Appendix C

Alternative proof of Proposition 1.2

Here, we showcase an alternative proof of Proposition 1.2. The advantage of this argument is that it does not make use of the explicit formula (2.2) for the density μ_t^s of an s-stable subordinator. The details go as follows.

Proof of Proposition 1.2. As in the proof of Lemma B.1, we denote by an additional subscript a the quantities related to the interval $(0, a)$. In particular, by (3.4) and (3.7),

$$\Phi_{D,a}^{x,y}(s,T) = a^{2s-1}\Phi_{D,1}^{x/a,y/a}\left(s, \frac{T}{a^{2s}}\right), \quad \Phi_{N,a}^{x,y}(s,T) = a^{2s-1}\Phi_{N,1}^{x/a,y/a}\left(s, \frac{T}{a^{2s}}\right). \tag{C.1}$$

From this, (B.5) and (B.6) (and the corresponding scaling properties for the Neumann case), we deduce that it suffices to establish Proposition 1.2 for $a := 1$.

Hence, let $x = y \in \Omega = (0,1)$. We have that

$$\begin{aligned}\frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}} &+ \frac{1-\exp(-T(\pi(2k+1))^{2s})}{(\pi(2k+1))^{2s}}\\ &\leqslant \frac{1}{(\pi k)^{2s}} + \frac{1}{(\pi(2k+1))^{2s}} \leqslant \frac{2}{(\pi k)^{2s}}\end{aligned}$$

and, as a result, we obtain that the series in Lemmas B.1 and B.2 converge absolutely for all $s \in (0,1)$ and $T > 0$ and uniformly in s in every set of the form $(s_0, 1)$ with $s_0 \in (0,1)$.

Consequently, the convergence or divergence of $\mathcal{E}(s,T)$ in this case is equivalent to that of $\Phi_D^{x,x}(s,T)$ or $\Phi_N^{x,x}(s,T)$, depending on the boundary conditions considered. Hence, when $s \in (0,1/2]$, for all $M \in \mathbb{N}$, we infer from (3.4) that

$$\begin{aligned}\Phi_D^{x,x}(s,T) &\geqslant 2\sum_{k=1}^{M}\int_0^T \sin^2(\pi k x)\exp(-t(\pi k)^{2s})dt\\ &= 2\sum_{k=1}^{M}\frac{\sin^2(\pi k x)(1-\exp(-T(\pi k)^{2s}))}{(\pi k)^{2s}}\end{aligned} \tag{C.2}$$

and from (3.7) that

$$\begin{aligned}\Phi_N^{x,x}(s,T) &\geqslant 2\sum_{k=1}^{M}\int_0^T \cos^2(\pi k x)\exp(-t(\pi k)^{2s})dt\\ &= 2\sum_{k=1}^{M}\frac{\cos^2(\pi k x)(1-\exp(-T(\pi k)^{2s}))}{(\pi k)^{2s}}.\end{aligned} \tag{C.3}$$

We now want to check the fact that, when $s \in (0, 1/2]$, the quantities in (C.2) and (C.3) are divergent as $M \to +\infty$. To this end, we need to estimate "how often" in k the functions $\sin^2(\pi k x)$ and $\cos^2(\pi k x)$ can get close to zero. This concept is formalized via the following claim.

Claim 1. Given $x \in (0, 1)$,

- there exist $\varepsilon_0 > 0$ and $K_0 \in \mathbb{N} \cap [1, +\infty)$ such that for every $k_0 \in \mathbb{N}$,
- there exists $k \in \{k_0, k_0 + 1, \dots, k_0 + K_0\}$ such that $\sin^2(\pi k x) \geqslant \varepsilon_0$.

To prove this, up to exchanging x with $1 - x$, we can suppose that $x \in (0, \frac{1}{2}]$. Thus, we argue by contradiction and we suppose that, for some $x \in (0, \frac{1}{2}]$, for every $\varepsilon > 0$, as small as we wish, and every $K \in \mathbb{N}$, as large as we wish, there exists $k_{\varepsilon,K} \in \mathbb{N}$ such that for all $k \in \{k_{\varepsilon,K}, k_{\varepsilon,K} + 1, \dots, k_{\varepsilon,K} + K\}$ we have that $\sin^2(\pi k x) < \varepsilon$.

This means that for all $k \in \{k_{\varepsilon,K}, k_{\varepsilon,K} + 1, \dots, k_{\varepsilon,K} + K\}$ the angle $\pi k x$ is sufficiently close to either 0 or π, modulo multiples of 2π. Hence, for concreteness, let us suppose that the angle $\pi k_{\varepsilon,K} x$ is sufficiently close to 0 modulo multiples of 2π, namely that

$$|\pi k_{\varepsilon,K} x + 2\pi J| < \delta := \arcsin \sqrt{\varepsilon}$$

for some $J \in \mathbb{N}$.

Therefore, for every $j \in \mathbb{N}$,

$$\pi(k_{\varepsilon,K} + j)\, x + 2\pi J \in (-\delta + \pi j x, \delta + \pi j x).$$

We also note that, if $j \leqslant \frac{\pi - 2\delta}{\pi x}$ and δ is sufficiently small, it follows that

$$(-\delta + \pi j x, \delta + \pi j x) \subseteq (-\delta, \pi - \delta).$$

Choosing

$$K \geqslant 1 + \frac{\pi - 2\delta}{\pi x},$$

we thus conclude that, for every $j \in \mathbb{N} \cap [0, \frac{\pi - 2\delta}{\pi x}]$,

$$\pi(k_{\varepsilon,K} + j)x + 2\pi J \in (-\delta, \delta).$$

Now, we remark that, for sufficiently small δ, we have

$$\frac{\pi - 2\delta}{\pi x} \geqslant \frac{2(\pi - 2\delta)}{\pi} \geqslant \frac{3}{2}.$$

In particular, we can find $j_\star \in \mathbb{N} \cap [\frac{\pi - 2\delta}{\pi x} - 1, \frac{\pi - 2\delta}{\pi x}]$. It thereby follows that

$$\begin{aligned} \delta > \pi(k_{\varepsilon,K} + j_\star)x + 2\pi J &= \pi k_{\varepsilon,K}\, x + 2\pi J + \pi j_\star\, x > -\delta + \pi j_\star x \\ &\geqslant -\delta + \pi x \left(\frac{\pi - 2\delta}{\pi x} - 1 \right) = \pi - 3\delta - \pi x \geqslant \frac{\pi}{2} - 3\delta > \delta, \end{aligned}$$

provided that δ is sufficiently small. This is a contradiction and Claim 1 is established.

Similarly, one can prove the following.

Claim 2. Given $x \in (0,1)$,

- there exist $\varepsilon_0 > 0$ and $K_0 \in \mathbb{N} \cap [1, +\infty)$ such that for every $k_0 \in \mathbb{N}$,
- there exists $k \in \{k_0, k_0+1, \dots, k_0+K_0\}$ such that $\cos^2(\pi k x) \geqslant \varepsilon_0$.

We now pick arbitrary integers $N, \bar{N} \in \mathbb{N}$ with $N < \bar{N}$ and take $M := \bar{N}(K_0+2)$ in (C.2). Thus, assuming N large enough such that $\exp(-T(\pi N)^{2s}) \leqslant \frac{1}{2}$ and using Claim 1, we conclude that

$$
\begin{aligned}
\Phi_D^{x,x}(s,T) &\geqslant 2 \sum_{k=N}^{\bar{N}(K_0+2)} \frac{\sin^2(\pi k x)(1-\exp(-T(\pi k)^{2s}))}{(\pi k)^{2s}} \\
&\geqslant \sum_{k=N}^{\bar{N}(K_0+2)} \frac{\sin^2(\pi k x)}{(\pi k)^{2s}} \\
&\geqslant \sum_{\ell=0}^{\bar{N}-1} \sum_{k=N+\ell K_0+\ell}^{N+(\ell+1)K_0+\ell} \frac{\sin^2(\pi k x)}{(\pi k)^{2s}} \\
&\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\bar{N}-1} \sum_{k=N+\ell K_0+\ell}^{N+(\ell+1)K_0+\ell} \frac{\sin^2(\pi k x)}{(N+(\ell+1)K_0+\ell)^{2s}} \\
&\geqslant \frac{1}{\pi^{2s}} \sum_{\ell=0}^{\bar{N}-1} \frac{\varepsilon_0}{(N+(\ell+1)K_0+\ell)^{2s}}.
\end{aligned}
\tag{C.4}
$$

Sending now $\bar{N} \to +\infty$ we conclude that, when $s \in (0, 1/2]$,

$$
\Phi_D^{x,x}(s,T) \geqslant \frac{\varepsilon_0}{\pi^{2s}} \sum_{\ell=0}^{+\infty} \frac{1}{(N+(\ell+1)K_0+\ell)^{2s}} = +\infty. \tag{C.5}
$$

Similarly, combining (C.3) and Claim 2, we find that, when $s \in (0, 1/2]$,

$$
\Phi_N^{x,x}(s,T) = +\infty.
$$

This and (C.5) yield that $\mathcal{E}(s,T) = +\infty$ for all $s \in (0, 1/2]$, as claimed in the statement of Proposition 1.2.

We now consider the case $s \in (1/2, 1]$. In this situation, it follows from (3.5) that, for every $x, y \in (0,1)$,

$$
\Phi_D^{x,y}(s,T) \leqslant 2 \sum_{k=1}^{+\infty} \frac{1-\exp(-T(\pi k)^{2s})}{(\pi k)^{2s}} \leqslant \frac{2}{\pi^{2s}} \sum_{k=1}^{+\infty} \frac{1}{k^{2s}} < +\infty. \tag{C.6}
$$

Similarly, using (3.8), for all $s \in (1/2, 1]$ and $x, y \in (0, 1)$,

$$\Phi_N^{x,y}(s, T) < +\infty.$$

From this estimate and (C.6) we infer that $\mathcal{E}(s, T) \in (0, +\infty)$ for all $s \in (1/2, 1)$, as desired. ∎