^2} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \ln\Big(\frac{\frac{3\pi}{2k} + 2\pi}{\frac{\pi}{2k} + 2\pi}\Big). \qquad (2.42)$$

As we observed in (2.39), one has that

$$\sum_{k=1}^{+\infty} \ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) = +\infty.$$

With reference to that, from (2.42) and the latter observation we obtain that

$$\begin{aligned}\lim_{k\to+\infty} \Phi_D^{x_k,y_k}\left(\frac{1}{2},T\right) &\geqslant \lim_{k\to+\infty} \frac{2c_1 e^{-c_2}}{\pi e^2} \sum_{j=\tilde{j}}^{j(\varepsilon_k)} \ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) \\ &= \frac{2c_1 e^{-c_2}}{\pi e^2} \sum_{j=\tilde{j}}^{+\infty} \ln\left(\frac{\frac{3\pi}{2k}+2\pi}{\frac{\pi}{2k}+2\pi}\right) \\ &= +\infty.\end{aligned}$$

This completes the proof of (2.33). ■

In the following result we give some upper bounds for the functional $\Phi_*^{x,y}(s,T)$. These estimates, together with the lower bound in (2.25), will turn out to be pivotal in order to determine the most rewarding search strategy in a regime where the initial position of the forager is close to the one of the prey, and thus prove Theorems 1.7 and 1.8.

In the Dirichlet framework, the behavior of the functional $\Phi_D^{x,y}(s,T)$ for x approaching y could be deduced from the already known estimates on the Green function $G_D^{\Omega}(x,y)$ of the Dirichlet spectral fractional Laplacian, see [34, Theorem 5.4].

Indeed, the Green function is given by

$$G_D^{\Omega}(x,y) := \int_0^{+\infty} r_D^s(t,x,y)\,dt,$$

for $x \neq y$, and therefore

$$\Phi_D^{x,y}(s,T) \leqslant G_D^{\Omega}(x,y),$$

for each $(x,y,T) \in \Omega \times \Omega \times (0,+\infty)$ with $x \neq y$ and $s \in (0,1)$.

Nevertheless, for our optimization purposes we need upper bounds where the dependence of the constants on the fractional exponent $s \in (0,1)$ is known. In this sense, the inequalities provided in the following result are more suitable in this context than the ones available in the literature for G_D^{Ω}.

Before stating and proving the theorem, we fix the following notation. For each $n \in \mathbb{N}$ and $s \in (0,1)$ we define the set

$$A_{n,s} := \left(0, 1+\frac{n}{2}-s\right) \cap \left[\frac{n}{2}-s, 1+\frac{n}{2}-s\right). \tag{2.43}$$

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Moreover, let $K \Subset \Omega$ be star-shaped with respect to some $x_0 \in K$. Then, for each $s \in (0,1)$ and $T \in (0,+\infty)$, there exists some $C_{*,K,T,\Omega} \in (0,+\infty)$ such that if $n \geqslant 3$, then*

$$\Phi_*^{x,y}(s,T) \leqslant \frac{C_{*,K,T,\Omega}}{|x-y|^{n-2s}} \quad \text{for all } (x,y) \in \mathscr{C} \cap (K \times K), \tag{2.44}$$

where $\mathscr{C}$ is given in (2.23).

Furthermore, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$ there exists some $C_{*,\mu,K,T,\Omega} \in (0,+\infty)$ such that

$$\Phi_*^{x,y}(s,T) \leqslant \frac{C_{*,\mu,K,T,\Omega}}{|x-y|^{2\mu}} \quad \text{for all } (x,y) \in \mathscr{C} \cap (K \times K), \tag{2.45}$$

where $A_{n,s}$ is defined (2.43).

Proof. We will first show the result for the Dirichlet case. To this aim, we recall the following identity

$$\int_0^{+\infty} r_D^s(t,x,y)\,dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\,dt \quad \text{for all } (x,y) \in \mathscr{C}, \tag{2.46}$$

see for instance [34, equation (2.4)]. For the convenience of the reader we give a proof of it in the appendix, see Proposition A.1.

We first prove (2.44). If $((x,y),T) \in \mathscr{C} \times (0,+\infty)$, thanks to the identity in (2.46) we have that

$$\begin{aligned}\Phi_D^{x,y}(s,T) &= \int_0^T r_D^s(t,x,y)\,dt \leqslant \int_0^{+\infty} r_D^s(t,x,y)\,dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1}\,dt.\end{aligned}$$

Using inequality (2.14) and the change of variable $a = \frac{|x-y|^2}{4t}$ we obtain that

$$\begin{aligned}\Phi_D^{x,y}(s,T) &\leqslant \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) dt \\ &= \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{1}{|x-y|^{n-2s}} \int_0^{+\infty} a^{\frac{n}{2}-1-s} e^{-a}\,da \\ &= \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \frac{\Gamma(\frac{n}{2}-s)}{|x-y|^{n-2s}}.\end{aligned} \tag{2.47}$$

Thus, by defining the constant

$$C_D := \sup_{s\in(0,1)} \frac{4^{-s}}{\pi^{\frac{n}{2}}\Gamma(s)} \Gamma\left(\frac{n}{2}-s\right),$$

we conclude the proof of (2.44) for the Dirichlet case.

Now, we prove (2.45). To this end, we observe that there exists some constant $c_3 \in (0, +\infty)$, depending on Ω, such that for all $\gamma \in [0, 1)$ it holds that

$$p_D^{\Omega}(t,x,y) \leqslant \frac{c_3}{t^{\frac{n}{2}+\gamma}} \exp\left(-\frac{|x-y|^2}{6t}\right) \quad \text{for all } (t,x,y) \in (0,+\infty) \times \Omega \times \Omega, \quad (2.48)$$

see for instance [9, Theorem 4.6.9].

Accordingly, using the identity given in equation (2.46) and the inequality in (2.48), we deduce that

$$\begin{aligned}
\Phi_D^{x,y}(s,T) &= \int_0^T r_D^s(t,x,y)\,dt \\
&\leqslant \int_0^{+\infty} r_D^s(t,x,y)\,dt \\
&= \frac{1}{\Gamma(s)} \int_0^{+\infty} p_D^{\Omega}(t,x,y) t^{s-1}\,dt \\
&\leqslant \frac{c_3}{\Gamma(s)} \int_0^{+\infty} \frac{1}{t^{\frac{n}{2}+\gamma-s+1}} \exp\left(-\frac{|x-y|^2}{6t}\right) dt \\
&= \frac{c_3}{\Gamma(s)} \int_0^{+\infty} \frac{(6\theta)^{\frac{n}{2}+\gamma-s-1}}{|x-y|^{n-2(s-\gamma)}} e^{-\theta}\,d\theta \\
&= \frac{C_{s,\gamma,n,\Omega}}{\Gamma(s)} \frac{1}{|x-y|^{n-2(s-\gamma)}},
\end{aligned} \qquad (2.49)$$

where we applied the change of variable $\theta = \frac{|x-y|^2}{4t}$ and we defined

$$C_{s,\gamma,n,\Omega} := c_3 6^{\frac{n}{2}+\gamma-s-1} \Gamma(\frac{n}{2}+\gamma-s),$$

for all $\gamma \in (s-\frac{n}{2}, 1) \cap [0, 1)$.

Now, we observe that if we define $\mu := \frac{n}{2}+\gamma-s$, then $\mu \in (0, 1+\frac{n}{2}-s) \cap [\frac{n}{2}-s, 1+\frac{n}{2}-s)$, and inequality (2.49) becomes

$$\Phi_D^{x,y}(s,T) \leqslant \frac{c_3}{\Gamma(s)} 6^{\mu-1} \frac{\Gamma(\mu)}{|x-y|^{2\mu}} \leqslant \frac{C_{D,\mu}}{|x-y|^{2\mu}}, \qquad (2.50)$$

where we defined

$$C_{D,\mu} := \sup_{s\in(0,1)} \frac{c_3}{\Gamma(s)} 6^{\mu-1} \Gamma(\mu).$$

This concludes the proof of (2.45) for the Dirichlet case.

Employing the result in Corollary 2.6 we prove now (2.44) and (2.45) for the Neumann case. Let $K \Subset \Omega$ and, up to a translation, let us assume that it is star-shaped with respect to

$$x_0 = 0.$$

Then, if $T \in (0,+\infty)$, $n \geqslant 3$ and $s \in (0,1)$, using equations (2.47) and (2.24) with $K' = K$ we obtain the existence of some $c_{K,\Omega}, C_{K,\Omega} \in (0,+\infty)$ and $\varepsilon_0 \in (0,1)$, depending on K and Ω, such that

$$\Phi_N^{x,y}(s,T) \leqslant C_{K,\Omega}\Phi_D^{x_\varepsilon,y_\varepsilon}(s,T) + c_{K,\Omega}T \leqslant C_{K,\Omega}\frac{C_D\varepsilon^{2s-n}}{|x-y|^{n-2s}} + c_{K,\Omega}T,$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in \mathcal{C} \cap (K \times K)$.

Consequently, if in the last equation we choose $\varepsilon_1 \in (0,\varepsilon_0)$ such that

$$\varepsilon_1 \leqslant \inf_{s\in(0,1)}\left(C_{K,\Omega}C_D\frac{d_K^{2s-n}}{c_{K,\Omega}T}\right)^{\frac{1}{n-2s}},$$

which depends on K, Ω and T, we obtain that for all $(x,y) \in \mathcal{C} \cap (K \times K)$ and $s \in (0,1)$ it holds that

$$\Phi_N^{x,y}(s,T) \leqslant \frac{C_{K,T,\Omega}}{|x-y|^{n-2s}},$$

with

$$C_{K,T,\Omega} := 2\sup_{s\in(0,1)} C_n\varepsilon_1^{2s-n}C_{K,\Omega}.$$

Analogously, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$, then one deduces from (2.50) and (2.24) that

$$\Phi_N^{x,y}(s,T) \leqslant C_{K,\Omega}\Phi_D^{x_\varepsilon,y_\varepsilon}(s,T) + c_{K,\Omega}T \leqslant C_{K,\Omega}\frac{C_{D,\mu}\varepsilon^{-2\mu}}{|x-y|^{2\mu}} + c_{K,\Omega}T,$$

for all $\varepsilon \in (0,\varepsilon_0)$ and $(x,y) \in \mathcal{C} \cap (K \times K)$.

As a result, if we choose some $\varepsilon_2 \in (0,\varepsilon_0)$ satisfying

$$\varepsilon_2^{2\mu} \leqslant C_{K,\Omega}C_{D,\mu}\frac{d_K^{-2\mu}}{c_{K,\Omega}T},$$

which depends on μ, K, Ω and T, we obtain that for all $(x,y) \in \mathcal{K}$ and $s \in (0,1)$ it holds that

$$\Phi_N^{x,y}(s,T) \leqslant \frac{C_{\mu,K,T,\Omega}}{|x-y|^{2\mu}},$$

where we set

$$C_{\mu,K,T,\Omega} := 2C_{K,\Omega}C_{\mu,\Omega}\varepsilon_2^{-2\mu}. \qquad \blacksquare$$

Remark 2.10. We note that for the Dirichlet case we obtained that the constants in equation (2.44) and (2.45) can be chosen independently of K and T. In particular, we have proved that if $n \geqslant 3$, then

$$\Phi_D^{x,y}(s,T) \leqslant \frac{4^{-s}}{\pi^{\frac{n}{2}}}\frac{\Gamma(\frac{n}{2}-s)}{|x-y|^{n-2s}} \tag{2.51}$$

for all $(x, y) \in \mathscr{C}$. If $n \leqslant 2$ and $\mu \in A_{n,s}$, where $A_{n,s}$ is given in (2.43), then

$$\Phi_D^{x,y}(s,T) \leqslant \frac{6^{\mu-1}c_3}{\Gamma(s)} \frac{\Gamma(\mu)}{|x-y|^{2\mu}}, \tag{2.52}$$

for all $(x, y) \in \mathscr{C}$.

We now turn our attention to the functional $\widetilde{\Phi}_*^{\Omega_1,\Omega_2}$ defined in (1.7). For this, it is convenient, for every bounded and measurable sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and each $s \in (0,1)$, to define

$$F^{\Omega_1,\Omega_2}(s) := \int_{\Omega_1\times\Omega_2} \frac{1}{|x-y|^{n-2s}}\,dx\,dy. \tag{2.53}$$

As a direct consequence of Lemma 2.7 and Theorem 2.9 we obtain the following upper and lower bounds for $\widetilde{\Phi}_*^{\Omega_1,\Omega_2}$. These bounds will play a crucial role in proving Theorems 1.15 and 1.16.

Corollary 2.11. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected, $K \Subset \Omega$ be star-shaped with respect to some $x_0 \in K$ and $\varepsilon_0 \in (0,1)$ be given as in Corollary 2.6. Then, for each $s \in (0,1)$ and $T \in (0,+\infty)$, if $n \geqslant 3$, we have that*

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{C_{*,K,T,\Omega}}{|\Omega_1||\Omega_2|} F^{\Omega_1,\Omega_2}(s) \quad \textit{for all } \Omega_1,\Omega_2 \subset K,$$

where F^{Ω_1,Ω_2} is given in (2.53).

Furthermore, if $n \leqslant 2$, $s \in (0,1)$ and $\mu \in A_{n,s}$, where $A_{n,s}$ is given in (2.43), one has that

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{C_{*,\mu,K,T,\Omega}}{|\Omega_1||\Omega_2|} F^{\Omega_1,\Omega_2}\Big(\frac{n-2\mu}{2}\Big) \quad \textit{for all } \Omega_1,\Omega_2 \subset K. \tag{2.54}$$

Moreover, for all $s \in (0,1]$ we have that

$$\widetilde{\Phi}_*^{\Omega_1,\Omega_2}(s,T) \geqslant \frac{C_{s,y,\Omega}}{|\Omega_1||\Omega_2|} F^{\Omega_1,\Omega_2}(s), \tag{2.55}$$

with $C_{s,y,\Omega}$ defined in (2.32).

The following result is devoted to the proof of the continuity of the functionals Φ_*, l_* and $\mathcal{A}_*$ with respect to the space, time and fractional variables. Also, we show that if $n < 2s$, then the limit in (2.33) is finite, and similarly $\Phi_*^{z,z}(s,T) < +\infty$ for each $z \in \Omega$.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open, bounded, smooth and connected. Then, $\Phi_*^{x,y}(s,T) \in (0,+\infty)$ for all $(s,(x,y),T) \in (0,1] \times \mathscr{C} \times (0,+\infty)$ and $\Phi_*^{x,y}(s,T) \in C((0,1] \times \mathscr{C} \times (0,+\infty))$, where $\mathscr{C}$ is given in (2.23).*

Also, if $n = 1$, then $\Phi_*^{x,y}(s,T) \in C((\frac{1}{2},1] \times \Omega \times \Omega \times (0,+\infty))$.

Moreover, for each $T \in (0, +\infty)$ *there exists some* $M \in (0, +\infty)$ *such that* $l_N^y(s, T)$, $\mathcal{A}_N^y(s, T) \in (0, M)$ *for all* $(s, y) \in (0, 1] \times \Omega$ *and* $l_*^y(s, T)$, $\mathcal{A}_*^y(s, T) \in C((0, 1] \times \Omega \times (0, +\infty))$.

Furthermore, there exists some $M \in (0, +\infty)$ *such that* $l_D^y(s, T)$, $\mathcal{A}_D^y(s, T) \in (0, M)$ *for all*

$$(s, y, T) \in (0, 1] \times \Omega \times (0, +\infty).$$

Proof. The positivity of the functionals follows from (1.3), (1.4) and the fact that $r_*^s(t, x, y)$ is strictly positive for all $(t, x, y) \in (0, +\infty) \times \Omega \times \Omega$, see for instance [10, Corollary 1] and [11, Corollary 1].

Now we establish the continuity statement. Thanks to equation (2.4) we have that

$$r_*^s(t, x, y) = \sum_{k=0}^{+\infty} \zeta_{*,k}(x)\zeta_{*,k}(y)\exp(-t\beta_{*,k}^s),$$

and each term of the series is continuous in $(s, t, x, y) \in (0, 1] \times (0, +\infty) \times \Omega \times \Omega$.

Furthermore, thanks to [11, Proposition 6 and Lemma 6], we have the existence of some $M \in \mathbb{N}$ such that for each $\varepsilon \in (0, 1]$ and $\delta \in (0, +\infty)$ it holds that

$$\begin{aligned}
&\sum_{k=0}^{+\infty} \|\zeta_{*,k}(x)\zeta_{*,k}(y)\exp(-t\beta_{*,k}^s)\|_{C^0(\Omega\times\Omega\times(\varepsilon,1)\times(\delta,+\infty))} \\
&\quad\leqslant \sum_{k=0}^{M} \|\zeta_{*,k}(x)\zeta_{*,k}(y)\exp(-t\beta_{*,k}^s)\|_{C^0(\Omega\times\Omega\times(\varepsilon,1)\times(\delta,+\infty))} \\
&\quad\quad + C_{*,m_0,\Omega,0}^2 \sum_{k=M}^{+\infty} \beta_{*,k}^{2\alpha(m_0)} \exp(-\delta\beta_{*,k}^\varepsilon) \\
&\quad< +\infty,
\end{aligned}$$

where $C_{*,m_0,\Omega,0}$ and $\alpha(m_0)$ are positive constants given in Proposition A.1.

Consequently, $r_*^s(t, x, y)$ is continuous for all

$$(s, t, x, y) \in (0, 1] \times (0, +\infty) \times \Omega \times \Omega.$$

Suppose now that $(\xi, y) \in \mathcal{C}$, and that $\{(s_k, T_k, y_k)\}_k \subset (0, 1] \times (0, +\infty) \times \Omega$ satisfies

$$(s_k, T_k, y_k) \to (s, T, y) \in (0, 1] \times (0, +\infty) \times \Omega.$$

Then, since $r_*^s(t, \xi, y)$ is continuous for all $(s, y) \in (0, 1] \times \Omega$, we have that

$$r_*^{s_k}(t, \xi, y_k)\chi_{(0,T_k)}(t) \to r_*^s(t, \xi, y)\chi_{(0,T)}(t),$$

for almost every $t \in (0, +\infty)$.

Moreover, if $\widetilde{T} := \sup_{k\in\mathbb{N}} T_k$, then using equations (2.1), (2.14), and (2.19) we obtain that

$$\begin{aligned}\chi_{T_k}(t)r_*^{s_k}(t,\xi,y_k) &\leqslant \chi_{\widetilde{T}}(t)\int_0^{+\infty} p_*^{\Omega}(l,\xi,y_k)\mu_t^{s_k}(l)\,dl \\ &\leqslant \chi_{\widetilde{T}}(t)\int_0^{+\infty} C_{*,l}\exp\Big(-\frac{|\xi-y_k|^2}{4l}\Big)\mu_t^{s_k}(l)\,dl \\ &\leqslant \chi_{\widetilde{T}}(t)M_*\int_0^{+\infty}\mu_t^{s_k}(l)\,dl \\ &= \chi_{\widetilde{T}}(t)M_*, \end{aligned} \tag{2.56}$$

where we defined

$$M_* := \sup_{l\in(0,+\infty)}\sup_{k\in\mathbb{N}} C_{*,l}\exp\Big(-\frac{|\xi-y_k|^2}{4l}\Big).$$

The last function in (2.56) is in $L^1((0,+\infty))$, and thus by the dominated convergence theorem we obtain that $\Phi_*^{\xi,y}(s,T)$ is continuous for all $(s,y,T)\in(0,1]\times(\Omega\setminus\{\xi\})\times(0,+\infty)$, and since it is symmetric with respect to the space variables, we deduce the continuity for all $(s,(\xi,y),T)\in(0,1]\times\mathscr{C}\times(0,+\infty)$.

If $n=1$ the eigenfunctions $\zeta_{*,k}$'s are uniformly bounded in $L^\infty(\Omega)$ and the eigenvalues $\beta_{*,k}$'s are proportional to k^2, for each $k\geqslant 1$. More precisely, there exist two positive constants $C_*, c_*>0$ such that

$$c_*k^2 \leqslant \beta_{*,k} \leqslant C_*k^2,$$

for each $k\geqslant 1$, see for instance [29].

Therefore, we have that

$$\begin{aligned} r_*^s(t,x,y) &= \sum_{k=0}^{+\infty}\zeta_{*,k}(x)\zeta_{*,k}(y)\exp(-t\beta_{*,k}^s) \\ &\leqslant \sum_{k=0}^{+\infty}\|\zeta_{*,k}\|^2_{L^\infty(\Omega)}\exp(-t\beta_{*,k}^s) \\ &\leqslant M_*^2\sum_{k=0}^{+\infty}\exp(-tc_*^sk^{2s}) \\ &=: f_{*,s}(t), \end{aligned} \tag{2.57}$$

where $\|\zeta_{*,k}\|_{L^\infty(\Omega)}\leqslant M_*$ for some $M_*\in(0,+\infty)$ and we adopted the convention

$$\zeta_{D,0} = 0 = \beta_{D,0}.$$

Thus, if $s, s_k \in (\frac{1}{2}, 1]$, then we can choose also $(\xi, y) \in \Omega \times \Omega$, and thanks to (2.57) we have that

$$\chi_{T_k}(t) r_*^{s_k}(t, \xi, y_k) \leqslant \chi_{\tilde{T}}(t) \inf_{k \in \mathbb{N}} f_{*,s_k}(t),$$

and the right-hand side is $L^1(0, +\infty)$.

Repeating the above reasoning, if $n = 1$, we obtain that

$$\Phi_*^{x,y}(s, T) \in C((\frac{1}{2}, 1] \times \Omega \times \Omega \times (0, +\infty)).$$

Now, we observe that

$$\begin{aligned} l_*^y(s, T) &= \int_0^T \int_\Omega |\xi - y| r_*^s(t, \xi, y)\, d\xi\, dt \\ &= \int_\Omega |\xi - y| \int_0^T r_*^s(t, \xi, y)\, dt\, d\xi \\ &= \int_\Omega |\xi - y| \Phi_*^{\xi,y}(s, T). \end{aligned} \tag{2.58}$$

Using this identity, the continuity of Φ_* and the estimates in Theorem 2.9, we conclude using the dominated convergence theorem. The proof of the continuity of $\mathcal{A}_*^y(s, T)$ is analogous

Also, if $n \geqslant 3$, from (2.58) and (2.51) we have that

$$l_D^y(s, T) \leqslant \frac{C_n}{\Gamma(s)} \int_\Omega |\xi - y|^{n-1}\, d\xi,$$

for some suitable C_n, which proves that l_D^y is uniformly bounded in $(0, 1] \times \Omega \times (0, +\infty)$ if $n \geqslant 3$.

The proof of the uniform boundedness in the case $n \leqslant 2$ is done similarly replacing (2.51) in the above equation with (2.52).

Finally, using [11, (28)] we obtain that

$$\begin{aligned} l_N^y(s, T) &= \int_0^T \int_\Omega |\xi - y| r_N^s(t, \xi, y)\, d\xi\, dt \\ &\leqslant d_\Omega \int_0^T \int_\Omega r_N^s(t, \xi, y)\, d\xi\, dt \\ &= d_\Omega T. \end{aligned}$$

The proof of the boundedness of $\mathcal{A}_*$ is analogous. ∎

In the following two lemmas we establish the limits as $s \searrow 0$ of the Dirichlet efficiency functionals given in (1.6) and (1.9).

We will show that Φ_D, l_D, $\mathcal{A}_D$, $\widetilde{\Phi}_D$, $\tilde{l}_D$, and $\widetilde{\mathcal{A}}_D$ all go to 0 linearly in s. Moreover, we will also determine the value of the limit for $\mathcal{E}_{2,D}$, $\mathcal{E}_{3,D}$, $\widetilde{\mathcal{E}}_{2,D}$, and $\widetilde{\mathcal{E}}_{3,D}$ so that we will be able to extend them by continuity in $[0, 1]$.

This asymptotic analysis is a fundamental tool in order to establish Theorems 1.6 and 1.14 and the claims in (1.17) and (1.24).

Lemma 2.13. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected, and let $\mathcal{C}$ be as in* (2.23). *Then, for all $((x, y), T) \in \mathcal{C} \times (0, +\infty)$, it holds that*

$$\lim_{s \searrow 0} \mathcal{E}^{x,y}_{1,D}(s, T) = 0, \tag{2.59}$$

$$\lim_{s \searrow 0} \mathcal{E}^{x,y}_{2,D}(s, T) = \frac{F_D(x, y)}{\int_\Omega |\xi - y| F_D(\xi, y)\, d\xi}, \tag{2.60}$$

$$\lim_{s \searrow 0} \mathcal{E}^{x,y}_{3,D}(s, T) = \frac{F_D(x, y)}{\int_\Omega |\xi - y|^2 F_D(\xi, y)\, d\xi}, \tag{2.61}$$

where we have defined

$$F_D(x, y) := \int_0^{+\infty} \frac{p^\Omega_D(l, x, y)}{l}\, dl \quad \text{for all } (x, y) \in \mathcal{C}. \tag{2.62}$$

Proof. Equation (2.59) is a direct consequence of (2.51) and (2.52), since $\Gamma(s) \to +\infty$ for $s \searrow 0$.

Now we focus on the proof of (2.60). For this, we claim that

$$\lim_{s \searrow 0} \frac{\Phi^{x,y}_D(s, T)}{s} = (1 - e^{-T}(T + 1)) F_D(x, y) \quad \text{for all } ((x, y), T) \in \mathcal{C} \times (0, +\infty). \tag{2.63}$$

Thanks to (2.9) and (2.14), if $s \in (0, \frac{1}{2})$ we have that

$$\frac{1}{s}\left| p^\Omega_D(l, x, y) \mu^s_t(l) \right| \leqslant \frac{t}{(4\pi l)^{\frac{n}{2}}} \exp\left(-\frac{|x - y|^2}{4l}\right) \frac{\Gamma(1 + s)}{l^{1+s}}. \tag{2.64}$$

This bound together with (2.1), (D.1) and the dominated convergence theorem yields to

$$\lim_{s \searrow 0} \frac{r^s_D(t, x, y)}{s} = t e^{-t} F_D(x, y), \tag{2.65}$$

for all $(t, (x, y)) \in (0, +\infty) \times \mathcal{C}$. Therefore, if $s \in (0, \frac{1}{2})$, from (2.1) and (2.64) we obtain that

$$\begin{aligned} \frac{r^s_D(t, x, y)}{s} &\leqslant \int_0^{+\infty} \frac{t\Gamma(1 + s)}{(4\pi)^{\frac{n}{2}} l^{\frac{n}{2}+s+1}} \exp\left(-\frac{|x - y|^2}{4l}\right) dl \\ &\leqslant \frac{C_0 t}{|x - y|^{n+2s}} \\ &=: f_{x,y}(t), \end{aligned} \tag{2.66}$$

where we defined

$$C_0 := \sup_{s \in (0, \frac{1}{2})} \frac{4^s \Gamma(1+s)}{\pi^{\frac{n}{2}}} \Gamma\Big(\frac{n}{2} + s\Big).$$

Now, clearly we have that

$$f_{x,y} \in L^1((0,T)), \tag{2.67}$$

and thus from (2.65), (2.66), and (2.67) we can apply the dominated convergence theorem to obtain that

$$\begin{aligned} \lim_{s \searrow 0} \frac{\Phi_D^{x,y}(s,T)}{s} &= \int_0^T te^{-t} F_D(x,y)\, dt \\ &= (1 - e^{-T}(T+1)) F_D(x,y). \end{aligned}$$

This concludes the proof of (2.63).

Note that using (2.51), (2.52), and (2.63), we obtain that

$$\begin{aligned} \lim_{s \searrow 0} \frac{l_D^y(s,T)}{s} &= \lim_{s \searrow 0} \frac{1}{s} \int_0^T \int_\Omega |\xi - y| r_D^s(t,\xi,y)\, d\xi\, dt \\ &= \lim_{s \searrow 0} \int_\Omega |\xi - y| \frac{\Phi_D^{\xi,y}(s,T)}{s}\, d\xi \\ &= (1 - e^{-T}(T+1)) \int_\Omega |\xi - y| F_D(\xi,y)\, d\xi, \end{aligned} \tag{2.68}$$

by means of the dominated convergence theorem. Finally, from (2.63) and (2.68) we deduce that

$$\begin{aligned} \lim_{s \searrow 0} \frac{\Phi_D^{x,y}(s,T)}{l_D^y(s,T)} &= \lim_{s \searrow 0} \frac{\int_0^T r_D^s(t,x,y)\, dt}{\int_0^T \int_\Omega |\xi - y| r_D^s(t,\xi,y)\, d\xi\, dt} \frac{s}{s} \\ &= \frac{F_D(x,y)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi}, \end{aligned}$$

which concludes the proof of (2.60).

It is left to show (2.61). To do so, we observe that applying the same reasoning we used to show (2.68), one can easily prove that

$$\lim_{s \searrow 0} \frac{\mathcal{A}_D^y(s,T)}{s} = (1 - e^{-T}(T+1)) \int_\Omega |\xi - y|^2 F_D(\xi,y)\, d\xi,$$

for all $(y,T) \in \Omega \times (0,+\infty)$. From this identity and (2.63) it is immediate to deduce (2.61). ■

The following result can be considered as the set functional version of Lemma 2.13.

Lemma 2.14. *Let Ω be bounded, smooth and connected and $\Omega_1, \Omega_2 \subset \Omega$ be smooth and disjoint. Then, for all $T \in (0, +\infty)$, it holds that*

$$\lim_{s \searrow 0} \widetilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}(s,T) = 0, \tag{2.69}$$

$$\lim_{s \searrow 0} \widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(s,T) = \frac{|\Omega_2| \widetilde{F}_D(\Omega_1,\Omega_2)}{\int_{\Omega_2 \times \Omega} |\xi - y| F_D(\xi, y)\, d\xi\, dy}, \tag{2.70}$$

$$\lim_{s \searrow 0} \widetilde{\mathcal{E}}_{3,D}^{\Omega_1,\Omega_2}(s,T) = \frac{|\Omega_2| \widetilde{F}_D(\Omega_1,\Omega_2)}{\int_{\Omega_2 \times \Omega} |\xi - y|^2 F_D(\xi, y)\, d\xi\, dy}, \tag{2.71}$$

where

$$\widetilde{F}_D(\Omega_1,\Omega_2) := \frac{1}{|\Omega_1||\Omega_2|} \int_{\Omega_1 \times \Omega_2} F_D(x,y)\, dx\, dy, \tag{2.72}$$

and F_D is given in equation (2.62).

Proof. We begin by proving (2.69). We have that

$$\frac{\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{s} = \frac{1}{|\Omega_1||\Omega_2|} \int_{\Omega_1 \times \Omega_2} \frac{\Phi_D^{x,y}(s,T)}{s}\, dx\, dy.$$

Thanks to equations (2.51) and (2.52), if $s \in (0, \frac{1}{2})$, there exists some constant $\widehat{C}_n$ depending on n such that

$$\frac{\Phi_D^{x,y}(s,T)}{s} \leqslant \frac{\widehat{C}_n}{s\Gamma(s)} \frac{1}{|x-y|^{n-2s}} \leqslant \frac{C_3}{|x-y|^n} =: g(x,y), \tag{2.73}$$

where C_3 depends only on Ω. If Ω_1, Ω_2 are smooth and disjoint, then $g \in L^1(\Omega_1 \times \Omega_2)$. Therefore, under these assumptions we can apply the dominated convergence theorem, which together with (2.63) yields to

$$\lim_{s \searrow 0} \frac{\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(s,T)}{s} = (1 - e^{-T}(T+1)) \widetilde{F}_D(\Omega_1,\Omega_2). \tag{2.74}$$

Also, thanks to Lemma D.4 and the hypotheses on Ω_1, Ω_2 we have that $\widetilde{F}_D(\Omega_1,\Omega_2)$ is finite. From this observation and (2.74) one readily deduces (2.69).

Now, we show (2.70). To do so, we claim that

$$\lim_{s \searrow 0} \frac{\widetilde{l}_D^{\Omega_2}(s,T)}{s} = \frac{(1 - e^{-T}(T+1))}{|\Omega_2|} \int_{\Omega_2 \times \Omega} |\xi - y| F_D(\xi, y)\, d\xi\, dy, \tag{2.75}$$

for all $T \in (0, +\infty)$. As a matter of fact

$$\frac{\widetilde{l}_D^{\Omega_2}(s,T)}{s} := \frac{1}{|\Omega_2|} \int_{\Omega_2} \frac{l_D^y(s,T)}{s}\, dy.$$

Hence, from (2.73) and the definition of $l_D^y(s,T)$ we infer the existence of some $C_4 \in (0,+\infty)$ such that

$$\frac{l_D^y(s,T)}{s} \leqslant C_4,$$

for all $s \in (0,\frac{1}{2})$. Therefore, by the dominated convergence theorem we can conclude the proof of (2.75). The limit in equation (2.70) follows easily from (2.74) and (2.75).

Following the same procedure adopted to prove (2.75), one obtains that

$$\lim_{s \searrow 0} \frac{\widetilde{\mathcal{A}}_D^{\Omega_2}(s,T)}{s} = \frac{(1-e^{-T}(T+1))}{|\Omega_2|} \int_{\Omega_2 \times \Omega} |\xi - y|^2 F_D(\xi, y)\, d\xi\, dy, \tag{2.76}$$

for all $T \in (0,+\infty)$. Thereby, the limit in equation (2.71) follows easily from (2.74) and (2.76). ∎

In the following lemma we study the asymptotic behavior of the Neumann functional $\Phi_N^{x,y}(s,T)$ for $s \searrow 0$. In particular, we observe that the limit substantially differs from the one of $\Phi_D^{x,y}$, which was indeed vanishing, see Lemma 2.13. With this result we establish also that the lim inf and lim sup of $\Phi_N^{x,y}(s,T)$ for $s \searrow 0$ are controlled by some quantities that do not depend on $x, y \in \Omega$. This feature will let us prove that if the forager starting position and target location are close enough, then the most rewarding search strategy for the Neumann functionals in equation (1.6) is not $s = 0$.

Lemma 2.15. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, there exist $h_1, h_2 \in C([0,+\infty))$ such that for each $T \in (0,+\infty)$ it holds that*

$$\frac{h_1(T)}{T} \leqslant \liminf_{s \searrow 0} \mathcal{E}_{1,N}^{x,y}(s,T) \leqslant \limsup_{s \searrow 0} \mathcal{E}_{1,N}^{x,y}(s,T) \leqslant \frac{h_2(T)}{T}, \tag{2.77}$$

$$\frac{h_1(T)}{h_2(T)M(y)} \leqslant \liminf_{s \searrow 0} \mathcal{E}_{2,N}^{x,y}(s,T) \leqslant \limsup_{s \searrow 0} \mathcal{E}_{2,N}^{x,y}(s,T) \leqslant \frac{h_2(T)}{M(y)h_1(T)}, \tag{2.78}$$

$$\frac{h_1(T)}{h_2(T)\widetilde{M}(y)} \leqslant \liminf_{s \searrow 0} \mathcal{E}_{3,N}^{x,y}(s,T) \leqslant \limsup_{s \searrow 0} \mathcal{E}_{3,N}^{x,y}(s,T) \leqslant \frac{h_2(T)}{\widetilde{M}(y)h_1(T)}, \tag{2.79}$$

for all $(x,y) \in \mathcal{C}$, where we set

$$M(y) := \int_\Omega |\xi - y|\, d\xi \quad \text{and} \quad \widetilde{M}(y) := \int_\Omega |\xi - y|^2\, d\xi. \tag{2.80}$$

Proof. Let $(x,y) \in \mathcal{C}$. Notice that if $t \in (0,+\infty)$ we can write

$$p_N^\Omega(t,x,y) = \frac{1}{|\Omega|} + \sum_{k=1}^{+\infty} \zeta_{N,k}(x)\zeta_{N,k}(y)\exp(-t\,\beta_{N,k}),$$

where $\zeta_{N,k}$'s and $\beta_{N,k}$'s are given in (2.3). Now, thanks to [11, Proposition 6 and Lemma 7], together with Weyl's law on the asymptotic behavior of the eigenvalues $\beta_{N,k}$'s (see for instance [29]), we have that

$$\lim_{t\to+\infty}\sum_{k=1}^{+\infty}\zeta_{N,k}(x)\zeta_{N,k}(y)\exp(-t\,\beta_{N,k})=0,$$

from which we deduce that

$$\lim_{l\to+\infty} p_N^{\Omega}(l,x,y)=\frac{1}{|\Omega|}.$$

Therefore, there exists some $t_0\in(1,+\infty)$ such that

$$\frac{1}{2|\Omega|}\leqslant p_N^{\Omega}(t,x,y)\quad \text{for all } t\in[t_0,+\infty).$$

Thus, using (2.1), we have that if $t_{1,s}=\max\{t_0,T^{\frac{1}{s}}\}$, we can apply Theorem 2.3 and obtain that

$$\begin{aligned}\Phi_N^{x,y}(s,T)&=\int_0^T\int_0^{+\infty}p_N^{\Omega}(l,x,y)\mu_t^s(l)\,dl\,dt\\&\geqslant\frac{1}{2|\Omega|}\int_0^T\int_{t_{1,s}}^{+\infty}\mu_t^s(l)\,dl\,dt\\&\geqslant\frac{C_1}{2\pi|\Omega|}\int_0^T\int_{t_{1,s}}^{+\infty}\frac{s\,t}{l^{1+s}}\,dl\,dt\\&\geqslant\frac{C_1}{2\pi|\Omega|}\int_0^T\frac{t}{t_{1,s}^s}\,dt\\&=\frac{C_1}{4\pi|\Omega|}\frac{T^2}{t_{1,s}^s}.\end{aligned}$$

Therefore, if $T\in(1,+\infty)$ from the above inequality we obtain

$$\limsup_{s\searrow 0}\Phi_N^{x,y}(s,T)\geqslant\liminf_{s\searrow 0}\Phi_N^{x,y}(s,T)\geqslant\frac{C_1}{4\pi|\Omega|}T, \tag{2.81}$$

while if $T\in(0,1]$ we have that

$$\limsup_{s\searrow 0}\Phi_N^{x,y}(s,T)\geqslant\liminf_{s\searrow 0}\Phi_N^{x,y}(s,T)\geqslant\frac{C_1}{4\pi|\Omega|}T^2. \tag{2.82}$$

Hence, we have just proved the left-hand side inequality in (2.77) with

$$h_1(T):=\begin{cases}\dfrac{C_1}{4\pi|\Omega|}T^2 & \text{if } T\in(0,1],\\[2mm] \dfrac{C_1}{4\pi|\Omega|}T & \text{if } T\in(1,+\infty).\end{cases} \tag{2.83}$$

Now we show the right-hand side inequality of (2.77). Using (2.19), we obtain that

$$\begin{aligned}\Phi_N^{x,y}(s,T) &= \int_0^T \int_0^{+\infty} p_N^{\Omega}(l,x,y)\mu_t^s(l)\,dl \\ &\leqslant \int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}}\exp\Big(-\frac{|x-y|^2}{6l}\Big)\mu_t^s(l)\,dl\,dt + \int_0^T \int_1^{+\infty} c_\Omega \mu_t^s(l)\,dl\,dt \\ &\leqslant \int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}}\exp\Big(-\frac{|x-y|^2}{6l}\Big)\mu_t^s(l)\,dl\,dt + c_\Omega T. \end{aligned} \tag{2.84}$$

Now, in view of (2.9) we have that

$$\left|\frac{c_\Omega}{l^{\frac{n}{2}}}\exp\Big(-\frac{|x-y|^2}{6l}\Big)\mu_t^s(l)\right| \leqslant \frac{t c_\Omega \Gamma(1+s)}{l^{\frac{n}{2}+1+s}}\exp\Big(-\frac{|x-y|^2}{6l}\Big),$$

and the function on the right-hand side in the above equation is in $L^1((0,T)\times(0,1))$.

Therefore, using also (2.9) we can apply the dominated convergence theorem and obtain the limit

$$\lim_{s\searrow 0}\int_0^T \int_0^1 \frac{c_\Omega}{l^{\frac{n}{2}}}\exp\Big(-\frac{|x-y|^2}{6l}\Big)\mu_t^s(l)\,dl\,dt = 0. \tag{2.85}$$

From this equation and (2.84), we can infer that if $T\in(1,+\infty)$

$$\liminf_{s\searrow 0}\Phi_N^{x,y}(s,T) \leqslant \limsup_{s\searrow 0}\Phi_N^{x,y}(s,T) \leqslant c_\Omega T. \tag{2.86}$$

Also, assuming that $T\in(0,1]$, from (2.9) we obtain that

$$\int_0^T \int_1^{+\infty} c_\Omega \mu_t^s(l)\,dl\,dt \leqslant c_\Omega \int_0^T \int_1^{+\infty} \frac{st\Gamma(1+s)}{l^{1+s}}\,dl\,dt = \frac{c_\Omega\Gamma(1+s)}{2}T^2.$$

Thus, from this latter observation, the limit in (2.85) and equation (2.84) we deduce that

$$\liminf_{s\searrow 0}\Phi_N^{x,y}(s,T) \leqslant \limsup_{s\searrow 0}\Phi_N^{x,y}(s,T) \leqslant \frac{c_\Omega}{2}T^2. \tag{2.87}$$

In light of (2.86) and (2.87), and defining

$$h_2(T) := \begin{cases} c_\Omega T^2 & \text{if } T\in(0,1], \\ c_\Omega T & \text{if } T\in(1,+\infty), \end{cases} \tag{2.88}$$

we conclude the proof of the right-hand side inequality of (2.77).

Now, we prove (2.78). To do so, we claim that

$$h_1(T)M(y) \leqslant \liminf_{s\searrow 0} l_N^y(s,T) \leqslant \limsup_{s\searrow 0} l_N^y(s,T) \leqslant h_2(T)M(y). \tag{2.89}$$

We recall that

$$l_N^y(s,T) = \int_\Omega |\xi - y|\Phi_N^{\xi,y}(s,T)\,d\xi,$$

with $(y,T) \in \Omega \times (0,+\infty)$. Then, using (2.81), (2.82) and Fatou's lemma we prove the left-hand side inequality of (2.78).

Now, we focus on the proof of the right-hand side inequality. Let $K \Subset \Omega$ be any compact such that it is star-shaped with respect to y and $y \in K^{\circ}$, and $d_K \leqslant 1$. Then, in view of (2.44), (2.45) and Proposition D.2 with $E = \Omega \setminus K$ and $F = y$, we evince the existence of some $u \in L^1(\Omega)$ such that

$$|\xi - y|\Phi_N^{\xi,y}(s,T) \leqslant u(\xi),$$

for all $\xi \in \Omega$. Thus, thanks to Fatou's lemma and (2.86) we obtain the right-hand side inequality of (2.78). Note that from (2.77) and (2.89) one evinces (2.78).

It is left to show (2.79). Reasoning analogously to the proof of claim (2.89), one obtains that

$$h_1(T)\widetilde{M}(y) \leqslant \liminf_{s\searrow 0} \mathcal{A}_N^y(s,T) \leqslant \limsup_{s\searrow 0} \mathcal{A}_N^y(s,T) \leqslant h_2(T)\widetilde{M}(y).$$

Making use of this two-sided inequality and (2.77) we conclude the proof of (2.79). ■

The following result is the Neumann counterpart of Lemma 2.14.

Lemma 2.16. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, for all $T \in (0,+\infty)$ and $\Omega_1, \Omega_2 \Subset \Omega$ smooth and disjoint, it holds that*

$$\frac{h_1(T)}{T} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{1,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{1,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{T}, \tag{2.90}$$

$$\frac{h_1(T)}{h_2(T)P(\Omega_2)} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{h_1(T)P(\Omega_2)}, \tag{2.91}$$

$$\frac{h_1(T)}{h_2(T)\widetilde{P}(\Omega_2)} \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{E}}_{3,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{E}}_{3,N}^{\Omega_1,\Omega_2}(s,T) \leqslant \frac{h_2(T)}{h_1(T)\widetilde{P}(\Omega_2)}, \tag{2.92}$$

where h_1 and h_2 are given respectively in (2.83) and (2.88), and we set

$$P(\Omega_2) := \frac{\|M\|_{L^1(\Omega_2)}}{|\Omega_2|} \quad \text{and} \quad \widetilde{P}(\Omega_2) := \frac{\|\widetilde{M}\|_{L^1(\Omega_2)}}{|\Omega_2|}, \tag{2.93}$$

where M and $\widetilde{M}$ are defined in (2.80).

Proof. We begin by proving (2.90). To do so, we notice that by definition we have

$$\widetilde{\Phi}_N^{\Omega_1,\Omega_2}(s,T) = \frac{1}{|\Omega_1||\Omega_2|}\int_{\Omega_1\times\Omega_2} \Phi_N^{x,y}(s,T)\,dx\,dy.$$

From Proposition 2.12 we know that $\Phi_N^{x,y}(s,T) \geqslant 0$. Thus, by Fatou's lemma, (2.81) and (2.82) we conclude the proof of the left-hand side inequality of (2.90).

Now, if $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \varnothing$, thanks to Proposition D.2 with $\Omega_1 = E$ and $\Omega_2 = F$, we easily obtain the right-hand side inequality of (2.90) using Fatou's lemma.

We assume now that $\bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \varnothing$. We claim that there exists some $z \in L^1(\Omega_1 \times \Omega_2)$ such that for all $s \in (0, \frac{1}{2})$ it holds that

$$\Phi_N^{x,y}(s,T) \leqslant z(x,y) \quad \text{for all } (x,y) \in \Omega_1 \times \Omega_2. \tag{2.94}$$

We prove claim (2.94). Thanks to the assumption $\bar{\Omega}_1 \cap \bar{\Omega}_2 \neq \varnothing$, the set $A := \partial\Omega_1 \cap \partial\Omega_2$ is nonempty.

Since $\Omega_1, \Omega_2 \Subset \Omega$ and A is compact, then we can choose $r > 0$ and $P_i \in A$ with $i \in \{1, \dots, N\}$ such that

$$A \subset B := \bigcup_{i=1}^{N} \overline{B_r(P_i)} \Subset \Omega.$$

If for some $i, j \in \{1, \dots, N\}$ it holds that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} \neq \varnothing$, then we can choose $K_{i,j} = \overline{B_r(P_i)} \cup \overline{B_r(P_j)}$ in (2.44) and (2.45) and deduce that

$$\Phi_N^{x,y}(s,T) \leqslant \frac{C}{|x-y|^n}, \tag{2.95}$$

for all $(x,y) \in K_{i,j} \times K_{i,j}$ with i, j such that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} \neq \varnothing$, where C depends on B, T, Ω. Moreover, we define the constant

$$\widetilde{C}_N := \max\{C_{B_r(P_i),B_r(P_j)} \text{ s.t. } \overline{B_r(P_i)} \cap \overline{B_r(P_j)} = \varnothing\}, \tag{2.96}$$

where $C_{B_r(P_i),B_r(P_j)}$ is given in (D.4) with $E = B_r(P_i)$ and $F = B_r(P_j)$. Therefore, if $x \in \Omega_1 \cap \overline{B_r(P_i)}$ and $y \in \Omega_2 \cap \overline{B_r(P_j)}$, such that $\overline{B_r(P_i)} \cap \overline{B_r(P_j)} = \varnothing$, then by (D.3) and (2.96) we see that

$$\Phi_N^{x,y}(s,T) \leqslant \widetilde{C}_N T. \tag{2.97}$$

Finally, if we set

$$\widehat{C}_{\Omega_1,\Omega_2} := \max\{C_{\Omega_1\cap K',\Omega_2\setminus K'}, C_{\Omega_1\setminus K',\Omega_2\cap K'}, C_{\Omega_1\setminus K',\Omega_2\setminus K'}\},$$

thanks to Proposition D.2, we obtain that

$$\Phi_N^{x,y}(s,T) \leqslant \widehat{C}_{\Omega_1,\Omega_2} T, \tag{2.98}$$

for all $(x, y) \in ((\Omega_1 \cap K') \times (\Omega_2 \setminus K')) \cup ((\Omega_1 \setminus K') \times (\Omega_2 \cap K)) \cup ((\Omega_1 \setminus K') \times (\Omega_2 \setminus K'))$.

Thanks to (2.95), (2.97), and (2.98) we conclude the proof of claim (2.94).

By that means, we can apply Fatou's lemma and using (2.77) we prove the right-hand side inequality in (2.90).

Now, we focus our attention on the proof of (2.91). In order to do so, we claim that

$$h_1(T)P(\Omega_2) \leqslant \liminf_{s\searrow 0} \tilde{l}_N^{\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \tilde{l}_N^{\Omega_2}(s,T) \leqslant h_2(T)P(\Omega_2). \tag{2.99}$$

We observe that

$$\tilde{l}_N^{\Omega_2} := \frac{1}{|\Omega_2|}\int_{\Omega_2} l_N^y(s,T)\,dy$$

and, since $l_N^y(s,T) \geqslant 0$, see Proposition 2.12, using Fatou's lemma and (2.89) we prove the left-hand side inequality of (2.99).

Furthermore, we discussed in Proposition 2.12 that $l_N^y(s,T)$ is uniformly bounded in $(s, y) \in (0, 1) \times \Omega$. Thus, we can apply again Fatou's lemma together with (2.89) and conclude the proof of the right-hand side inequality of (2.99). The inequalities in (2.90) and (2.99) yields to (2.91).

It is left to show (2.92). To do so, it is enough to show that

$$h_1(T)\widetilde{P}(\Omega_2) \leqslant \liminf_{s\searrow 0} \widetilde{\mathcal{A}}_N^{\Omega_2}(s,T) \leqslant \limsup_{s\searrow 0} \widetilde{\mathcal{A}}_N^{\Omega_2}(s,T) \leqslant h_2(T)\widetilde{P}(\Omega_2). \tag{2.100}$$

From this and (2.90) it is easy to deduce (2.92). The proof of (2.100) is analogous to the one of (2.99), and thus it is omitted. ∎