Appendix B

Some explicit formulas for the one-dimensional functionals

Lemma B.1. *We have that*

$$l_D^y(s,T) = 2a^{1+2s}\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k a}+\frac{y(-1)^k}{\pi k a}-\frac{2}{(\pi k)^2}\sin\left(\frac{\pi k y}{a}\right)\right)$$
$$\times\sin\left(\frac{\pi k y}{a}\right)\left(\frac{1-\exp(-T(\frac{\pi k}{a})^{2s})}{(\pi k)^{2s}}\right), \tag{B.1}$$

$$\mathcal{A}_D^y(s,T)$$
$$= 2a^{2+2s}\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k a^2}+\frac{y^2}{\pi k a^2}-\frac{2y(-1)^k}{\pi k a}\right)$$
$$\times\sin\left(\frac{\pi k y}{a}\right)\left(\frac{1-\exp\left(-T(\frac{\pi k}{a})^{2s}\right)}{(\pi k)^{2s}}\right). \tag{B.2}$$

Proof. The gist to obtain explicit formulas for the average distance $l_D^y(s,T)$ and the mean square displacement $\mathcal{A}_D^y(s,T)$ is to compute the $L^2((0,a))$ components of the decomposition in eigenfunctions of the functions $|x-y|$ and $(x-y)^2$. For this, it is first useful to consider the case $a := 1$ and then to reduce to it via a scaling argument. Thus, we first suppose that $a = 1$ and note that

$$\int_0^1 |x-y|\sin(\pi k x)dx = \frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi k y)}{(\pi k)^2},$$
$$\int_0^1 (x-y)^2\sin(\pi k x)dx = \frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}$$
$$+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}.$$

Therefore, $l_D^y(s,T)$ and $\mathcal{A}_D^y(s,T)$ take the form

$$l_D^y(s,T) = 2\int_0^T\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi k y)}{(\pi k)^2}\right)$$
$$\times\sin(\pi k y)\exp(-(\pi k)^{2s}t)dt$$
$$= 2\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k}+\frac{y(-1)^k}{\pi k}-2\frac{\sin(\pi k y)}{(\pi k)^2}\right)$$
$$\times\sin(\pi k y)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right) \tag{B.3}$$

and

$$\begin{aligned}
&\mathcal{A}_D^y(s,T)\\
&= 2\int_0^1 \sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}\right)\\
&\quad\times \sin(\pi k y)\exp(-(\pi k)^{2s}t)dt\\
&= 2\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k}+\frac{2(-1)^k}{(\pi k)^3}-\frac{2}{(\pi k)^3}-\frac{y^2(-1)^k}{\pi k}+\frac{y^2}{\pi k}-\frac{2y(-1)^k}{\pi k}\right)\\
&\quad\times \sin(\pi k y)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right),
\end{aligned} \tag{B.4}$$

which is the desired result for $a = 1$.

Now we address the case of a general $a > 0$. To this end, we denote with an additional subscript a the quantities related to the interval $(0, a)$ (and, consistently, with an additional subscript 1 the quantities related to the interval $(0, 1)$). With this notation, we infer from (3.1), (3.2) and (3.3) that

$$\phi_{k,a}(x) = \frac{1}{\sqrt{a}}\phi_{k,1}\left(\frac{x}{a}\right), \quad \lambda_{k,a} = \frac{\lambda_{k,1}}{a^2}, \quad r^s_{D,a}(t,x,y) = \frac{1}{a}\, r^s_{D,1}\left(\frac{t}{a^{2s}}, \frac{x}{a}, \frac{y}{a}\right).$$

As a consequence, by (1.3),

$$\begin{aligned}
l^y_{D,a}(s,T) &= \int_0^T\int_0^a |\zeta - y| r^s_{D,a}(t,\zeta,y)d\zeta\, dt\\
&= \int_0^T\int_0^a \left|\frac{\zeta}{a}-\frac{y}{a}\right| r^s_{D,1}\left(\frac{t}{a^{2s}},\frac{\zeta}{a},\frac{y}{a}\right)d\zeta\, dt\\
&= a^{1+2s}\int_0^{T/a^{2s}}\int_0^1 \left|\tilde\zeta-\frac{y}{a}\right| r^s_{D,1}\left(\tilde t,\tilde\zeta,\frac{y}{a}\right)d\tilde\zeta\, d\tilde t\\
&= a^{1+2s}\, l^{y/a}_{D,1}\left(s,\frac{T}{a^{2s}}\right).
\end{aligned} \tag{B.5}$$

This and (B.3) yield that

$$\begin{aligned}
l^y_{D,a}(s,T) = 2a^{1+2s}\sum_{k=1}^{+\infty}&\left(\frac{(-1)^{k+1}}{\pi k}+\frac{y}{\pi k a}+\frac{y(-1)^k}{\pi k a}-\frac{2}{(\pi k)^2}\sin\left(\frac{\pi k y}{a}\right)\right)\\
&\times \sin\left(\frac{\pi k y}{a}\right)\left(\frac{1-\exp(-T(\frac{\pi k}{a})^{2s})}{(\pi k)^{2s}}\right)
\end{aligned}$$

and this gives (B.1), as desired.

Furthermore, by (1.4),

$$\begin{aligned}\mathcal{A}^y_{D,a}(s,T) &= \int_0^T \int_0^a |\zeta - y|^2\, r^s_{D,a}(t,\zeta,y)\,d\zeta\,dt\\ &= \frac{1}{a}\int_0^T \int_0^a |\zeta - y|^2\, r^s_{D,1}\left(\frac{t}{a^{2s}},\frac{\zeta}{a},\frac{y}{a}\right)d\zeta\,dt\\ &= a^{2+2s}\int_0^{T/a^{2s}} \int_0^1 \left|\tilde{x} - \frac{y}{a}\right|^2 r^s_{D,1}\left(\tilde{t},\tilde{x},\frac{y}{a}\right)d\tilde{x}\,d\tilde{t}\\ &= a^{2+2s}\,\mathcal{A}^{y/a}_{D,1}\left(s,\frac{T}{a^{2s}}\right).\end{aligned} \tag{B.6}$$

Thus, recalling (B.4),

$$\begin{aligned}&\mathcal{A}^y_{D,a}(s,T)\\ &= 2a^{2+2s}\sum_{k=1}^{+\infty}\left(\frac{(-1)^{k+1}}{\pi k} + \frac{2(-1)^k}{(\pi k)^3} - \frac{2}{(\pi k)^3} - \frac{y^2(-1)^k}{\pi k a^2} + \frac{y^2}{\pi k a^2} - \frac{2y(-1)^k}{\pi k a}\right)\\ &\quad\times \sin\left(\frac{\pi k y}{a}\right)\left(\frac{1-\exp\big(-T(\frac{\pi k}{a})^{2s}\big)}{(\pi k)^{2s}}\right),\end{aligned}$$

which proves (B.2), as desired. ■

Additionally, the Neumann counterpart of Lemma B.1 reads as follows.

Lemma B.2. *We have that*

$$\begin{aligned}l^y_N(s,T) &= aT\left(\frac{1}{2} + \frac{y^2}{a^2} - \frac{y}{a}\right)\\ &\quad + 2a^{1+2s}\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2} - \frac{2}{(\pi k)^2}\cos\left(\frac{\pi k y}{a}\right) + \frac{(-1)^k}{(\pi k)^2}\right)\\ &\quad\times\cos\left(\frac{\pi k y}{a}\right)\frac{1-\exp\big(-T(\frac{\pi(2k+1)}{a})^{2s}\big)}{(\pi(2k+1))^{2s}},\\ \mathcal{A}^y_N(s,T) &= a^2 T\left(\frac{1}{3} + \frac{y^2}{a^2} - \frac{y}{a}\right)\\ &\quad + 2a^{2+2s}\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2} - \frac{2y}{a}\Big(\frac{(-1)^k}{(\pi k)^2} - \frac{1}{(\pi k)^2}\Big)\right)\\ &\quad\times\cos\left(\frac{\pi k y}{a}\right)\frac{1-\exp\big(-T(\frac{\pi k}{a})^{2s}\big)}{(\pi k)^{2s}}.\end{aligned}$$

Proof. As in the proof of Lemma B.1, we can focus on the case $a := 1$, since the general case then would follow from scaling. Thus, we consider the coefficients of the

$L^2((0,1))$ expansion of the functions $|x-y|$ and $(x-y)^2$ in terms of the Neumann eigenfunctions, thus finding that

$$\int_0^1 |x-y|\cos(\pi k x)dx = \begin{cases} \frac{1}{2} + y^2 - y & \text{if } k = 0, \\ \frac{1}{(\pi k)^2} - 2\frac{\cos(\pi k y)}{(\pi k)^2} + \frac{(-1)^k}{(\pi k)^2} & \text{if } k \neq 0, \end{cases}$$

$$\int_0^1 (x-y)^2\cos(\pi k x)dx = \begin{cases} \frac{1}{3} - y + y^2 & \text{if } k = 0, \\ \frac{2}{(\pi k)^2} - 2y\left(\frac{(-1)^k}{(\pi k)^2} - \frac{1}{(\pi k)^2}\right) & \text{if } k \neq 0. \end{cases}$$

Therefore, $l_N^y(s,T)$ and $\mathcal{A}_N^y(s,T)$ take the form

$$\begin{aligned} l_N^y(s,T) &= \int_0^T \left(\frac{1}{2} + y^2 - y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2} - 2\frac{\cos(\pi k y)}{(\pi k)^2} + \frac{(-1)^k}{(\pi k)^2}\right) \\ &\quad \times \cos(\pi k y)\exp(-t(\pi(2k+1))^{2s})dt \\ &= T\left(\frac{1}{2} + y^2 - y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{1}{(\pi k)^2} - 2\frac{\cos(\pi k y)}{(\pi k)^2} + \frac{(-1)^k}{(\pi k)^2}\right) \\ &\quad \times \cos(\pi k y)\left(\frac{1-\exp(-T(\pi(2k+1))^{2s})}{(\pi(2k+1))^{2s}}\right), \\ \mathcal{A}_N^y(s,T) &= \int_0^T 2\left(\frac{1}{3} + y^2 - y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2} - 2y\left(\frac{(-1)^k}{(\pi k)^2} - \frac{1}{(\pi k)^2}\right)\right) \\ &\quad \times \cos(\pi k y)\exp(-t(\pi k)^{2s})dt \\ &= T\left(\frac{1}{3} + y^2 - y\right) + 2\sum_{k=1}^{+\infty}\left(\frac{2}{(\pi k)^2} - 2y\left(\frac{(-1)^k}{(\pi k)^2} - \frac{1}{(\pi k)^2}\right)\right) \\ &\quad \times \cos(\pi k y)\left(\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}}\right), \end{aligned}$$

as claimed. ∎