Chapter 3

Proof of the main results

This chapter is devoted to the proofs of the main results discussed in the introduction. It is divided into two main parts.

In Section 3.1 we prove the results stated in Section 1.2. Namely, we analyze the environmental scenario where the target location coincides with the forager starting point.

In Section 3.2 we instead discuss the best search strategy when the prey is in a small neighborhood of the seeker initial position. In particular, we prove all the results contained in Sections 1.3 and 1.4.

3.1 Proof of the results in Section 1.2

To prove the results presented in Section 1.2, we consider $\Omega = (0, a)$ for some $a \in (0, +\infty)$. The normalized eigenfunctions of the Laplacian in $(0, a)$ with Dirichlet datum as defined in (2.3) are

$$\zeta_{D,k}(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi k x}{a}\right) \tag{3.1}$$

and the corresponding eigenvalues are

$$\beta_{D,k} = \left(\frac{\pi k}{a}\right)^2. \tag{3.2}$$

As a consequence, recalling (2.4), the Dirichlet spectral fractional heat kernel reads as

$$r_D^s(t, x, y) = \frac{2}{a} \sum_{k=1}^{+\infty} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right). \tag{3.3}$$

This and (1.5) lead to

$$\Phi_D^{x,y}(s, T) = \frac{2}{a} \int_0^T \sum_{k=1}^{+\infty} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right) dt \tag{3.4}$$

and accordingly, if $s \in (\frac{1}{2}, 1)$,

$$\Phi_D^{x,y}(s, T) = \frac{2a^{2s-1}}{\pi^{2s}} \sum_{k=1}^{+\infty} \frac{1}{k^{2s}} \sin\left(\frac{\pi k y}{a}\right) \sin\left(\frac{\pi k x}{a}\right) \left[1 - \exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)\right]. \tag{3.5}$$

We can also compute explicitly the average distance $l_D^y(s,T)$ and the mean square displacement $\mathcal{A}_D^y(s,T)$ as a series, as showed in detail in Appendix B.

The normalized eigenfunctions of the Laplacian in $(0,a)$ under Neumann conditions as defined in (2.3) take the form

$$\begin{cases} \zeta_{N,k}(x) = \sqrt{\dfrac{2}{a}}\cos\left(\dfrac{\pi k x}{a}\right) & \text{if } k \in \{1,2,3,\dots\}, \\ \zeta_{N,0}(x) = \dfrac{1}{\sqrt{a}} & \text{if } k = 0, \end{cases}$$

and the corresponding eigenvalues are

$$\beta_{N,k} = \left(\frac{\pi k}{a}\right)^2 \quad \text{if } k \in \{0,1,2,3,\dots\}. \tag{3.6}$$

Therefore, in view of (2.4), the Neumann spectral fractional heat kernel reads as

$$r_N^s(t,x,y) = \frac{1}{a} + \frac{2}{a}\sum_{k=1}^{+\infty}\cos\left(\frac{\pi k x}{a}\right)\cos\left(\frac{\pi k y}{a}\right)\exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right).$$

Hence, by (1.5),

$$\Phi_N^{x,y}(s,T) = \frac{T}{a} + \frac{2}{a}\sum_{k=1}^{+\infty}\int_0^T \cos\left(\frac{\pi k x}{a}\right)\cos\left(\frac{\pi k y}{a}\right)\exp\left(-t\left(\frac{\pi k}{a}\right)^{2s}\right)dt \tag{3.7}$$

and, as a result, when $s \in (\frac{1}{2},1)$,

$$\begin{aligned} &\Phi_N^{x,y}(s,T) \\ &= \frac{T}{a} + \frac{2a^{2s-1}}{\pi^{2s}}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\cos\left(\frac{\pi k x}{a}\right)\cos\left(\frac{\pi k y}{a}\right)\left[1-\exp\left(-T\left(\frac{\pi k}{a}\right)^{2s}\right)\right]. \end{aligned} \tag{3.8}$$

Thanks to these preliminary observations, we are now in the position of proving the results presented in Section 1.2. We begin by showing Proposition 1.1.

We recall that we adopt the subscript $*$ every time that a functional refers to both the Dirichlet and the Neumann case.

Proof of Proposition 1.1. Let $x \in \Omega$ and $T \in (0,+\infty)$. Then, thanks to Theorem 2.8 we know that if either $n \geqslant 2$ or $n = 1$ and $s \in (0,\frac{1}{2}]$ it holds that

$$\Phi_*^{x,x}(s,T) = +\infty. \tag{3.9}$$

Furthermore, from Proposition 2.12 we have that

$$l_*^x(s,T) \in (0,+\infty) \quad \text{and} \quad \mathcal{A}_*^x(s,T) \in (0,+\infty), \tag{3.10}$$

for all $s \in (0, 1]$. Therefore, as a direct consequence of (3.9) and (3.10) we obtain the desired claim. ∎

The proof of Proposition 1.2 that we present here below is a consequence of Theorem 2.8. For the sake of completeness, in Appendix C we also provide an alternative proof of Proposition 1.2 which employs directly the spectral structure of the efficiency functionals.

Proof of Proposition 1.2. Let $x \in \Omega = (0, a)$, for some $a \in (0, +\infty)$. Then, thanks to Theorem 2.8 we have that for each $s \in (0, \frac{1}{2}]$ and $T \in (0, +\infty)$ the statement in (3.9) holds true. Also, if $s \in (\frac{1}{2}, 1]$, in view of Proposition 2.12 one has that

$$\Phi_*^{x,x}(s, T) \in (0, +\infty). \tag{3.11}$$

Furthermore, from Proposition 2.12 we know that for each $s \in (0, 1]$ and $T \in (0, +\infty)$ the statement in (3.10) holds true as well. Therefore, using equation (3.11), in the notation of Proposition 1.2, we conclude that

$$\mathcal{E}_{*,j}(s, T) \in (0, +\infty) \quad \text{for all } s \in \left(\frac{1}{2}, 1\right],$$

for all $j \in \{1, 2, 3\}$.

Hence, to complete the proof of Proposition 1.2, it is only left to show the continuity statement. Thanks to Proposition 2.12 we have that, for each $x \in \Omega$ and $T \in (0, +\infty)$, the functional $\Phi_*^{x,x}(\cdot, T)$ is continuous with respect to $s \in (\frac{1}{2}, 1]$. Also, the continuity with respect to $s \in (0, 1]$ of the functionals $\mathcal{A}_*^x(s, T)$ and $l_*^x(s, T)$ was already established in Proposition 2.12.

As a consequence, recalling (3.10) we conclude that the functionals in (1.6) are continuous in $s \in (\frac{1}{2}, 1]$ for $x = y$. ∎

Now we prove Theorem 1.3. Here we establish that

$$s = \frac{1}{2}$$

is the best search strategy in $(\frac{1}{2}, 1]$ when the forager initial point coincide with the target location.

Proof of Theorem 1.3. We point out that, in order to prove Theorem 1.3, it suffices to establish (1.10). Indeed, once (1.10) is proved, we already know from Proposition 1.2 that $\mathcal{E}_{*,j}(s, T) \in (0, +\infty)$ for all $s \in (\frac{1}{2}, 1]$ and $j \in \{1, 2, 3\}$ and accordingly the supremum over $s \in (\frac{1}{2}, 1)$ of $\mathcal{E}_{j,*}$ is attained at $s = 1/2$.

Furthermore, thanks to (C.1) it is enough to show (1.10) for $a := 1$. To prove it, we observe that all the denominators in (1.6) satisfy (3.10). Consequently, the claim in (1.10) is equivalent to

$$\lim_{s \searrow \frac{1}{2}} \Phi_*^{x,x}(s, T) = +\infty. \tag{3.12}$$

Thus, from now on we focus on the proof of the claims in (3.12). We establish the claim for the Dirichlet case, since the Neumann one follows from the Dirichlet one and (2.22).

For this, we recall (C.4) and we see that there exist K_0, $N \geqslant 1$ such that, for every $\bar{N} \in \mathbb{N}$,

$$\Phi_D^{x,x}(s,T) \geqslant \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\bar{N}-1} \frac{\varepsilon_0}{(N+3\ell K_0)^{2s}}.$$

We now pick $L > 0$, to be taken as large as we wish in what follows, such that $e^L \in \mathbb{N}$, and we choose $\bar{N} := e^{2L} + 1$. In this way, we find that

$$\begin{aligned}
\Phi_D^{x,x}(s,T) &\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=\exp(L)+1}^{\exp(2L)} \frac{\varepsilon_0}{(N+3\ell K_0)^{2s}} \geqslant \frac{1}{\pi^{2s}} \sum_{\ell=\exp(L)+1}^{\exp(2L)} \frac{\varepsilon_0}{(4\ell K_0)^{2s}} \\
&= \frac{\varepsilon_0}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \sum_{\ell=\exp(L+j-1)+1}^{\exp(L+j)} \frac{1}{\ell^{2s}} \\
&\geqslant \frac{\varepsilon_0}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \sum_{\ell=\exp(L+j-1)+1}^{\exp(L+j)} \frac{1}{\exp(2s(L+j))} \\
&= \frac{\varepsilon_0(e-1)}{(4\pi K_0)^{2s}} \sum_{j=1}^{L} \frac{\exp(L+j-1)}{\exp(2s(L+j))} \\
&= \frac{\varepsilon_0(e-1)}{(4\pi K_0)^{2s}\exp((2s-1)L+1)} \sum_{j=1}^{L} \frac{1}{\exp((2s-1)j)} \\
&= \frac{\varepsilon_0(e-1)}{(4\pi K_0)^{2s}\exp((2s-1)L+1)} \times \frac{\exp(1-2s)(1-\exp((1-2s)L))}{1-\exp(1-2s)} \\
&= \frac{\varepsilon_0(e-1)}{(4\pi e K_0)^{2s}\exp(2(2s-1)L)} \times \frac{(\exp((2s-1)L)-1)}{1-\exp(1-2s)}.
\end{aligned}$$

In particular, we can choose $L \in [\frac{1}{2s-1}, \frac{2}{2s-1}]$ such that $e^L \in \mathbb{N}$ and deduce from the above estimate that

$$\Phi_D^{x,x}(s,T) \geqslant \frac{\varepsilon_0(e-1)^2}{(4\pi e K_0)^{2s} e^4} \times \frac{1}{1-\exp(1-2s)}.$$

Sending now $s \searrow \frac{1}{2}$ we see that

$$\lim_{s \searrow \frac{1}{2}} \Phi_D^{x,x}(s,T) = +\infty,$$

proving the claim in (3.12) for the Dirichlet case, as desired. ∎

Finally, we prove Theorem 1.4. In this result, we discuss the impact of some geometrical properties of the domain, such as the size of it, on the monotonicity of the efficiency functionals in (1.5) with respect to the fractional exponent.

Proof of Theorem 1.4. We prove the monotonicity properties of Φ_*. To this end, in the Dirichlet case, when $a \in (0, \pi]$ the first eigenvalue of the Laplacian is less than or equal to 1, thanks to (3.2); hence, we capitalize on [10, Theorem 7] and we conclude that, for all $s_0 \in (0,1)$ and $s_1 \in (s_0, 1)$, we have that, for every $x \in (0, a)$,

$$r_D^{s_0}(t,x,x) > r_D^{s_1}(t,x,x). \tag{3.13}$$

Similarly, in the Neumann case, when $a \in (0, \pi]$ the first nontrivial eigenvalue of the Laplacian is less than or equal to 1, due to (3.6). This allows us to use [11, Theorem 7] and obtain that, for all $s_0 \in (0,1)$, $s_1 \in (s_0, 1)$ and $x \in (0, a)$,

$$r_N^{s_0}(t,x,x) > r_N^{s_1}(t,x,x). \tag{3.14}$$

Now, from (1.5), (3.13), and (3.14) it follows that, for all $s_0 \in (0,1)$, $s_1 \in (s_0, 1)$ and $x \in (0, a)$,

$$\Phi_*^{x,x}(s_0, T) > \Phi_*^{x,x}(s_1, T). \tag{3.15}$$

From (3.15) we obtain the desired monotonicity property when $a \in (0, \pi]$, as stated in formula (1.11) of Theorem 1.4.

Now we deal with the case in which a is sufficiently large and we prove (1.12) and (1.13). To this end, we start with the Dirichlet case, utilize (3.5) with the notation $\alpha := \frac{a}{\pi}$ and deduce that, for every $T \in (0, +\infty)$ and $x \in (0, a)$,

$$\begin{aligned}
\frac{a}{2}\partial_s \Phi_D^{x,x}(s,T) &= \frac{\partial}{\partial s}\left[\alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]\right] \\
&= 2\alpha^{2s}\ln\alpha\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right] \\
&\quad - 2\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right] \\
&\quad + 2T\alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}\ln\left(\frac{k}{\alpha}\right) \\
&= 2\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right] \\
&\quad - 2T\alpha^{2s}\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\sin^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}
\end{aligned}$$

$$= 2\alpha^{2s} \sum_{k=1}^{+\infty} \frac{\ln \alpha - \ln k}{k^{2s}} \sin^2\left(\frac{kx}{\alpha}\right) \times \left[1 - \exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1 + T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]. \tag{3.16}$$

These observations lead to

$$\begin{aligned} &\frac{a}{4\alpha^{2s}} \partial_s \Phi_D^{x,x}(s,T) \\ &\quad = \ln \alpha \sin^2\left(\frac{x}{\alpha}\right)\left[1 - \exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1 + \frac{T}{\alpha^{2s}}\right)\right] \\ &\qquad + \sum_{k=2}^{+\infty} \frac{\ln \alpha - \ln k}{k^{2s}} \sin^2\left(\frac{kx}{\alpha}\right)\left[1 - \exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1 + T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]. \end{aligned}$$

We also observe that, if $f(\tau) := 1 - e^{-\tau}(1+\tau)$, we have that $f'(\tau) = \tau e^{-\tau} > 0$ for all $\tau > 0$. Accordingly, we see that $1 - e^{-\tau}(1+\tau) > f(0) = 0$ for all $\tau > 0$. In addition, we have that $f(\tau) \leqslant 1$ for all $\tau > 0$. As a result,

$$\begin{aligned} &\sum_{k=2}^{+\infty} \frac{\ln \alpha - \ln k}{k^{2s}} \sin^2\left(\frac{kx}{\alpha}\right)\left[1 - \exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1 + T\left(\frac{k}{\alpha}\right)^{2s}\right)\right] \\ &\quad \geqslant \sum_{k \in \mathbb{N} \cap (\alpha, +\infty)} \frac{\ln \alpha - \ln k}{k^{2s}} \sin^2\left(\frac{kx}{\alpha}\right)\left[1 - \exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(1 + T\left(\frac{k}{\alpha}\right)^{2s}\right)\right] \\ &\quad \geqslant - \sum_{k \in \mathbb{N} \cap (\alpha, +\infty)} \frac{\ln k}{k^{2s}}. \end{aligned}$$

From these remarks, we arrive at

$$\frac{a}{4\alpha^{2s}} \partial_s \Phi_D^{x,x}(s,T) \geqslant \ln \alpha \sin^2\left(\frac{x}{\alpha}\right)\left[1 - \exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1 + \frac{T}{\alpha^{2s}}\right)\right] - \sum_{k \in \mathbb{N} \cap (\alpha, +\infty)} \frac{\ln k}{k^{2s}}.$$

Now, if $T \in [\nu a^{2s}, +\infty) = [\nu \pi^{2s} \alpha^{2s}, +\infty)$, then

$$1 - \exp\left(-\frac{T}{\alpha^{2s}}\right)\left(1 + \frac{T}{\alpha^{2s}}\right) = f\left(\frac{T}{\alpha^{2s}}\right) \geqslant f(\nu \pi^{2s}) \geqslant f(\nu).$$

Hence, in this situation,

$$\frac{a}{4\alpha^{2s}} \partial_s \Phi_D^{x,x}(s,T) \geqslant \ln \alpha \sin^2\left(\frac{x}{\alpha}\right) f(\nu) - \sum_{k \in \mathbb{N} \cap (\alpha, +\infty)} \frac{\ln k}{k^{2s}}.$$

We also recall that

$$\int_{\alpha-2}^{+\infty} \frac{\ln \tau}{\tau^{2s}}\, d\tau = \frac{1+(2s-1)\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}}$$

and therefore, if α is large enough,

$$\sum_{k\in\mathbb{N}\cap(\alpha,+\infty)} \frac{\ln k}{k^{2s}} \leqslant \frac{1+(2s-1)\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}} \leqslant \frac{2s\ln(\alpha-2)}{(2s-1)^2(\alpha-2)^{2s-1}}$$
$$\leqslant \frac{2s\ln\alpha}{(2s-1)^2(\alpha-2)^{2s-1}}.$$

Besides, if $x\in(\nu a,(1-\nu)a)=(\nu\alpha\pi,(1-\nu)\alpha\pi)$ we have that

$$\left|\sin\left(\frac{x}{\alpha}\right)\right| > \sin(\varepsilon\pi). \tag{3.17}$$

These observations lead to

$$\frac{a}{4\alpha^{2s}\ln\alpha}\partial_s\Phi_D^{x,x}(s,T) \geqslant \sin^2(\nu\pi)f(\nu) - \frac{2s}{(2s-1)^2(\alpha-2)^{2s-1}}$$
$$\geqslant \sin^2(\nu\pi)f(\nu) - \frac{2}{\nu^2(\alpha-2)^{\nu}} > 0,$$

as long as α (whence a) is sufficiently large, possibly in dependence of ν.

This establishes (1.12) in the Dirichlet case and we now focus on the proof of (1.13) in the Neumann case. In this situation, recalling (3.8),

$$\frac{a}{2}\Phi_N^{x,x}(s,T) = \frac{T}{2} + \alpha^{2s}\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]$$

and therefore

$$\frac{a}{4\alpha^{2s}}\partial_s\Phi_N^{x,x}(s,T) = \ln\alpha\sum_{k=1}^{+\infty}\frac{1}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]$$
$$-\sum_{k=1}^{+\infty}\frac{\ln k}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\left[1-\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\right]$$
$$-T\sum_{k=1}^{+\infty}\frac{\ln\alpha-\ln k}{k^{2s}}\cos^2\left(\frac{kx}{\alpha}\right)\exp\left(-T\left(\frac{k}{\alpha}\right)^{2s}\right)\left(\frac{k}{\alpha}\right)^{2s}.$$

This puts us in the same position as in (3.16), but with the sine replaced by the cosine. Hence, in this case, we only need to detect the analog of (3.17). For this, we observe that if $x\in(0,\frac{(1-\nu)a}{2})\cup(\frac{(1+\nu)a}{2},a)=(0,\frac{(1-\nu)\alpha\pi}{2})\cup(\frac{(1+\nu)\alpha\pi}{2},\alpha\pi)$ we have that

$$\left|\cos\left(\frac{x}{\alpha}\right)\right| > \cos\left(\frac{(1-\nu)\pi}{2}\right).$$

Thus, the same argument as in the Dirichlet case leads to (1.13). ∎

3.2 Proof of the results in Sections 1.3 and 1.4

In this section we prove the results stated in Section 1.3. Here we discuss the optimal search strategy when the forager starting position $y \in \Omega$ is sufficiently close to the prey location $x \in \Omega$, but does not coincide with it.

We recall that we adopt the subscript $*$ every time that we refer to both the Dirichlet and the Neumann case.

We start this section by showing that all the functionals defined in (1.6) are continuous with respect to $s \in (0, 1]$.

Proof of Proposition 1.5. Let $(x, y, T) \in \Omega \times \Omega \times (0, +\infty)$ be such that $x \neq y$. Then, thanks to Proposition 2.12 we have that

$$\Phi_*^{x,y}(s, T) \in (0, +\infty),$$

and also

$$l_*^y(s, T) \in (0, +\infty) \quad \text{and} \quad \mathcal{A}_*^y(s, T) \in (0, +\infty), \tag{3.18}$$

for each $s \in (0, 1]$. These considerations give that $\mathcal{E}_{j,*}^{x,y}(s, T)$ for all $s \in (0, 1]$.

Now, from Proposition 2.12 and (3.18), we deduce that the functionals $\mathcal{E}_{j,*}^{x,y}(\cdot, T)$ are continuous with respect to $s \in (0, 1]$. ∎

Now we prove Theorem 1.6. We show that $s = 0$ is a global minimizer for $\mathcal{E}_{1,D}^{x,y}(\cdot, T)$ in $(0, 1)$ for each $x, y \in \Omega$ such that $x \neq y$ and for all $T \in (0, +\infty)$. Moreover, we discuss the existence of the limit for $s \searrow 0$ of $\mathcal{E}_{2,D}$ and $\mathcal{E}_{3,D}$.

Proof of Theorem 1.6. Let $x, y \in \Omega$ such that $x \neq y$ and $T \in (0, +\infty)$. Then, thanks to Lemma 2.13 we have that

$$\lim_{s \searrow 0} \mathcal{E}_{1,D}^{x,y}(s, T) = 0.$$

Since $\Phi_D^{x,y}(s, T) \in (0, +\infty)$ for each $s \in (0, 1]$, see Proposition 2.12, we establish (1.14). We point out that the existence of the limits in (1.15) was already obtained in Lemma 2.13.

Besides, making use of the maximum principle for the heat equation, we see that

$$F_D(z, w) > 0 \quad \text{for all } z, w \in \Omega, \tag{3.19}$$

and so the right-hand sides of the expressions in (2.60) and (2.61) are nonnegative. Also, using (D.6) and (3.19) we deduce that the limits in (2.60) and (2.61) are also positive and finite. ∎

We prove now Theorems 1.7 and 1.8. We recall that this result states that if the forager starting position is close enough to the target location, then the optimal search strategy for the functionals in equation (1.6) is in a small neighborhood of $s = 0$.

Proof of Theorems 1.7 *and* 1.8. Let $(y,T)\in\Omega\times(0,+\infty)$. We recall the limit in (2.59) and we observe that

$$\sup_{s\in(0,1)} \mathcal{E}^{x,y}_{1,D}(s,T)=\mathcal{E}^{x,y}_{1,D}(s^{(1)}_{x,y,T},T) \quad \text{with } s^{(1)}_{x,y,T}\in(0,1], \tag{3.20}$$

for each $x\in\Omega\setminus\{y\}$.

Also, From Lemma 2.15 and equation (2.25), we evince that if $s_0\in(0,\frac{1}{2})$ there exists some $\beta=\beta_{s_0,y,T,\Omega}\in(0,\widehat{\delta})$ such that, if $x\in B_\beta(y)\setminus\{y\}$, then

$$\limsup_{s\searrow 0} \mathcal{E}^{x,y}_{j,N}(s,T)\leqslant \mathcal{E}^{x,y}_{j,N}(s_0,T),$$

for all $j\in\{1,2,3\}$, where $\widehat{\delta}$ is provided in (2.27).

Thus, we deduce that there exists some $\widehat{\beta}=\widehat{\beta}_{y,T,\Omega}$ such that if $x\in B_{\widehat{\beta}}(y)\setminus\{y\}$, then

$$\sup_{s\in(0,1)} \mathcal{E}^{x,y}_{j,N}(s,T)=\mathcal{E}^{x,y}_{1,D}(s^{(j)}_{x,y,T},T) \quad \text{with } s^{(j)}_{x,y,T}\in(0,1], \tag{3.21}$$

for all $j\in\{1,2,3\}$.

Let us first study the case $n\leqslant 2$. We recall that thanks to Lemma 2.7, for each $s_0\in(0,\frac{1}{2})$ we have the existence of some $\widehat{\delta}=\widehat{\delta}_{s_0,y,T,\Omega}$, given in (2.27), such that, for each $x\in B_{\widehat{\delta}}(y)\setminus\{y\}$, one has that

$$\Phi^{x,y}_{*}(s_0,T)\geqslant \frac{C_{s_0,y,\Omega}}{|x-y|^{n-2s_0}}, \tag{3.22}$$

where $C_{s_0,y,\Omega}$ is provided in (2.32). Also, for each $s\in(0,1)$ and $\mu\in A_{n,s}$, where $A_{n,s}$ is given in (2.43), thanks to (2.45) we have the existence of some constant $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ such that

$$\Phi^{x,y}_{*}(s,T)\leqslant \frac{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2\mu}}, \tag{3.23}$$

for each $x\in B_{\widehat{\delta}}(y)\setminus\{y\}$.

Consequently, from the last two inequalities we obtain that if $s_0\in(0,\frac{1}{2})$, $s_1\in(s_0,1)$ and $\mu\in A_{n,s_1}$, then

$$\frac{\mathcal{E}^{x,y}_{1,*}(s_0,T)}{\mathcal{E}^{x,y}_{1,*}(s_1,T)}=\frac{\Phi^{x,y}_{*}(s_0,T)}{\Phi^{x,y}_{*}(s_1,T)}\geqslant \frac{C_{*,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{n-2s_0-2\mu}}, \tag{3.24}$$

for all $x\in B_{\widehat{\delta}}(y)\setminus\{y\}$, where we set

$$C_{*,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega}:=\frac{C_{s_0,y,\Omega}}{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}. \tag{3.25}$$

As a result, for each $\varepsilon \in (0,1)$, by choosing $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n-\varepsilon)/2$ in (3.24), and recalling also (3.20) and (3.21), we infer the existence of some $\delta^{(1)} = \delta^{(1)}_{\varepsilon,y,T,\Omega} \in (0, \widehat{\delta})$ such that for each $x \in B_{\delta^{(1)}}(y) \setminus \{y\}$ it holds that

$$\sup_{s\in(0,1)} \mathcal{E}^{x,y}_{1,*}(s,T) = \mathcal{E}^{x,y}_{1,*}(s^{(1)}_{*,x,y,T}, T) \quad \text{with } s^{(1)}_{*,x,y,T} \in (0,\varepsilon).$$

This concludes the proof of (1.16) and (1.18) with $j = 1$.

Let us now prove (1.17) for the functional $\mathcal{E}_{2,D}$. To this end, let $d_y := \frac{d(y,\partial\Omega)}{2}$ and $B_y := B_{d_y}(y)$. Then, thanks to equation (D.5) in Lemma D.4 we have that there exists a constant $\tilde{c}_{B_y,\Omega}$ such that for each $x \in B_y \setminus \{y\}$ it holds that

$$\mathcal{E}^{x,y}_{2,D}(0,T) = \frac{F_D(x,y)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \geqslant \frac{\tilde{c}_{B_y,\Omega}}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{1}{|x-y|^n}. \tag{3.26}$$

Therefore, using (3.23) and the estimates in (3.26), if $s \in (0,1)$, $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$ and μ is given as in (2.43), we obtain that

$$\begin{aligned} \frac{\mathcal{E}^{x,y}_{2,D}(0,T)}{\mathcal{E}^{x,y}_{2,D}(s,T)} &= \frac{F_D(x,y)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{l^y_D(s,T)}{\Phi^{x,y}_D(s,T)} \\ &\geqslant \frac{\Gamma(s) l^y_D(s,T)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{\tilde{c}_{B_y,\Omega}}{C_{D,\mu,B_{\widehat{\delta}}(y),T,\Omega}} \frac{1}{|x-y|^{n-2\mu}}. \end{aligned} \tag{3.27}$$

Now, using (2.68) and the limit

$$\lim_{s\searrow 0} s\Gamma(s) = 1,$$

we obtain that

$$\lim_{s\searrow 0} \frac{\Gamma(s) l^y_D(s,T)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} = 1 - e^{-T}(T+1).$$

Thanks to this observation and Proposition 2.12, we can define the positive constant

$$C_{\mu,y,B_{\widehat{\delta}}(y),T,\Omega} := \inf_{s\in(0,1)} \frac{\Gamma(s) l^y_D(s,T)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{\tilde{c}_{B_y,\Omega}}{C_{D,\mu,B_{\widehat{\delta}}(y),T,\Omega}} > 0.$$

Then, we obtain from (3.27) that for each $x \in B_y \setminus \{y\}$, $s \in (0,1)$ and μ as in (2.43) it holds that

$$\frac{\mathcal{E}^{x,y}_{2,D}(0,T)}{\mathcal{E}^{x,y}_{2,D}(s,T)} \geqslant \frac{C_{\mu,y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{n-2\mu}}. \tag{3.28}$$

Therefore, for each $\varepsilon \in (0,1)$ by taking $s \in (\varepsilon,1)$ and choosing $\mu := (n-\varepsilon)/2$ in (3.28), we deduce that there exists some $\delta^{(2)} = \delta^{(2)}_{\varepsilon,y,T,\Omega}$ such that for each $x \in B_{\delta^{(2)}}(y) \setminus \{y\}$ it holds that

$$\mathcal{E}^{x,y}_{2,D}(0,T) \geqslant \sup_{s\in(\varepsilon,1)} \mathcal{E}^{x,y}_{2,D}(s,T).$$

The proof of (1.17) for $\mathcal{E}_{3,D}$ is analogous to the one for $\mathcal{E}_{2,D}$ and therefore it will be omitted. This last step concludes the proof of Theorem 1.7 for $n \leqslant 2$.

Now we show (1.18) when $n \leqslant 2$ for $\mathcal{E}_{2,N}$. To do so, thanks to Proposition 2.12 and (2.89) we can define the positive constant

$$\widetilde{C}_{y,T,\Omega} := \inf_{\substack{s_0 \in (0,1) \\ s_1 \in (0,1)}} \frac{l_N^y(s_1,T)}{l_N^y(s_0,T)} > 0.$$

Then, if $s_0 \in (0, \frac{1}{2})$, $s_1 \in (s_0, 1)$ and $\mu \in A_{n,s_1}$, thanks to equations (3.22) and (3.23) we have that

$$\frac{\mathcal{E}_{2,N}^{x,y}(s_0,T)}{\mathcal{E}_{2,N}^{x,y}(s_1,T)} = \frac{\Phi_N^{x,y}(s_0,T)}{\Phi_N^{x,y}(s_1,T)} \frac{l_N^y(s_1,T)}{l_N^y(s_0,T)} \geqslant \frac{\widetilde{C}_{\mu,s_0,y,K,T,\Omega}}{|x-y|^{n-2s_0-2\mu}}, \tag{3.29}$$

for all $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, where we defined

$$\widetilde{C}_{\mu,s_0,y,K,T,\Omega} := C_{N,s_0,y,\mu,B_{\widehat{\delta}}(y),T,\Omega} \widetilde{C}_{y,T,\Omega}. \tag{3.30}$$

Therefore, for each $\varepsilon \in (0,1)$, by choosing $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n-\varepsilon)/2$ in (3.29), and recalling (3.21), we deduce the existence of some

$$\delta^{(2)} = \delta^{(2)}_{\varepsilon,y,T,\Omega} \in (0, \widehat{\beta})$$

such that for each $x \in B_{\delta^{(2)}}(y) \setminus \{y\}$ it holds that

$$\sup_{s \in (0,1)} \mathcal{E}_{2,N}^{x,y}(s,T) = \mathcal{E}_{2,N}^{x,y}(s^{(2)}_{x,y,T},T) \quad \text{with } s^{(2)}_{x,y,T} \in (0,\varepsilon).$$

This concludes the proof of (1.18) for $\mathcal{E}_{2,N}$. The proof of (1.18) for $\mathcal{E}_{3,N}$ is analogous to the one for $\mathcal{E}_{2,N}$.

It is left to prove Theorems 1.7 and 1.8 when $n \geqslant 3$.

If $n \geqslant 3$, we just have to replace the inequality (3.23) with the one in (2.44). Thus, repeating the above procedure with this change, the inequalities in (3.24) and (3.29) become

$$\frac{\mathcal{E}_{*}^{x,y}(s_0,T)}{\mathcal{E}_{*}^{x,y}(s_1,T)} \geqslant \frac{C_{*,s_0,y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2(s_1-s_0)}}, \tag{3.31}$$

for all $s_0 \in (0, \frac{1}{2})$, $s_1 \in (s_0, 1)$ and $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, where we denoted by $\mathcal{E}_*$ any of the functionals $\mathcal{E}_{1,D}$, $\mathcal{E}_{1,N}$ and $\mathcal{E}_{2,N}$.

The constant $C_{*,s_0,y,B_{\widehat{\delta}}(y),T,\Omega}$ is obtained substituting the constant $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ with $C_{*,B_{\widehat{\delta}}(y),T,\Omega}$ in (3.25) for $\mathcal{E}_{1,D}$ and $\mathcal{E}_{1,N}$, and in (3.30) for $\mathcal{E}_{2,N}$.

Analogously, equation (3.28) becomes

$$\frac{\mathcal{E}_{2,D}^{x,y}(0,T)}{\mathcal{E}_{2,D}^{x,y}(s_1,T)} \geqslant \frac{C^{(1)}_{y,B_{\widehat{\delta}}(y),T,\Omega}}{|x-y|^{2s_1}} \tag{3.32}$$

for all $s \in (0,1)$ and $x \in B_{\widehat{\delta}}(y) \setminus \{y\}$, where we defined

$$C^{(1)}_{y,B_{\widehat{\delta}}(y),T,\Omega} := \inf_{s\in(0,1)} \frac{\Gamma(s) l^y_D(s,T)}{\int_\Omega |\xi - y| F_D(\xi,y)\, d\xi} \frac{\tilde{c}_{B_y,\Omega}}{C_{D,B_{\widehat{\delta}}(y),T,\Omega}}.$$

Therefore, for each $\varepsilon \in (0,1)$, by choosing $s_0 := \frac{\varepsilon}{2}$ in (3.31) and $s_1 \in (\varepsilon,1)$ in (3.31) and (3.32), we obtain (1.16), (1.17) for $\mathcal{E}_{2,D}$ and (1.18) for both $\mathcal{E}_{1,N}$ and $\mathcal{E}_{2,N}$ when $n \geqslant 3$.

The proof of (1.17) and (1.18) respectively for $\mathcal{E}_{3,D}$ and $\mathcal{E}_{3,N}$ are analogous to the one of $\mathcal{E}_{2,D}$ and $\mathcal{E}_{2,N}$ when $n \geqslant 3$ and are therefore omitted. ∎

Now, we prove Corollary 1.12. Namely, we establish in the one-dimensional framework, and under suitable geometric assumptions on the domain, that if the target location $x \in \Omega$ is close enough to the forager starting position $y \in \Omega$, then there exists a local maximizer for the functionals $\mathcal{E}^{x,y}_{1,D}$ and $\mathcal{E}^{x,y}_{1,N}$ in a neighborhood of the local Brownian strategy $s = 1$.

Proof of Corollary 1.12. We will only prove (1.19), since the proof of (1.20) is analogous. For this, let $\nu \in (0, \frac{1}{2})$. Then, thanks to Theorem 1.4, we have that there exists some $a_\nu \in (\pi, +\infty)$ such that, for all $a \in (a_\nu, +\infty)$, $T \in [\nu a^{2s}, +\infty)$ and $y \in (\nu a, \nu(1-a))$, it holds that

$$\Phi^{y,y}_D(s_0,T) < \Phi^{y,y}_D(s_1,T), \tag{3.33}$$

for all $s_0 \in (\frac{1+\nu}{2}, 1]$ and $s_1 \in (s_0, 1]$.

Now, for any $\varepsilon \in (\frac{1+\nu}{2}, 1)$ we define the positive quantity

$$\widetilde{\delta} = \widetilde{\delta}_{\varepsilon,\nu,y,T} := \Phi^{y,y}_D(1,T) - \Phi^{y,y}_D(1-\varepsilon,T).$$

Also, thanks to the continuity of $\Phi^{x,y}_D(s,T)$ with respect to $(s,x,y) \in (\frac{1}{2},1] \times \Omega \times \Omega$ stated in Proposition 2.12, we can define $\delta_{\varepsilon,\nu y,T,\Omega} \in (0,+\infty)$, such that

$$|\Phi^{x,y}_D(s,T) - \Phi^{y,y}_D(s,T)| \leqslant \frac{\widetilde{\delta}}{4} \quad \text{for all } s \in \left(\frac{1+\nu}{2}, 1\right].$$

Thus, using the monotonicity of $\Phi^{y,y}_D$ in (3.33), we obtain that, for each $x \in \Omega$ and $s_0 \in (\frac{1+\nu}{2}, 1-\varepsilon)$,

$$\begin{aligned}
\Phi^{x,y}_D(1,T) - \Phi^{x,y}_D(s_0,T) &= \Phi^{x,y}_D(1,T) - \Phi^{y,y}_D(1,T) + \Phi^{y,y}_D(1,T) \\
&\quad - \Phi^{y,y}_D(s_0,T) + \Phi^{y,y}_D(s_0,T) - \Phi^{x,y}_D(s_0,T) \\
&> \Phi^{x,y}_D(1,T) - \Phi^{y,y}_D(1,T) + \widetilde{\delta} \\
&\quad + \Phi^{y,y}_D(s_0,T) - \Phi^{x,y}_D(s_0,T) \\
&\geqslant -\frac{\widetilde{\delta}}{4} + \widetilde{\delta} - \frac{\widetilde{\delta}}{4} = \frac{\widetilde{\delta}}{2}.
\end{aligned}$$

From this, we infer that

$$\sup_{s\in(\frac{1+\nu}{2},1)} \mathcal{E}^{x,y}_{1,D}(s,T) = \mathcal{E}^{x,y}_{1,D}(s^*_{x,y,T},T) \quad \text{with } s^*_{x,y,T} \in (1-\varepsilon,1],$$

which proves (1.19). ■

We now prove Proposition 1.13 and establish the continuity with respect to the fractional exponent of the set functionals in (1.9).

Proof of Proposition 1.13. Since the proof for the Dirichlet and Neumann case are analogous, we focus on the Dirichlet framework.

We already established in Proposition 2.12 that for all $y \in \Omega$ and $s \in (0,1)$ one has that (3.18) holds, and also the functionals in (1.3) and (1.4) are uniformly bounded in $(0,1] \times \Omega$.

Therefore, by definition we obtain that, for all $\Omega_2 \subset \Omega$,

$$\tilde{l}^{\Omega_2}_D(s,T) \in (0,+\infty) \quad \text{and} \quad \tilde{\mathcal{A}}^{\Omega_2}_D(s,T) \in (0,+\infty), \tag{3.34}$$

for all $s \in (0,1]$ and $T \in (0,+\infty)$.

Besides, thanks to Proposition 2.12 we know that $l^y_D(\cdot,T)$ and $\mathcal{A}^y_D(\cdot,T)$ are continuous in $(0,1]$. Thus, by the dominated convergence theorem we obtain that $\tilde{l}^{\Omega_2}_D(\cdot,T)$ and $\tilde{\mathcal{A}}^{\Omega_2}_D(\cdot,T)$ are continuous in $(0,1]$.

Now, we observe that

$$\tilde{\Phi}^{\Omega_1,\Omega_2}_D(s,T) = \frac{1}{|\Omega_1||\Omega_2|}\int_0^T \int_{\Omega_1\times\Omega_2} r^s_D(t,x,y)\,dx\,dy\,dt.$$

Therefore, thanks to [10, Theorem 6] we obtain that

$$\tilde{\Phi}^{\Omega_1,\Omega_2}_D(s,T) \in (0,+\infty). \tag{3.35}$$

Also, $r^s_D(t,x,y)$ is continuous for $s \in (0,1]$ for all $(t,x,y) \in (0,+\infty)\times\Omega\times\Omega$, see, e.g., [10, Theorem 5]. Thanks to [11, Proposition 6 and Lemma 6], we have that, for each $t \in (0,+\infty)$ and $\varepsilon \in (0,1)$, the kernel $r^s_D(t,x,y)$ is uniformly bounded in $(s,x,y) \in (\varepsilon,1]\times\Omega\times\Omega$. Thus, as a consequence of the dominated convergence theorem we obtain that

$$f(s,t) := \int_{\Omega_1\times\Omega_2} r^s_D(t,x,y)\,dx\,dy$$

is continuous in $s \in (0,1]$.

Additionally, in view of [10, Theorem 6], we see that

$$|f(s,t)| \leqslant |\Omega_2| \quad \text{for all } (s,t) \in (0,1)\times(0,+\infty),$$

and therefore by the dominated convergence theorem we obtain the continuity of $\widetilde{\Phi}_D^{\Omega_1,\Omega_2}$ for $s \in (0,1]$.

Finally, the continuity of the functionals in (1.9) with respect to $s \in (0,1]$ follows from (3.34) and the fact that $\widetilde{\Phi}_D^{\Omega_1,\Omega_2}(\cdot, T) \in C((0,1])$ and $\widetilde{l}_D^{\Omega_2}(\cdot, T)$, $\widetilde{\mathcal{A}}_D^{\Omega_2}(\cdot, T) \in C((0,1])$. ∎

Now we prove Theorem 1.14. In this result we show that $s = 0$ is a minimizer for the functional $\widetilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}$, where Ω_1 and Ω_2 are disjoint and smooth. Also, we show that $\widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}$ and $\widetilde{\mathcal{E}}_{3,D}^{\Omega_1,\Omega_2}$ admit a positive and finite limit for $s \searrow 0$.

Proof of Theorem 1.14. Let $T \in (0,+\infty)$ and $\Omega_1, \Omega_2 \Subset \Omega$ be disjoint and smooth. Then, thanks to Lemma 2.14, we obtain that

$$\lim_{s \searrow 0} \widetilde{\mathcal{E}}_{1,D}^{\Omega_1,\Omega_2}(s,T) = 0.$$

Furthermore, thanks to (3.35), we see that $\mathcal{E}_{1,D}^{\Omega_1,\Omega_2}(s,T) \in (0,+\infty)$ for all $s \in (0,1]$. This latter observation together with the above limit lead to (1.21).

Now we prove (1.22). The existence of the limits in (1.22) was already established in Lemma 2.14. Using the fact that Ω_1 and Ω_2 are disjoint and smooth, together with the inequality in (D.6) and also (3.19), we evince that

$$\widetilde{F}_D(\Omega_1,\Omega_2) \in (0,+\infty), \qquad \int_{\Omega\times\Omega_2} |\xi - y| F_D(\xi, y)\, d\xi\, dy \in (0,+\infty)$$

and

$$\int_{\Omega\times\Omega_2} |\xi - y|^2 F_D(\xi, y)\, d\xi\, dy \in (0,+\infty),$$

where F_D and $\widetilde{F}_D$ are given respectively in (2.62) and (2.72). Therefore, from (2.70), (2.71), and these considerations we conclude the proof of (1.22). ∎

Now we focus our attention on Theorems 1.15 and 1.16. To prove these results, it is useful to state and prove the following proposition regarding a monotonicity property with respect to s and a scaling property for the functional F^{Ω_1,Ω_2} introduced in (2.53). In what follows we denote by d_B the diameter of B for each bounded set $B \subset \mathbb{R}^n$.

Proposition 3.1. *Let $K \subset \mathbb{R}^n$ be a compact set and $\Omega_1, \Omega_2 \subset K$ be measurable sets such that $\Omega_1 \cap \Omega_2 = \varnothing$. Then, if $d_K \leqslant 1$, we have that*

$$\frac{d}{ds} F^{\Omega_1,\Omega_2}(s) \leqslant 0 \quad \textit{for all } s \in (0,1). \tag{3.36}$$

Moreover, for each $r \in (0,+\infty)$ and $y \in \mathbb{R}^n$, it holds that

$$F^{r_y\Omega_1, r_y\Omega_2}(s) = r^{n+2s} F^{\Omega_1,\Omega_2}(s). \tag{3.37}$$

Proof. We observe that, thanks to the dominated convergence theorem,

$$\frac{d}{ds}F^{\Omega_1,\Omega_2}(s) = 2\int_{\Omega_1\times\Omega_2} \frac{\ln|x-y|}{|x-y|^{n-2s}}\,dx\,dy.$$

Hence, if $d_K \leqslant 1$, then

$$\frac{d}{ds}F^{\Omega_1,\Omega_2}(s) \leqslant 0,$$

which proves (3.36).

Now we show the scaling property in (3.37). Let $r\in(0,+\infty)$ and, up to a translation, assume that $y=0$. Then, applying the change of variable

$$(x,y)=(rX,rY)$$

we obtain that

$$\begin{aligned} F^{r\Omega_1,r\Omega_2}(s) &= \int_{r\Omega_1\times r\Omega_2} \frac{1}{|x-y|^{n-2s}}\,dx\,dy \\ &= \int_{\Omega_1\times\Omega_2} \frac{r^{2n}}{r^{n-2s}|X-Y|^{n-2s}}\,dX\,dY \\ &= r^{n+2s}F^{\Omega_1,\Omega_2}(s), \end{aligned}$$

which completes the proof. ∎

With this preliminary work, we can now prove Theorems 1.15 and 1.16. We recall that the aim of this result is to show that if $\Omega_1,\Omega_2\subset\Omega$ are disjoint, smooth and close enough, then the best search strategy for the set efficiency functionals provided in (1.9) is in a small neighborhood of $s=0$.

Proof of Theorems 1.15 and 1.16. Let $(y,T)\in\Omega\times(0,+\infty)$. If $\Omega_1,\Omega_2\subset\Omega$ are smooth and disjoint, then thanks to Theorem 1.14 we have that

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,D}(s^{(1)}_{\Omega_1,\Omega_2,T},T) \quad \text{with } s^{(1)}_{\Omega_1,\Omega_2,T}\in(0,1]. \tag{3.38}$$

Moreover, If P and $\widetilde{P}$ are given as in (2.93), we observe that

$$\inf_{\Omega_2\subset\Omega} P(\Omega_2)\in(0,+\infty) \quad \text{and} \quad \inf_{\Omega_2\subset\Omega} \widetilde{P}(\Omega_2)\in(0,+\infty).$$

Now, using (2.55) we have that, for $s_0\in(0,\frac{1}{2})$ and $r\in(0,\widehat{\delta})$, where $\widehat{\delta}=\widehat{\delta}_{s_0,y,T,\Omega}$ has been given in (2.27), then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_N(s_0,T) \geqslant \frac{C_{s_0,y,\Omega}}{(2r)^{n-2s_0}},$$

for all $\Omega_1,\Omega_2\subset B_r(y)$, where $C_{s_0,y,\Omega}$ is given (2.32).

Consequently, using also (2.90), (2.91), and (2.92), we deduce that there exists some $\beta = \beta_{y,T,\Omega} \in (0,1)$ such that if $\Omega_1, \Omega_2 \subset B_\beta(y)$ are smooth and disjoint, then

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{j,N}(s^{(j)}_{\Omega_1,\Omega_2,T},T) \quad \text{with } s^{(j)}_{\Omega_1,\Omega_2,T} \in (0,1], \tag{3.39}$$

for all $j \in \{1,2,3\}$.

We will first prove the results for $n \leqslant 2$.

We recall that, by Corollary 2.11, if $s_1 \in (0,1)$ and $\mu \in A_{n,s_1}$, where A_{n,s_1} is given in (2.43), then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_*(s_1,T) \leqslant \frac{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}{|\Omega_1||\Omega_2|} F^{\Omega_1,\Omega_2}\Big(\frac{n-2\mu}{2}\Big), \tag{3.40}$$

for all $\Omega_1, \Omega_2 \Subset B_{\widehat{\delta}}(y)$, where $C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}$ is introduced in Theorem 2.9.

Also, in light of (2.55) we deduce that if $s_0 \in (0,\frac{1}{2})$ and $\Omega_1, \Omega_2 \Subset B_{\widehat{\delta}}(y)$, then

$$\widetilde{\Phi}^{\Omega_1,\Omega_2}_*(s_0,T) \geqslant \frac{C_{s_0,y,\Omega}}{|\Omega_1||\Omega_2|} F^{\Omega_1,\Omega_2}(s_0). \tag{3.41}$$

Now, we define

$$\delta_0 := \min\Big\{\widehat{\delta}, \frac{1}{2}\Big\},$$

and we consider $\Omega_1, \Omega_2 \subset B_{\delta_0}(y)$ smooth and such that

$$\Omega_1 \cap \Omega_2 = \varnothing.$$

Thus, from (1.9), (3.36), (3.37), (3.40), and (3.41) we deduce that if $r \in (0,1)$, $s_0 \in (0,\frac{1}{2})$, $s_1 \in (s_0,1)$ and $\mu \in (0,\frac{n}{2}-s_0) \cap A_{n,s_1}$, where A_{n,s_1} is given as in (2.43), it holds that

$$\begin{aligned}
\frac{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{1,*}(s_0,T)}{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{1,*}(s_1,T)} &= \frac{\widetilde{\Phi}^{r_y\Omega_1,r_y\Omega_2}_*(s_0,T)}{\widetilde{\Phi}^{r_y\Omega_1,r_y\Omega_2}_*(s_1,T)} \\
&\geqslant C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega} \frac{F^{r_y\Omega_1,r_y\Omega_2}(s_0)}{F^{r_y\Omega_1,r_y\Omega_2}(\frac{n-2\mu}{2})} \\
&\geqslant C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega} \frac{r^{n+2s_0} F^{\Omega_1,\Omega_2}(s_0)}{r^{2n-2\mu} F^{\Omega_1,\Omega_2}(\frac{n-2\mu}{2})} \\
&\geqslant \frac{C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega}}{r^{n-2s_0-2\mu}},
\end{aligned} \tag{3.42}$$

where we defined

$$C^{(1)}_{s_0,*,\mu,B_{\widehat{\delta}}(y),y,T,\Omega} := \frac{C_{s_0,y,\Omega}}{C_{*,\mu,B_{\widehat{\delta}}(y),T,\Omega}}.$$

We recall that in writing $r_y\Omega_1$ and $r_y\Omega_2$ we adopted the notation in (1.23).

As a result, for all $\varepsilon \in (0,1)$, by choosing for instance $s_0 := \frac{\varepsilon}{4}$ and $\mu := (n-\varepsilon)/2$ in (3.42), and using also (3.38) and (3.39), we infer that there exists some $r^{(1)} = r^{(1)}_{\varepsilon,y,T,\Omega}$ such that if $\Omega_1, \Omega_2 \Subset B_{r^{(1)}\delta_0}(y)$ are smooth and satisfy

$$\Omega_1 \cap \Omega_2 = \varnothing,$$

then

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,*}(s,T) = \widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{1,*}(s^{(1)}_{*,\Omega_1,\Omega_2,T},T) \quad \text{with } s^{(1)}_{*,\Omega_1,\Omega_2,T} \in (0,\varepsilon).$$

We now focus on the proof of (1.24) for $\widetilde{\mathcal{E}}_{2,D}$. Let $K \Subset \Omega$ and assume that $\Omega_1, \Omega_2 \subset K$. Then, thanks to equations (2.70) and (D.5) we have that

$$\widetilde{\mathcal{E}}^{\Omega_1,\Omega_2}_{2,D}(0,T) \geqslant \frac{\tilde{c}_{K,\Omega}}{\int_{\Omega\times\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy}\,\frac{F^{\Omega_1,\Omega_2}(0)}{|\Omega_1|},$$

where F_D and $\tilde{c}_{K,\Omega}$ are given respectively in (2.62) and (D.7). Then, in light of (2.54) and (3.40), if $s \in (0,1)$ and $\mu \in A_{n,s} \cap (0,1)$, where $A_{n,s}$ is given in (2.43), we have that, for each $r \in (0,1)$,

$$\begin{aligned}
&\frac{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{2,D}(0,T)}{\widetilde{\mathcal{E}}^{r_y\Omega_1,r_y\Omega_2}_{2,D}(s,T)} \\
&\quad= \frac{|r_y\Omega_2|\widetilde{F}_D(r_y\Omega_1,r_y\Omega_2)}{\int_{\Omega\times r_y\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy}\,\frac{\tilde{l}^{r_y\Omega_2}_D(s,T)}{\widetilde{\Phi}^{r_y\Omega_1,r_y\Omega_2}_D(s,T)} \\
&\quad\geqslant \frac{|r_y\Omega_2|\tilde{c}_{K,\Omega}\Gamma(s)}{\int_{\Omega\times r_y\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy}\,\frac{\tilde{l}^{r_y\Omega_2}_D(s,T)}{C_{D,\mu,K,T,\Omega}}\,\frac{F^{r_y\Omega_1,r_y\Omega_2}(0)}{F^{r_y\Omega_1,r_y\Omega_2}(\frac{n-2\mu}{2})},
\end{aligned} \tag{3.43}$$

where $C_{D,\mu,K,T,\Omega}$ was introduced in Theorem 2.9.

Now, we observe that thanks to the limit in equation (2.75) one has that

$$\begin{aligned}
&\lim_{s\searrow 0} \tilde{l}^{r_y\Omega_2}_D(s,T)\Gamma(s) \\
&\qquad= \lim_{s\searrow 0} \frac{\tilde{l}^{r_y\Omega_2}_D(s,T)}{s}\Gamma(s)s \\
&\qquad= \frac{(1-e^{-T}(T+1))}{|r_y\Omega_2|} \int_{\Omega\times r_y\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy.
\end{aligned} \tag{3.44}$$

Let us set the notation

$$C_{\mu,K,T,r_y\Omega_2,\Omega} := \inf_{s\in(0,1)} \frac{|r_y\Omega_2|\tilde{c}_{K,\Omega}\Gamma(s)}{\int_{\Omega\times r_y\Omega_2} |\xi-y| F_D(\xi,y)\,d\xi\,dy}\,\frac{\tilde{l}^{r_y\Omega_2}_D(s,T)}{C_{D,\mu,K,T,\Omega}}.$$

In view of (3.44), we see that if such infimum is attained at $s = 0$, then it does not depend on $r_y\Omega_2$.

If the infimum is attained for some $\hat{s} \in (0, 1]$, then using Proposition 2.12 and Lemma D.3 with

$$f(y) := l_D^y(\hat{s}, T) \quad \text{and} \quad g(y) = \int_\Omega |\xi - y| F_D(\xi, y)\, d\xi,$$

we obtain that

$$C_{\mu,K,T,\Omega} := \inf_{\substack{r\in(0,1)\\ \Omega_2\subset K}} C_{\mu,K,T,r_y\Omega_2,\Omega} > 0.$$

As a result, using equation (3.43) and Proposition 3.1, we deduce that if $d_K \leqslant 1$, then

$$\frac{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(0,T)}{\widetilde{\mathcal{E}}_{2,D}^{r_y\Omega_1,r_y\Omega_2}(s,T)} \geqslant C_{\mu,K,T,\Omega} \frac{F^{r_y\Omega_1,r_y\Omega_2}(0)}{F^{r_y\Omega_1,r_y\Omega_2}(\frac{n-2\mu}{2})} \geqslant \frac{C_{\mu,K,T,\Omega}}{r^{n-2\mu}}. \tag{3.45}$$

Therefore, for all $\varepsilon \in (0, 1)$ and $K \Subset \Omega$ that are start-shaped with respect to $y \in K$, by choosing $s \in (\varepsilon, 1)$ and $\mu := (n - \varepsilon)/2$ in (3.45), we deduce the existence of some $r^{(2)} = r^{(2)}_{\varepsilon,K,T,\Omega}$ such that if $\Omega_1, \Omega_2 \subset r_y^{(1)} K$ satisfy

$$\Omega_1 \cap \Omega_2 = \varnothing$$

and are smooth, then

$$\widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(0,T) \geqslant \sup_{s\in(\varepsilon,1)} \widetilde{\mathcal{E}}_{2,D}^{\Omega_1,\Omega_2}(s,T).$$

This concludes the proof of (1.24) for $\widetilde{\mathcal{E}}_{2,D}$. The proof of (1.24) for $\widetilde{\mathcal{E}}_{3,D}$ will be omitted, being analogous to the one for $\widetilde{\mathcal{E}}_{2,D}$.

We now prove (1.25) for $\widetilde{\mathcal{E}}_{2,N}$. To do so, we fix some $s_0 \in (0, \frac{1}{2})$, and, in light of Proposition 2.12, we define the positive constant

$$C_{s_0,T,\Omega_2,\Omega} := \inf_{s_1\in(s_0,1)} \frac{\tilde{l}_N^{\Omega_2}(s_1,T)}{\tilde{l}_N^{\Omega_2}(s_0,T)}.$$

Also, if the above infimum is attained for some $\hat{s} \in [s_0, 1]$, using Lemma D.3 with

$$f(y) = l_N^y(\hat{s}, T) \quad \text{and} \quad g(y) = l_N^y(s_0, T),$$

we set

$$C_{s_0,T,\Omega} := \inf_{\Omega_2\Subset\Omega} C_{s_0,T,\Omega_2,\Omega} > 0.$$

Thus, making use of equations (3.36), (3.37), (3.40), and (3.41), we deduce that if $\Omega_1, \Omega_2 \Subset B_{\delta_0}(y)$, and $K \supset B_{\delta_0}(y)$, $r \in (0, 1)$, $s_0 \in (0, \frac{1}{2})$, $s_1 \in (s_0, 1)$, and

$\mu \in A_{n,s_1} \cap (0, \frac{n}{2} - s_0)$, we have that

$$\frac{\widetilde{\mathcal{E}}_{2,N}^{r_y\Omega_1,r_y\Omega_2}(s_0,T)}{\widetilde{\mathcal{E}}_{2,N}^{r_y\Omega_1,r_y\Omega_2}(s_1,T)} = \frac{\widetilde{\Phi}_N^{r_y\Omega_1,r_y\Omega_2}(s_0,T)}{\tilde{l}_N^{r_y\Omega_2}(s_0,T)} \frac{\tilde{l}_N^{r_y\Omega_2}(s_1,T)}{\widetilde{\Phi}_N^{r_y\Omega_1,r_y\Omega_2}(s_1,T)}$$
$$\geqslant C^{(1)}_{\mu,s_0,y,K,T,\Omega} \frac{F^{r_y\Omega_1,r_y\Omega_2}(s_0)}{F^{r_y\Omega_1,r_y\Omega_2}(\frac{n-2\mu}{2})}$$
$$= C^{(1)}_{\mu,s_0,y,K,T,\Omega} \frac{r^{n+2s_0} F^{\Omega_1,\Omega_2}(s_0)}{r^{2n-2\mu} F^{\Omega_1,\Omega_2}(\frac{n-2\mu}{2})}$$
$$\geqslant \frac{C^{(1)}_{\mu,s_0,y,K,T,\Omega}}{r^{n-2s_0-2\mu}}, \tag{3.46}$$

where we defined

$$C^{(1)}_{\mu,s_0,y,K,T,\Omega} := \frac{C_{s_0,y,\Omega}}{C_{N,\mu,B_{\hat{\delta}}(y),T,\Omega}} C_{s_0,T,\Omega}.$$

Therefore, for each $\varepsilon \in (0,1)$, by choosing for instance $s_0 := \frac{\varepsilon}{4}$, $s_1 \in (\varepsilon, 1)$ and $\mu := (n-\varepsilon)/2$ in (3.46), and also thanks to (3.39), we deduce that there exists some $r^{(2)} = r^{(2)}_{\varepsilon,y,T,\Omega} \in (0,\beta)$ such that, for each $\Omega_1, \Omega_2 \subset B_{r^{(2)}\delta_0}(y)$ smooth and disjoint,

$$\sup_{s\in(0,1)} \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s,T) = \widetilde{\mathcal{E}}_{2,N}^{\Omega_1,\Omega_2}(s^{(2)}_{\Omega_1,\Omega_2,T},T) \quad \text{with } s^{(2)}_{\Omega_1,\Omega_2,T} \in (0,\varepsilon).$$

This concludes the proof of (1.25) for $\widetilde{\mathcal{E}}_{2,N}$. The proof of (1.25) for $\widetilde{\mathcal{E}}_{3,N}$ is analogous to the one for $\widetilde{\mathcal{E}}_{2,N}$ just concluded and therefore it will be omitted.

This concludes the proof of Theorems 1.15 and 1.16 for $n \leqslant 2$.

Few changes are in order to show Theorems 1.15 and 1.16 also for $n \geqslant 3$. In particular, we have to repeat the above arguments by replacing (3.40) with the inequality in (2.54). The procedure will determine changes only on the constants involved, in the same fashion of the proof of Theorems 1.7 and 1.8 for $n \geqslant 3$. ∎