Appendix A

Green function for the Dirichlet spectral fractional Laplacian

Here, we give a proof of a well-known identity for the Green function $G^s_D(x, y)$ of the Dirichlet spectral fractional Laplacian. The Green function is given by

$$G^s_D(x, y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} p^{\Omega}_D(t, x, y) t^{s-1} dt,$$

see also [1]. Before we state the following result, let us recall the notation

$$\mathcal{C} = \{(x, y) \in \Omega \times \Omega \text{ s.t. } x \neq y\}.$$

Proposition A.1. *Let $\Omega \subset \mathbb{R}^n$ be bounded, smooth and connected. Then, for each $(x, y) \in \mathcal{C}$ it holds that*

$$\int_0^{+\infty} r^s_D(t, x, y) dt = \frac{1}{\Gamma(s)} \int_0^{+\infty} p^{\Omega}_D(t, x, y) t^{s-1} dt. \tag{A.1}$$

Proof. Given $x, y \in \mathcal{C}$, we let

$$\mathcal{I}(x, y) := \int_0^{+\infty} r^s_D(t, x, y) dt,$$
$$\mathcal{J}(x, y) := \frac{1}{\Gamma(s)} \int_0^{+\infty} p^{\Omega}_D(t, x, y) t^{s-1} dt.$$

Now, let $\{\phi_k\}_k$ be an orthonormal basis of $L^2(\Omega)$ made of eigenfunctions of the Laplacian with Dirichlet boundary conditions, ordered such that if λ_k's are the corresponding eigenvalues, then $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ (see, for instance, [18]). In view of [10, Theorem 5] we know that

$$r^s_D(t, x, y) = \sum_{k=1}^{+\infty} \phi_k(x)\phi_k(y) \exp(-t\lambda^s_k)$$

for each $(t, x, y) \in (0, +\infty) \times \Omega \times \Omega$. In order to prove (A.1), we first show that $\mathcal{I}(x, y)$ and $\mathcal{J}(x, y)$ are both continuous in $\mathcal{C}$. Thanks to Theorem 2.8 we know that

$$\int_0^T r^s_D(t, x, y) dy < +\infty$$

for each $T \in (0, +\infty)$, $s \in (0, 1]$ and $x \neq y$. Moreover, thanks to [11, Proposition 6] we observe that for each $t > T$ and $s \in (0, 1]$ it holds that

$$\begin{aligned} r_D^s(t, x, y) &= \exp(-t\lambda_1^s) \sum_{k=1}^{+\infty} \phi_k(x)\phi_k(y) \exp(-t(\lambda_k^s - \lambda_1^s)) \\ &\leqslant c_{m_0,\Omega,0} \exp(-t\lambda_1^s) \sum_{k=1}^{+\infty} \lambda_k^{2\alpha(m_0)} \exp(-T(\lambda_k^s - \lambda_1^s)) \\ &\leqslant C_{T,s,\Omega} \exp(-t\lambda_1^s), \end{aligned} \tag{A.2}$$

where the last inequality is a consequence of [11, Lemma 7], and $C_{T,s,\Omega} > 0$ is a constant depending on $T > 0$, $s \in (0, 1]$ and Ω. The constants $\alpha(m_0)$ and $c_{m_0,\Omega,0}$ have been explicitly defined in [11, Proposition 6]. Therefore, if we call

$$g_D^s(t, x, y) := \begin{cases} r_D^s(t, x, y) & \text{for all } (t, x, y) \in (0, T] \times \mathcal{C}, \\ C_{T,s,\Omega} \exp(-t\lambda_1^s) & \text{for all } (t, x, y) \in (T, +\infty) \times \mathcal{C}, \end{cases}$$

we obtain that

$$g_D^s(t, x, y) \in L^1(0, +\infty)$$

for each $(x, y) \in \mathcal{C}$, and also

$$r_D^s(t, x, y) \leqslant g_D^s(t, x, y)$$

for each $(t, x, y) \in (0, +\infty) \times \Omega \times \Omega$ and $s \in (0, 1]$. Therefore, thanks to the continuity of the kernel r_D^s discussed in [10, Lemma 2], we conclude by the dominated convergence theorem that $\mathcal{I}(\cdot, \cdot)$ is continuous in $\mathcal{C}$.

Furthermore, thanks to the inequalities in (2.14) and (A.2) we have that if we define

$$f_D(t, x, y) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\dfrac{|x - y|^2}{4t} \right) & \text{for all } (t, x, y) \in (0, T] \times \mathcal{C}, \\ C_{T,1,\Omega} \exp(-t\lambda_1) & \text{for all } (t, x, y) \in (T, +\infty) \times \mathcal{C}, \end{cases}$$

then we get that

$$f_D(t, x, y)t^{s-1} \in L^1(0, +\infty)$$

for each $(x, y) \in \mathcal{C}$, and also

$$p_D^\Omega(t, x, y)t^{s-1} \leqslant f_D(t, x, y)t^{s-1}$$

for each $(t, x, y) \in (0, +\infty) \times \mathcal{C}$. Thanks to the continuity of p_D^Ω (see for instance [10, Lemma 2]) and the last observations we can apply the dominated convergence theorem and conclude that $\mathcal{J}(\cdot, \cdot) \in C(\mathcal{C})$.

Now, let $f \in C_c^\infty(\Omega)$ such that $f \geqslant 0$. Then, for each $x \in \Omega$ we compute

$$\begin{aligned}\int_\Omega \mathcal{I}(x,y) f(y)dy &= \int_\Omega \int_0^{+\infty} r_D^s(t,x,y) f(y)dt\, dy \\ &= \int_0^{+\infty} \int_\Omega r_D^s(t,x,y) f(y) dy\, dt \\ &= \int_0^{+\infty} \sum_{k=1}^{+\infty} f_k \phi_k(x) \exp(-t\lambda_k^s) dt \\ &= \sum_{k=1}^{+\infty} \frac{f_k \phi_k(x)}{\lambda_k^s}. \end{aligned} \tag{A.3}$$

In the above computation, we denoted

$$f_k := \int_\Omega f(y)\phi_k(y)dy,$$

and the identity between the first and the second line, as well as between the second and the third, are due to [11, Lemma 6]; in addition, the estimates on the coefficients f_k given in [11, Proposition 7].

Similarly, we also observe that

$$\begin{aligned}\int_\Omega \mathcal{J}(x,y) f(y)dy &= \frac{1}{\Gamma(s)} \int_\Omega \int_0^{+\infty} p_D^\Omega(t,x,y) t^{s-1} f(y) dt\, dy \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} \int_\Omega p_D^\Omega(t,x,y) f(y) dy\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^{+\infty} \sum_{k=1}^{+\infty} f_k \phi_k(x) \exp(-t\lambda_k) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \sum_{k=1}^{+\infty} f_k \phi_k(x) \int_0^{+\infty} \exp(-t\lambda_k) t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \sum_{k=1}^{+\infty} f_k \phi_k(x) \frac{\Gamma(s)}{\lambda_k^s} = \sum_{k=1}^{+\infty} \frac{f_k \phi_k(x)}{\lambda_k^s}. \end{aligned} \tag{A.4}$$

Therefore, from equations (A.3) and (A.4) we deduce that for each $x \in \Omega$ and $f \in C_c^\infty(\Omega)$ such that $f \geqslant 0$ it holds

$$\int_\Omega (\mathcal{I}(x,y) - \mathcal{J}(x,y)) f(y) dy = 0.$$

Thanks to this latter identity and the fact that $\mathcal{J}, \mathcal{I} \in C(\mathscr{C})$ we conclude the proof of (A.1). ∎