Appendix D

Some technical results

In this chapter we collect some technical results which have been used throughout the memoir.

Proposition D.1. *Let* $(l, t) \in (0, +\infty) \times (0, +\infty)$. *Then,*

$$\lim_{s \searrow 0} \frac{\mu_t^s(l)}{s} = \frac{te^{-t}}{l}. \tag{D.1}$$

Proof. Thanks to (2.2), we have that

$$\begin{aligned} \lim_{s \searrow 0} \frac{\mu_t^s(l)}{s} &= \lim_{s \searrow 0} \frac{1}{\pi} \int_0^{+\infty} e^{-lu} e^{-tu^s \cos(\pi s)} tu^s \frac{\sin(tu^s \sin(\pi s))}{tu^s s}\, du \\ &= \int_0^{+\infty} \lim_{s \searrow 0} e^{-lu-tu^s \cos(\pi s)} tu^s \frac{\sin(tu^s \sin(\pi s))}{tu^s s}\, du \\ &= te^{-t} \int_0^{+\infty} e^{-lu}\, du \\ &= \frac{te^{-t}}{l}, \end{aligned} \tag{D.2}$$

where we have used the fact that for each $s \in (0, \frac{1}{2})$ it holds that

$$\begin{aligned} &\left| e^{-lu-tu^s \cos(\pi s)} tu^s \frac{\sin(tu^s \sin(\pi s))}{tu^s s} \right| \\ &\qquad \leqslant te^{-lu} \big(\chi_{(0,1)}(u) + \chi_{(1,+\infty)}(u) u^{\frac{1}{2}} \big) \in L^1((0, +\infty)) \end{aligned}$$

in order to apply the dominated convergence theorem in (D.2). ∎

Proposition D.2. *Let* $\Omega \subset \mathbb{R}^n$ *be bounded, smooth and connected. Then, if* $E, F \subset \Omega$ *and* $\bar{E} \cap \bar{F} = \varnothing$, *there exists some constant* $C_{E,F} \in (0, +\infty)$, *depending only on* E *and* F, *such that for all* $(s, T) \in (0, 1) \times (0, +\infty)$ *it holds that*

$$\Phi_N^{x,y}(s, T) \leqslant C_{E,F} T \quad \textit{for all } (x, y) \in E \times F. \tag{D.3}$$

Proof. Thanks to the hypothesis $\bar{E} \cap \bar{F} = \varnothing$, we can define the positive constant

$$\bar{d} := \inf_{\substack{x \in E \\ y \in F}} |x - y|.$$

Then, by the definition of Φ_N and the upper bound in (2.19), we obtain that

$$\begin{aligned}\Phi_N^{x,y}(s,T) :=& \int_0^T \int_0^{+\infty} p_N^{\Omega}(l,x,y)\mu_t^s(l)dl\,dt \\ \leqslant& \int_0^T \int_0^{+\infty} c_\Omega \max\left\{\frac{1}{l^{\frac{n}{2}}},1\right\}\exp\left(-\frac{\bar{d}^2}{6l}\right)\mu_t^s(l)dl\,dt \\ \leqslant& \int_0^T \int_0^{+\infty} C_{E,F}\,\mu_t^s(l)dl\,dt \\ =& C_{E,F}T,\end{aligned}$$

where we set

$$C_{E,F} := \sup_{l\in(0,+\infty)} c_\Omega \max\left\{\frac{1}{l^{\frac{n}{2}}},1\right\}\exp\left(-\frac{\bar{d}^2}{6l}\right). \tag{D.4}$$

This establishes the desired result. ∎

Lemma D.3. *Let $\Omega\subset\mathbb{R}^n$ be bounded and $f,g\in C(\overline{\Omega})$ be strictly positive in a compact set $K\Subset\Omega$. Then,*

$$\inf_{\Omega_2\subset K}\frac{\int_{\Omega_2} f(x)dx}{\int_{\Omega_2} g(x)dx}\in(0,+\infty).$$

Proof. We set

$$m := \min_{x\in K} f(x)\in(0,+\infty) \quad \text{and} \quad M = \max_{x\in K} g(x)\in(0,+\infty).$$

Then,

$$\inf_{\Omega_2\subset K}\frac{\int_{\Omega_2} f(x)dx}{\int_{\Omega_2} g(x)dx}\geqslant\frac{m}{M}\in(0,+\infty).$$ ∎

We give some lower and upper bounds for the function $F_D(x,y)$ defined in equation (2.62). This result is applied several times, when proving Theorem 1.7.

Lemma D.4. *Let $\Omega\subset\mathbb{R}^n$ be bounded, smooth and connected. Then, for each $K\Subset\Omega$ there exists some constant $\tilde{c}_{K,\Omega}\in(0,+\infty)$ such that*

$$F_D(x,y)\geqslant\frac{\tilde{c}_{K,\Omega}}{|x-y|^n} \quad \textit{for all } (x,y)\in\mathcal{C}\cap(K\times K), \tag{D.5}$$

where $\mathcal{C}$ has been defined in (2.23).

Furthermore, it holds that

$$F_D(x,y)\leqslant\frac{C_n}{|x-y|^n} \quad \textit{for all } (x,y)\in\mathcal{C}, \tag{D.6}$$

for some $C_n\in(0,+\infty)$.

Proof. We first prove (D.5). Thanks to equations (2.15) and (2.16) we observe that there exists two constants c_1, c_2 and some $T_{K,\Omega} \in (0, +\infty)$ depending on Ω and K, such that

$$p_D^\Omega(t,x,y) \geqslant \frac{c_1}{t^{\frac{n}{2}}} \exp\left(- \frac{c_2|x-y|^2}{t}\right) \quad \text{for all } (t,x,y) \in (0,+\infty) \times K \times K.$$

Therefore, thanks to equation (2.62) we deduce that for each $(x,y) \in \mathcal{C} \cap (K \times K)$ it holds that

$$\begin{aligned} F_D(x,y) &= \int_0^{+\infty} \frac{p_D^\Omega(l,x,y)}{l}\, dl \\ &\geqslant \int_0^{T_{K,\Omega}} \frac{c_1}{l^{\frac{n}{2}+1}} \exp\left(- \frac{c_2|x-y|^2}{l}\right) dl \\ &= \frac{c_1 c_2^{-\frac{n}{2}}}{|x-y|^n} \int_{\frac{c_2|x-y|^2}{T_{K,\Omega}}}^{+\infty} a^{\frac{n}{2}-1} e^{-a}\, da \\ &\geqslant \frac{\tilde{c}_{K,\Omega}}{|x-y|^n}, \end{aligned}$$

where, by calling as usual d_K the diameter of K, we defined

$$\tilde{c}_{K,\Omega} := c_1 c_2^{-\frac{n}{2}} \int_{\frac{c_2 d_K^2}{T_{K,\Omega}}}^{+\infty} a^{\frac{n}{2}-1} e^{-a}\, da. \tag{D.7}$$

This concludes the proof of (D.5).

We now show (D.6). By equation (2.14) and the change of variable $\theta = \frac{|x-y|^2}{4l}$ we obtain that

$$\begin{aligned} F_D(x,y) &= \int_0^{+\infty} \frac{p_D^\Omega(l,x,y)}{l}\, dl \\ &\leqslant \frac{1}{(4\pi)^{\frac{n}{2}}} \int_0^{+\infty} \frac{1}{l^{\frac{n}{2}+1}} \exp\left(- \frac{|x-y|^2}{4l}\right) dl \\ &\leqslant \frac{1}{\pi^{\frac{n}{2}}|x-y|^n} \int_0^{+\infty} \theta^{\frac{n}{2}-1} e^{-\theta}\, d\theta \\ &= \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}} \frac{1}{|x-y|^n}. \end{aligned}$$

Therefore, (D.6) is proved with $C_n := \frac{\Gamma(\frac{n}{2})}{\pi^{\frac{n}{2}}}$. ∎