Chapter 1

Introduction

Our core technical contribution is a geometric structural result (stochastic clustering) for subsets of finite dimensional normed spaces. It provides new links between nonlinear questions in metric geometry and volumetric issues in convex geometry. An unexpected aspect of our statement is that it contradicts an impossibility result of the well-known work [76] by Charikar, Chekuri, Goel, Guha and Plotkin in the computer science literature, thus leading to bounds that were previously thought to be impossible. This is reconciled in Section 1.7, where we explain the source of the error in [76].

The aforementioned link opens up a vista that allows one to apply the extensive literature on the linear theory to important and well-studied nonlinear questions. It also raises new fundamental issues within the linear theory that we will only begin to address here. So, in order to fully explain both the history and the ideas and their consequences, we will start with a quick overview of some of our main results that assumes familiarity with standard concepts in the respective areas. We will then present a gradual and complete introduction to our work that specifies all of the necessary background.

1.1 Brief highlights of main results

Associate to every separable complete metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ two bi-Lipschitz invariants $\mathsf{e}(\mathfrak{M}), \mathsf{SEP}(\mathfrak{M}) \in (0, \infty]$ called, respectively, the *Lipschitz extension modulus of* $\mathfrak{M}$ and the *separation modulus of* $\mathfrak{M}$, that are defined as follows. The Lipschitz extension modulus of $\mathfrak{M}$ is the infimum over those $L \in (0, \infty]$ such that for every Banach space $\mathbf{Z}$ and every subset $\mathcal{C} \subseteq \mathfrak{M}$, every 1-Lipschitz function $f : \mathcal{C} \to \mathbf{Z}$ can be extended to a $\mathbf{Z}$-valued L-Lipschitz function that is defined on all of $\mathfrak{M}$. The separation modulus of $\mathfrak{M}$ is the infimum over those $\sigma \in (0, \infty]$ such that for any $\Delta > 0$ there is a distribution over random partitions[1] of $\mathfrak{M}$ into clusters of diameter at most Δ such that for every two points $x, y \in \mathfrak{M}$ the probability that they belong to different clusters is at most $\sigma d_{\mathfrak{M}}(x, y)/\Delta$.

The question of estimating the Lipschitz extension modulus received great scrutiny over the past century; see Section 1.3 for an indication of (a small part of) the

[1] We are suppressing here measurability issues that are addressed in Section 1.7 and Section 3.1.

extensive knowledge on this topic. The separation modulus was introduced by Bartal in the mid-1990s and received a lot of attention in the computer science literature due to its algorithmic applications; see Section 1.7.3 for the history. Its connection to Lipschitz extension was found by Lee and the author [171, 173], who proved that

$$\mathsf{e}(\mathfrak{M}) \lesssim \mathsf{SEP}(\mathfrak{M}).$$

By a well-known theorem of Johnson, Lindenstrauss and Schechtman [140], every normed space $\mathbf{X}$ satisfies $\mathsf{e}(\mathbf{X}) = O(\dim(\mathbf{X}))$. Here we obtain a power-type lower bound on $\mathsf{e}(\mathbf{X})$ in terms of $\dim(\mathbf{X})$.

Theorem 1. *There is a universal constant $c > 0$ such that $\mathsf{e}(\mathbf{X}) \geqslant \dim(\mathbf{X})^c$ for every normed space $\mathbf{X}$.*

Theorem 1 improves over the previously best-available bound

$$\mathsf{e}(\mathbf{X}) \geqslant e^{c\sqrt{\log \dim(\mathbf{X})}};$$

see Remark 98 for the history of this question. Despite substantial efforts, the asymptotic growth rate (as $\dim(\mathbf{X}) \to \infty$) of $\mathsf{e}(\mathbf{X})$ was not previously known (even up to lower order factors) for *any* sequence of normed spaces.

Theorem 2. *For every $n \in \mathbb{N}$ we have*[2] $\mathsf{e}(\ell_\infty^n) \asymp \sqrt{n}$.

The previously best-known upper bound on $\mathsf{e}(\ell_\infty^n)$ was nothing better than the aforementioned general $O(n)$ bound of [140]. Theorem 2 is just one instance of our asymptotically improved upper bounds on the Lipschitz extension moduli of many normed spaces of interest; we also get, e.g., the best-known bound when $\mathbf{X} = \ell_p^n$ for any $p > 2$. Nevertheless, currently ℓ_∞^n is essentially[3] the only normed space whose Lipschitz extension modulus is known up to lower order factors (by Theorem 2), and the same question even for the Euclidean space ℓ_2^n remains a well-known longstanding open problem; see Section 1.3 for more on this.

All of the upper bounds on the Lipschitz extension modulus that we obtain herein use the upper bound on the separation modulus that appears in Theorem 3 below. This theorem also contains a new lower bound on the separation modulus, which we

[2] We use the following conventions for asymptotic notation, in addition to the usual $O(\cdot), o(\cdot), \Omega(\cdot)$ notation. Given $a, b > 0$, by writing $a \lesssim b$ or $b \gtrsim a$ we mean that $a \leqslant Cb$ for some universal constant $C > 0$, and $a \asymp b$ stands for $(a \lesssim b) \wedge (b \lesssim a)$. If we need to allow for dependence on parameters, we indicate it by subscripts. For example, in the presence of an auxiliary parameter q, the notation $a \lesssim_q b$ means that $a \leqslant C(q)b$, where $C(q) > 0$ may depend only on q, and similarly for $a \gtrsim_q b$ and $a \asymp_q b$.

[3] The proof of Theorem 2 artificially gives more such spaces, e.g., $\ell_\infty^n \oplus \ell_2^n$, or $\ell_\infty^n \oplus \mathbf{X}$ for any normed space $\mathbf{X}$ with $\dim(\mathbf{X}) \leqslant \sqrt{n}$.

will see shows that in several cases of interest our results are a sharp evaluation of the asymptotic growth rate of the separation modulus.[4]

Theorem 3. *Let* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ *and* $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ *be normed spaces whose unit balls satisfy* $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$. *Then*

$$\mathrm{vr}(\mathbf{X}^*)\sqrt{n} \lesssim \mathsf{SEP}(\mathbf{X}) \lesssim \frac{\mathrm{diam}_{\mathbf{X}^*}(\Pi B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})}. \tag{1.1}$$

In the left-hand side of (1.1), $\mathrm{vr}(\mathbf{X}^*)$ is the *volume ratio* [293, 294] of the dual $\mathbf{X}^*$, i.e., it is the nth root of the ratio of the volume of $B_{\mathbf{X}^*}$ and maximal volume of an ellipsoid that is contained in $B_{\mathbf{X}^*}$. In the right-hand side of (1.1), $\Pi B_{\mathbf{Y}}$ is the *projection body* [251] of $B_{\mathbf{Y}}$, and $\mathrm{diam}_{\mathbf{X}^*}(\cdot)$ denotes diameter with respect to the metric on $\mathbb{R}^n$ that is induced by $\mathbf{X}^*$. We will recall the definition of a projection body later[5] and it suffices to mention now that the mapping $K \mapsto \Pi K$, which is of central importance in convex geometry (see [47,102,190,282] for an indication of the extensive literature on this topic), associates to every convex body $K \subseteq \mathbb{R}^n$ a convex body $\Pi K \subseteq \mathbb{R}^n$ that encodes isoperimetric properties of K.

A key contribution of Theorem 3 is the role of the auxiliary normed space $\mathbf{Y}$, which appears despite the fact that we are interested in the separation modulus of $\mathbf{X}$. By substituting $\mathbf{Y} = \mathbf{X}$ into the right-hand side of (1.1) one *does* get a meaningful estimate, and in particular the resulting bound is $O(n)$, i.e., (1.1) implies the bound of [140]. However, we will see that by introducing a suitable perturbation $\mathbf{Y}$ of $\mathbf{X}$, the second inequality in (1.1) can sometimes be significantly stronger than the special case $\mathbf{Y} = \mathbf{X}$. We will exploit this powerful degree of freedom heavily; its geometric significance is discussed in Section 1.4.

The previously best-known upper and lower estimates on the separation moduli of normed spaces are due to [76], where it was proved that

$$\mathsf{SEP}(\ell_1^n) \asymp n \quad \text{and} \quad \mathsf{SEP}(\ell_2^n) \asymp \sqrt{n}.$$

By bi-Lipschitz invariance, this implies that any n-dimensional normed space $\mathbf{X}$ satisfies

$$\frac{n}{d_{\mathrm{BM}}(\ell_1^n, \mathbf{X})} \lesssim \mathsf{SEP}(\mathbf{X}) \lesssim d_{\mathrm{BM}}\big(\ell_2^n, \mathbf{X}\big)\sqrt{n}, \tag{1.2}$$

[4]Our approach also pertains to subsets of normed spaces, e.g., we will prove that for any $p \in [1,\infty]$, $n \in \mathbb{N}$ and $r \in \{1,\ldots,n\}$, the separation modulus of the set of n-by-n matrices of rank at most r, equipped with the Schatten–von Neumann-p norm, is equal up to lower order factors to $\max\{\sqrt{r}, r^{1/p}\}\sqrt{n}$, which is new even in the Euclidean (Hilbert–Schmidt) setting $p = 2$. However, for the purpose of this initial overview we will restrict attention to bounds for the entire space $\mathbf{X}$.

[5]By [187, 188] the mapping that assigns a convex body $K \subseteq \mathbb{R}^n$ to its projection body ΠK is characterized axiomatically as the unique (up to scaling) translation-invariant $\mathsf{SL}_n(\mathbb{R})$-contravariant Minkowski valuation.

where $d_{\mathrm{BM}}(\cdot,\cdot)$ denotes the Banach–Mazur distance. Both of the bounds in (1.2) can be inferior to those that follow from Theorem 3. For example, suppose that $n=m^2$ for some $m\in\mathbb{N}$ and consider $\mathbf{X}=\ell_\infty^m(\ell_1^m)$. Then,

$$d_{\mathrm{BM}}(\mathbf{X},\ell_1^n)\asymp d_{\mathrm{BM}}(\mathbf{X},\ell_2^n)\asymp\sqrt{n}$$

by the work [163] of Kwapień and Schütt. Therefore, in this case (1.2) becomes the estimates $\sqrt{n}\lesssim\mathsf{SEP}(\mathbf{X})\lesssim n$, while we will see that (1.1) implies that $\mathsf{SEP}(\mathbf{X})\asymp n^{3/4}$.

The following corollary collects examples of applications of Theorem 3 that we will deduce herein.

Corollary 4 (Examples of consequences of Theorem 3). *The following statements hold for any $n\in\mathbb{N}$.*

- *For any $p\geqslant 1$, the separation modulus of ℓ_p^n satisfies*

$$\mathsf{SEP}\big(\ell_p^n\big)\asymp n^{\max\{\frac12,\frac1p\}}. \tag{1.3}$$

  More generally, let $(\mathbf{E},\|\cdot\|_{\mathbf{E}})$ be any n-dimensional normed space with a 1-symmetric basis $e_1,\ldots,e_n$. Then, $\mathsf{SEP}(\mathbf{E})$ is equal to the following quantity up to lower order factors:

$$\|e_1+\cdots+e_n\|_{\mathbf{E}}\bigg(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\bigg).$$

- *For any $p\geqslant 1$, the separation modulus of the Schatten–von Neumann trace class S_p^n on $\mathsf{M}_n(\mathbb{R})$ is*

$$\mathsf{SEP}\big(\mathsf{S}_p^n\big)=n^{\max\{1,\frac12+\frac1p\}+o(1)}=\dim\big(\mathsf{S}_p^n\big)^{\max\{\frac12,\frac14+\frac{1}{2p}\}+o(1)}. \tag{1.4}$$

  More generally, let $(\mathbf{E},\|\cdot\|_{\mathbf{E}})$ be any n-dimensional normed space with a 1-symmetric basis $e_1,\ldots,e_n$ and denote its unitary ideal by $\mathsf{S}_{\mathbf{E}}=(\mathsf{M}_n(\mathbb{R}),\|\cdot\|_{\mathsf{S}_{\mathbf{E}}})$. Then, $\mathsf{SEP}(\mathsf{S}_{\mathbf{E}})$ is equal to the following quantity up to lower order factors:

$$\|e_1+\cdots+e_n\|_{\mathbf{E}}\bigg(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\bigg)\sqrt{n}.$$

- *For any $p,q\geqslant 1$, the separation modulus of the $\ell_p^n(\ell_q^n)$ norm on $\mathsf{M}_n(\mathbb{R})$ is*

$$\begin{aligned}\mathsf{SEP}\big(\ell_p^n(\ell_q^n)\big)&\asymp n^{\max\{1,\frac1p+\frac1q,\frac12+\frac1p,\frac12+\frac1q\}}\\&=\dim\big(\ell_p^n(\ell_q^n)\big)^{\max\{\frac12,\frac{1}{2p}+\frac{1}{2q},\frac14+\frac{1}{2p},\frac14+\frac{1}{2q}\}}.\end{aligned} \tag{1.5}$$

- *For any $p,q\geqslant 1$, the separation modulus of $\mathsf{M}_n(\mathbb{R})$ equipped with the operator norm $\|\cdot\|_{\ell_p^n\to\ell_q^n}$ from ℓ_p^n to ℓ_q^n is equal to the following quantity up to lower order* *factors:*

$$\begin{cases} n^{\frac{3}{2}-\frac{1}{\min\{p,q\}}} & \text{if } p,q \geqslant 2,\\ n^{\frac{1}{2}+\frac{1}{\max\{p,q\}}} & \text{if } p,q \leqslant 2,\\ n & \text{if } p \leqslant 2 \leqslant q,\\ n^{\max\{1,\frac{1}{q}-\frac{1}{p}+\frac{1}{2}\}} & \text{if } q \leqslant 2 \leqslant p. \end{cases}$$

- *For any $p,q \geqslant 1$, the separation modulus of the projective tensor product $\ell_p^n \widehat{\otimes} \ell_q^n$, i.e., the norm on $\mathsf{M}_n(\mathbb{R})$ whose unit ball is the convex hull of the set*

$$\{(x_i y_j) \in \mathsf{M}_n(\mathbb{R});\ (x_1,\ldots,x_n) \in B_{\ell_p^n} \ \wedge\ (y_1,\ldots,y_n) \in B_{\ell_q^n}\},$$

is equal to the following quantity up to lower order factors:

$$\begin{cases} n^{\frac{3}{2}} & \text{if } \max\{p,q\} \geqslant 2,\\ n^{1+\frac{1}{\max\{p,q\}}} & \text{if } \max\{p,q\} \leqslant 2. \end{cases}$$

All of the results in Corollary 4 are new, except for the range $1 \leqslant p \leqslant 2$ of (1.3), which is due to [76]. The range $p \in (2,\infty]$ of (1.3) is $\mathsf{SEP}(\ell_p^n) \asymp \sqrt{n}$, which is incompatible with the statement $\mathsf{SEP}(\ell_p^n) \asymp n^{1-1/p}$ of [76]. We will explain the reason why the latter assertion of [76] is erroneous in Remark 78.

The wealth of knowledge that is available on the volumetric quantities that appear in (1.1) leads to new estimates that relate the separation modulus of an n-dimensional normed space $\mathbf{X}$ to classical invariants of $\mathbf{X}$. We will derive several such results herein, without attempting to be encyclopedic. As a noteworthy example, we will deduce from the first inequality in (1.1) that if $B_{\mathbf{X}}$ is a polytope with ρn vertices, then

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \frac{n}{\sqrt{\log \rho}}. \tag{1.6}$$

We will also deduce that if $T_2(\mathbf{X})$ denotes the type 2 constant of $\mathbf{X}$ (see (1.77) or the survey [203]), then

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \max\big\{\sqrt{\dim(\mathbf{X})}, T_2(\mathbf{X})^2\big\}. \tag{1.7}$$

We will see that both (1.6) and (1.7) are sharp for the entire range of the relevant parameters (e.g., in the two extremes, the case $\mathbf{X} = \ell_1^n$ corresponds to $\rho = O(1)$ and $T_2(\mathbf{X}) \asymp \sqrt{n}$ in (1.6) and (1.7), respectively, and the case when $\mathbf{X}$ is $O(1)$-isomorphic to ℓ_2^n corresponds to $\log \rho \asymp n$ and $T_2(\mathbf{X}) = O(1)$ in (1.6) and (1.7), respectively).

1.1.1 A conjectural isomorphic reverse isoperimetric phenomenon

The lower bound on $\mathsf{SEP}(\mathbf{X})$ in Theorem 3 is not always sharp. Indeed, consider the space $\mathbf{X} = \ell_1^n \oplus \ell_2^n$ for which $\mathsf{SEP}(\mathbf{X}) \asymp n$ yet $\mathrm{vr}(\mathbf{X}^*)\sqrt{\dim(\mathbf{X})} \asymp n^{3/4}$. It could be, however, that the upper bound on $\mathsf{SEP}(\mathbf{X})$ in Theorem 3 is optimal for every $\mathbf{X}$.

Question 5. Is the separation modulus of any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ bounded above and below by some universal constant multiples of the minimum of the quantity $\mathrm{diam}_{\mathbf{X}^*}(\Pi B_{\mathbf{Y}})/\operatorname{vol}_n(B_{\mathbf{Y}})$ over all those normed spaces $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ that satisfy $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$?

See Remark 23 for an explanation why the minimum that is described in Question 5 is affine invariant, which is necessary for Question 5 to make sense, since the separation modulus is a bi-Lipschitz invariant.

For sufficiently symmetric spaces, we expect that the lower bound on $\mathsf{SEP}(\mathbf{X})$ in Theorem 3 is sharp.

Conjecture 6. Every finite dimensional normed space $\mathbf{X}$ with enough symmetries satisfies

$$\mathsf{SEP}(\mathbf{X}) \asymp \mathrm{vr}(\mathbf{X}^*)\sqrt{\dim(\mathbf{X})}. \tag{1.8}$$

The notion of having enough symmetries was introduced in [103]; its definition is recalled in Section 1.6.2. We prefer to formulate Conjecture 6 using this notion at the present introductory juncture even though weaker requirements are needed for our purposes because it is a standard assumption in Banach space theory and it suffices for all of the most pressing applications that we have in mind.

The upper bound on $\mathsf{SEP}(\mathbf{X})$ in (1.8) implies by [173] that

$$\mathsf{e}(\mathbf{X}) \lesssim \mathrm{vr}(\mathbf{X}^*)\sqrt{\dim(\mathbf{X})},$$

which would be a valuable Lipschitz extension theorem due to the fact that estimating the volume ratio is typically tractable given the variety of tools and extensive knowledge that are available in the literature. For example, Milman and Pisier [219] proved (improving by lower-order factors over a major theorem of Bourgain and Milman [49, 50]; see also [217]), that any finite dimensional normed space $\mathbf{X}$ satisfies

$$\mathrm{vr}(\mathbf{X}) \lesssim C_2(\mathbf{X})\big(1 + \log C_2(\mathbf{X})\big), \tag{1.9}$$

where $C_2(\mathbf{X})$ is the cotype 2 constant of $\mathbf{X}$ (see (1.77) or the survey [203]). Therefore, if (1.8) holds, then

$$\mathsf{e}(\mathbf{X}) \lesssim C_2(\mathbf{X})\big(1 + \log C_2(\mathbf{X})\big)\sqrt{\dim(\mathbf{X})}, \tag{1.10}$$

which would be a remarkable generalization of the bound $\mathsf{e}(\ell_2^n) \lesssim \sqrt{n}$ of [173].

We expect that Theorem 3 already implies Conjecture 6, as expressed in the following conjecture which would yield a positive answer to Question 5 for normed spaces with enough symmetries.

Conjecture 7. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space with enough symmetries, then there is a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ that satisfies

$$B_{\mathbf{Y}} \subseteq B_{\mathbf{X}} \quad \text{and} \quad \frac{\mathrm{diam}_{\mathbf{X}^*}(\Pi B_{\mathbf{Y}})}{\operatorname{vol}_n(B_{\mathbf{Y}})} \lesssim \mathrm{vr}(\mathbf{X}^*)\sqrt{n}.$$

As an illustrative example of Conjecture 7, consider the space $\mathbf{X} = \ell_\infty^n$. We then have $\mathrm{vr}((\ell_\infty^n)^*) = \mathrm{vr}(\ell_1^n) = O(1)$. One can compute that $\Pi B_{\ell_\infty^n} = 2^{n-1} B_{\ell_\infty^n}$. Therefore, $\mathrm{diam}_{\ell_1^n}(\Pi B_{\ell_\infty^n})/\operatorname{vol}_n(B_{\ell_\infty^n}) \asymp n$, so taking $\mathbf{Y} = \ell_\infty^n$ in Theorem 3 only gives the bound $\mathsf{SEP}(\ell_\infty^n) \lesssim n$. However, we will prove that there exists a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ with $B_{\mathbf{Y}} \subseteq B_{\ell_\infty^n}$ for which $\mathrm{diam}_{\ell_1^n}(\Pi B_{\mathbf{Y}})/\operatorname{vol}_n(B_{\mathbf{Y}}) \lesssim \sqrt{n}$. More generally, we will prove that Conjecture 7 (hence also Conjecture 6, by Theorem 3) holds for any normed space for which the standard basis of $\mathbb{R}^n$ is 1-symmetric, and we will also see that Conjecture 7 holds up to a logarithmic factor for its unitary ideal.

The minimization in Question 5 can be viewed as a shape optimization problem [130] that could potentially be approached using calculus of variations. Given an origin-symmetric convex body $K \subseteq \mathbb{R}^n$, it asks for the minimum of the affine invariant functional $L \mapsto \mathsf{outradius}_{K^\circ}(\Pi L)/\operatorname{vol}_n(L)$ over all origin-symmetric convex bodies $L \subseteq K$, where for any two origin-symmetric convex bodies $A, B \subseteq \mathbb{R}^n$ we denote the minimum radius of a dilate of A that circumscribes B by

$$\mathsf{outradius}_A(B) = \min\{r \geqslant 0 : B \subseteq rA\}$$

and

$$K^\circ = \{y \in \mathbb{R}^n : \sup_{x \in K} \langle x, y\rangle \leqslant 1\}$$

is the polar of K. Conjecture 7 asserts that if K has enough symmetries, then this minimum is bounded above and below by universal constant multiples of $\mathrm{vr}(K^\circ)\sqrt{n}$.

The minimization problem in Question 5 also has an isoperimetric flavor. As such, its investigation led us to formulate the following conjectural twist of Ball's reverse isoperimetric phenomenon [22], which we think is a fundamental geometric open question and it would be valuable to understand it even without its consequences that we derive herein.

The *isoperimetric quotient* of a convex body $K \subseteq \mathbb{R}^n$ is defined (see [126, p. 269] or [286]) to be

$$\mathrm{iq}(K) = \frac{\operatorname{vol}_{n-1}(\partial K)}{\operatorname{vol}_n(K)^{\frac{n-1}{n}}}. \tag{1.11}$$

Using this notation, the classical Euclidean isoperimetric theorem states that

$$\mathrm{iq}(K) \geqslant \mathrm{iq}\big(B_{\ell_2^n}\big) = \frac{n\sqrt{\pi}}{\Gamma\big(\frac{n}{2}+1\big)^{\frac{1}{n}}} \asymp \sqrt{n}. \tag{1.12}$$

The following theorem of Ball [22] shows that a judicious choice of the scalar product on $\mathbb{R}^n$ ensures that the isoperimetric quotient of a convex body can also be bounded from above.

Theorem 8 (Ball's reverse isoperimetric theorem [22]). *For every $n \in \mathbb{N}$ and every origin-symmetric convex body $K \subseteq \mathbb{R}^n$ there exists a linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$ such that* $\mathrm{iq}(SK) \leqslant 2n = \mathrm{iq}([-1,1]^n)$.

We expect that in the isomorphic regime (i.e., permitting non-isometric $O(1)$ perturbations), origin-symmetric convex bodies have asymptotically better reverse isoperimetric properties than what is guaranteed by Theorem 8. In fact, we conjecture that if in addition to passing from K to SK for some $S \in \mathsf{SL}_n(\mathbb{R})$, a $O(1)$-perturbation of SK is allowed, then the isoperimetric quotient can be decreased to be of the same order of magnitude as that of the Euclidean ball.

Conjecture 9 (Isomorphic reverse isoperimetry). There is a universal constant $c > 0$ with the following property. For every $n \in \mathbb{N}$ and every origin-symmetric convex body $K \subseteq \mathbb{R}^n$, there exist a linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$ and an origin-symmetric convex body $L \subseteq \mathbb{R}^n$ with $cSK \subseteq L \subseteq SK$ and $\mathrm{iq}(L) \lesssim \sqrt{n}$.

Conjecture 9 can be restated analytically as the assertion that any n-dimensional normed space is at Banach–Mazur distance $O(1)$ from a normed space whose unit ball has isoperimetric quotient $O(\sqrt{n})$. We will prove that Conjecture 9 holds when K is the unit ball of ℓ_p^n for any $p \in [1, \infty]$ and $n \in \mathbb{N}$, and we will also see that Conjecture 9 holds up to lower-order factors for any Schatten–von Neumann trace class.

The requirement $L \supseteq cSK$ of Conjecture 9 implies that $\sqrt[n]{\mathrm{vol}_n(L)} \geqslant c\sqrt[n]{\mathrm{vol}_n(K)}$. So, the following weaker conjecture is implied by Conjecture 9; we will prove it for any 1-unconditional body.

Conjecture 10 (Weak isomorphic reverse isoperimetry). For every $n \in \mathbb{N}$ and every origin-symmetric convex body $K \subseteq \mathbb{R}^n$ there exist a linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$ and an origin-symmetric convex body $L \subseteq SK$ that satisfies $\sqrt[n]{\mathrm{vol}_n(L)} \gtrsim \sqrt[n]{\mathrm{vol}_n(K)}$ and $\mathrm{iq}(L) \lesssim \sqrt{n}$.

In Section 1.6 we will elucidate the relation between the task of bounding from above the rightmost quantity in (1.1) and isomorphic reverse isoperimetry. While Conjecture 9 is the strongest version of the isomorphic reverse isoperimetric phenomenon that we expect holds in full generality, we will see that it would suffice to prove its weaker variant Conjecture 10 for the purpose of using Theorem 3. In particular, consider the following symmetric version of Conjecture 10, which we will prove in Section 1.6 implies Conjecture 7 (hence, using Theorem 3, it also implies Conjecture 6).

Conjecture 11 (Symmetric version of Conjecture 10). For every $n \in \mathbb{N}$ and every normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ with enough symmetries whose isometry group is a subgroup of the orthogonal group $\mathsf{O}_n \subseteq \mathsf{GL}_n(\mathbb{R})$, there is a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ with $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ and $\sqrt[n]{\mathrm{vol}_n(B_{\mathbf{Y}})} \gtrsim \sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})}$ such that $\mathrm{iq}(B_{\mathbf{Y}}) \lesssim \sqrt{n}$.

The only difference between Conjecture 10 and Conjecture 11 is that if we impose the further requirement that K is the unit ball of a normed space with enough symmetries whose isometry group consists only of orthogonal matrices, then we are naturally conjecturing that S can be taken to be the identity matrix, i.e., there is no need to change the standard Euclidean structure on $\mathbb{R}^n$.

We will prove Conjecture 11 for various spaces, including $\ell_p^n(\ell_q^n)$ for any $p,q \geqslant 1$ and $n \in \mathbb{N}$, and any finite dimensional space with a 1-symmetric basis. Also, we will show that Conjecture 11 holds up to a factor of $O(\sqrt{\log n})$ for any unitarily invariant norm on $\mathsf{M}_n(\mathbb{R})$. In general, an argument that was shown to us by B. Klartag and E. Milman and is included in Section 7 (see also Section 1.6.3) shows that Conjecture 10 and Conjecture 11 hold up to a factor of $O(\log n)$. We will see that these results lead to Corollary 4, and in general we will deduce that Conjecture 7, and hence, thanks to Theorem 3, also Conjecture 6, hold up to lower order factors. Thus, we will obtain the following theorem.

Theorem 12. $\mathsf{SEP}(\mathbf{X}) \asymp \mathrm{vr}(\mathbf{X}^*)\dim(\mathbf{X})^{\frac{1}{2}+o(1)}$ *for any normed space* $\mathbf{X}$ *with enough symmetries.*

Assuming Conjecture 11, it is possible to compute the exact asymptotic growth rate of the separation moduli of several important matrix spaces. For example, if Conjecture 11 holds for S_∞^n, then we will see that the $o(1)$ term in (1.4) could be removed altogether, i.e.,

$$\forall (p,n) \in [1,\infty] \times \mathbb{N}, \quad \mathsf{SEP}\big(\mathsf{S}_p^n\big) \asymp n^{\max\{1,\frac{1}{2}+\frac{1}{p}\}}. \tag{1.13}$$

Also, assuming Conjecture 11 the lower order factors in the last two statements of Corollary 4 could be removed, namely we will see that Conjecture 11 implies that the separation modulus of $\mathsf{M}_n(\mathbb{R})$ equipped with the operator norm $\|\cdot\|_{\ell_p^n\to\ell_q^n}$ from ℓ_p^n to ℓ_q^n satisfies

$$\mathsf{SEP}\big(\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\ell_p^n\to\ell_q^n}\big) \asymp \begin{cases} n^{\frac{3}{2}-\frac{1}{\min\{p,q\}}} & \text{if } p,q \geqslant 2,\\ n^{\frac{1}{2}+\frac{1}{\max\{p,q\}}} & \text{if } p,q \leqslant 2,\\ n & \text{if } p \leqslant 2 \leqslant q,\\ n^{\max\{1,\frac{1}{q}-\frac{1}{p}+\frac{1}{2}\}} & \text{if } q \leqslant 2 \leqslant p, \end{cases} \tag{1.14}$$

and the separation modulus of the projective tensor product $\ell_p^n \widehat{\otimes} \ell_q^n$ satisfies

$$\mathsf{SEP}\big(\ell_p^n \widehat{\otimes} \ell_q^n\big) \asymp \begin{cases} n^{\frac{3}{2}} & \text{if } \max\{p,q\} \geqslant 2,\\ n^{1+\frac{1}{\max\{p,q\}}} & \text{if } \max\{p,q\} \leqslant 2. \end{cases} \tag{1.15}$$

Remark 174 describes ramifications of these conjectural statements to norms of algorithmic importance.

Roadmap. The rest of the Introduction effectively restarts the description of the present work, with many more details/definitions/background/ideas of proofs, than what we have included above. We organized the introductory material in this way because this work pertains to multiple mathematical disciplines, including notably Banach spaces, convex geometry, nonlinear functional analysis, metric embeddings, extension of functions, and theoretical computer science. The backgrounds of potential readers are therefore varied, so even though the above overview achieves the goal of presenting the main results quickly, it inevitably includes terminology that is not familiar to some. The aforementioned organizational choice makes the ensuing discussion accessible. Additional background can be found in the monographs [181,220, 305] (Banach space theory), [36] (nonlinear functional analysis), [201, 244] (metric embeddings), [64] (extension of functions), as well as the references that are cited throughout.

While the ensuing extended introductory text is not short, it achieves more than merely a description of the results, history, concepts and methods: it also contains groundwork that is needed for the subsequent sections. Thus, reading the Introduction will lead to a thorough conceptual understanding of the contents, leaving to the remaining sections considerations that are for the most part more technical.

We will start by focusing on the classical Lipschitz extension problem because it is more well known than the stochastic clustering issues that lead to most of our new results on Lipschitz extension, and also because it requires less technicalities (e.g., a suitable measurability setup) than our subsequent treatment of stochastic clustering. Throughout the Introduction (and beyond), we will formulate conjectures and questions that are valuable even without the links to Lipschitz extension and clustering that are derived herein. After the Introduction, the rest of this work will be organized thematically as follows. Section 2 is devoted to proofs of our various lower bounds, namely impossibility results that rule out the existence of extensions and clusterings with certain properties. Section 3 and Section 4 deal with positive results about random partitions. Specifically, Section 3 is of a more foundational nature as it describes the concepts, basic constructions, and proofs of measurability statements that are needed for later applications in the infinitary setting (of course, measurability can be ignored for statements about finite sets). Section 4 analyses in the case of normed spaces a periodic version of a commonly used randomized partitioning technique called *iterative ball partitioning*, and computes optimally (up to universal constant factors) the probabilities of its separation and padding events. Section 5 shows how to pass from random partitions to Lipschitz extension, by adjusting to the present setting the method that was developed in [173]. Section 5 also contains further foundational results on Lipschitz extension, as well questions and conjectures that are of independent interest. Section 6 contains a range of volume and surface area estimates that are needed in conjunction with the theorems of the preceding sections in order to deduce new Lipschitz extension and stochastic clustering results for various normed spaces and their subsets. Section 7 proves that Conjecture 10 and Conjecture 11 hold up to a factor of $O(\log n)$, and also shows that the approach that leads to this result cannot fully resolve Conjecture 11.

1.2 Basic notation

Given a metric space $(\mathfrak{M}, d_\mathfrak{M})$, a point $x \in \mathfrak{M}$ and a radius $r \geqslant 0$, the corresponding *closed* ball is denoted $B_\mathfrak{M}(x,r) = \{y \in \mathfrak{M} : d_\mathfrak{M}(y,x) \leqslant r\}$. If $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ is a Banach space (in this work, all vector spaces are over the real scalars unless stated otherwise), then denote by $B_\mathbf{X}$ the unit ball centered at the origin. Under this notation we have $B_\mathbf{X} = B_\mathbf{X}(0,1)$ and $B_\mathbf{X}(x,r) = x + rB_\mathbf{X}$ for every $x \in X$ and $r \geqslant 0$.

If $(\mathfrak{M}, d_\mathfrak{M}), (\mathfrak{N}, d_\mathfrak{N})$ are metric spaces and $\psi : \mathfrak{M} \to \mathfrak{N}$, then for $\mathcal{C} \subseteq \mathfrak{M}$ the Lipschitz constant of ψ on $\mathcal{C}$ is denoted $\|\psi\|_{\mathrm{Lip}(\mathcal{C};\mathfrak{N})} \in [0,\infty]$. Thus, if $\mathcal{C}$ contains at least two points, then

$$\|\psi\|_{\mathrm{Lip}(\mathcal{C};\mathfrak{N})} \stackrel{\mathrm{def}}{=} \sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \frac{d_\mathfrak{N}\big(\psi(x),\psi(y)\big)}{d_\mathfrak{M}(x,y)}.$$

In the special case $\mathfrak{N} = \mathbb{R}$ we will use the simpler notation $\|\psi\|_{\mathrm{Lip}(\mathcal{C};\mathbb{R})} = \|\psi\|_{\mathrm{Lip}(\mathcal{C})}$.

If $(\mathbf{X}, \|\cdot\|_\mathbf{X}), (\mathbf{Y}, \|\cdot\|_\mathbf{Y})$ are isomorphic Banach spaces, then their Banach–Mazur distance $d_{\mathrm{BM}}(\mathbf{X},\mathbf{Y})$ is the infimum of the products of the operator norms $\|T\|_{\mathbf{X}\to\mathbf{Y}}$ and $\|T^{-1}\|_{\mathbf{Y}\to\mathbf{X}}$ over all possible (surjective) linear isomorphisms

$$T : \mathbf{X} \to \mathbf{Y}.$$

The (bi-Lipschitz) distortion of a metric space $(\mathfrak{M}, d_\mathfrak{M})$ into a metric space $(\mathfrak{N}, d_\mathfrak{N})$, denoted $\mathsf{c}_{(\mathfrak{N},d_\mathfrak{N})}(\mathfrak{M}, d_\mathfrak{M})$ or $\mathsf{c}_\mathfrak{N}(\mathfrak{M})$ if the underlying metrics are clear from the context, is the infimum over those $D \in [1,\infty]$ for which there exists a mapping $\phi : \mathfrak{M} \to \mathfrak{N}$ and (a scaling factor) $\lambda > 0$ such that

$$\forall x,y \in \mathfrak{M}, \quad \lambda d_\mathfrak{M}(x,y) \leqslant d_\mathfrak{N}\big(\phi(x),\phi(y)\big) \leqslant D\lambda d_\mathfrak{M}(x,y). \tag{1.16}$$

Fix $n \in \mathbb{N}$. Throughout what follows, $\mathbb{R}^n$ will be always be endowed with its standard Euclidean structure, i.e., with the scalar product $\langle x,y\rangle = x_1y_1 + \cdots + x_ny_n$ for $x = (x_1,\ldots,x_n), y = (y_1,\ldots,y_n) \in \mathbb{R}^n$. Given $z \in \mathbb{R}^n \smallsetminus \{0\}$, the orthogonal projection onto its orthogonal hyperplane $z^\perp = \{x \in \mathbb{R}^n : \langle x,z\rangle = 0\}$ will be denoted $\mathsf{Proj}_{z^\perp} : \mathbb{R}^n \to \mathbb{R}^n$. For $0 < s \leqslant n$, the s-dimensional Hausdorff measure of a closed subset $A \subseteq \mathbb{R}^n$ is denoted $\mathrm{vol}_s(A)$. Integration with respect to the s-dimensional Hausdorff measure is indicated by $\mathrm{d}x$. If $0 < \mathrm{vol}_s(A) < \infty$ and $f : A \to \mathbb{R}$ is continuous, then write $⨍_A f(x)\,\mathrm{d}x = \mathrm{vol}_s(A)^{-1}\int_A f(x)\,\mathrm{d}x$.

Given a normed space $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ and $p \in [1,\infty]$, $\ell_p^n(\mathbf{X})$ is the vector space $\mathbf{X}^n$ equipped with the norm

$$\forall x = (x_1,\ldots,x_n) \in \mathbf{X}^n, \quad \|x\|_{\ell_p^n(\mathbf{X})} = \big(\|x_1\|_\mathbf{X} + \cdots + \|x_n\|_\mathbf{X}\big)^{\frac{1}{p}},$$

where for $p = \infty$ this is understood to be $\|x\|_{\ell_\infty^n(\mathbf{X})} = \max_{j\in\{1,\ldots,n\}}\|x_j\|_\mathbf{X}$. It is common to use the simpler notation $\ell_p^n = \ell_p^n(\mathbb{R})$ and we write as usual $S^{n-1} = \partial B_{\ell_2^n}$.

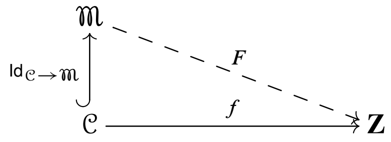

Figure 1.1. Given $K \geqslant 1$, the assertion that the Lipschitz extension modulus of a metric space $\mathfrak{M}$ satisfies $\mathsf{e}(\mathfrak{M}) < K$ means that for *all* subsets $\mathcal{C} \subseteq \mathfrak{M}$, *all* Banach spaces $\mathbf{Z}$ and *all* 1-Lipschitz mappings $f : \mathcal{C} \to \mathbf{Z}$, there is a K-Lipschitz mapping $F : \mathfrak{M} \to \mathbf{Z}$ such that the above diagram commutes, where $\mathsf{Id}_{\mathcal{C}\to\mathfrak{M}} : \mathcal{C} \to \mathfrak{M}$ is the formal inclusion.

The Schatten–von Neumann trace class S_p^n is the (n^2-dimensional) space of all n by n real matrices $\mathsf{M}_n(\mathbb{R})$, equipped with the norm that is defined by

$$\forall T \in \mathsf{M}_n(\mathbb{R}), \quad \|T\|_{\mathsf{S}_p^n} = \Big(\mathbf{Tr}\big((TT^*)^{\frac{p}{2}}\big)\Big)^{\frac{1}{p}} = \Big(\mathbf{Tr}\big((T^*T)^{\frac{p}{2}}\big)\Big)^{\frac{1}{p}},$$

where $\|T\|_{\mathsf{S}_\infty^n} = \|T\|_{\ell_2^n \to \ell_2^n}$ is the operator norm of T when it is viewed as a linear operator from ℓ_2^n to ℓ_2^n.

1.3 Lipschitz extension

As we recalled in Section 1.1, one associates to every metric space $(\mathfrak{M}, d_\mathfrak{M})$ a bi-Lipschitz invariant[6], called the Lipschitz extension modulus of $(\mathfrak{M}, d_\mathfrak{M})$ and denoted $\mathsf{e}(\mathfrak{M}, d_\mathfrak{M})$ or $\mathsf{e}(\mathfrak{M})$ if the metric is clear from the context, by defining it to be the infimum over those $K \in [1, \infty]$ with the property that for *every* nonempty subset $\mathcal{C} \subseteq \mathfrak{M}$, *every* Banach space $(\mathbf{Z}, \|\cdot\|_\mathbf{Z})$ and *every* Lipschitz function $f : \mathcal{C} \to \mathbf{Z}$ there is a mapping $F : \mathfrak{M} \to \mathbf{Z}$ that extends f, i.e., $F(x) = f(x)$ whenever $x \in \mathcal{C}$, and $\|F\|_{\mathrm{Lip}(\mathfrak{M},\mathbf{Z})} \leqslant K\|f\|_{\mathrm{Lip}(\mathcal{C},\mathbf{Z})}$; see Figure 1.1. All of the ensuing extension theorems hold for a larger class of target metric spaces that need not necessarily be Banach spaces, including Hadamard spaces and Busemann nonpositively curved spaces [57], or more generally spaces that posses a conical geodesic bicombing (see, e.g., [86]). This greater generality will be discussed in Section 5, but we prefer at this introductory juncture to focus on the more classical and highly-studied setting of Banach space targets.

When $(\mathbf{X}, \|\cdot\|_\mathbf{X})$ is a finite dimensional normed space, the currently best-available general bounds on the quantity $\mathsf{e}(\mathbf{X})$ in terms of $\dim(\mathbf{X})$ are contained the following theorem.

[6]The assertion that $\mathsf{e}(\mathfrak{M})$ is a bi-Lipschitz invariant refers to the fact that the definition immediately implies that if $(\mathfrak{N}, d_\mathfrak{N})$ is another metric space into which $(\mathfrak{M}, d_\mathfrak{M})$ admits a bi-Lipschitz embedding, then $\mathsf{e}(\mathfrak{M}) \leqslant \mathsf{c}_\mathfrak{N}(\mathfrak{M})\mathsf{e}(\mathfrak{N})$.

Theorem 13. *There is a universal constant $c > 0$ such that for any finite dimensional normed space* $\mathbf{X}$,

$$\dim(\mathbf{X})^c \lesssim \mathsf{e}(\mathbf{X}) \lesssim \dim(\mathbf{X}). \tag{1.17}$$

The bound $\mathsf{e}(\mathbf{X}) \lesssim \dim(\mathbf{X})$ in (1.17) is a famous result of Johnson, Lindenstrauss and Schechtman [140], which they proved by cleverly refining the classical extension method of Whitney [312]; different proofs of this estimate were found by Lee and the author [173] as well as by Brudnyi and Brudnyi [61] (see also the discussion in the paragraph following equation (1.37) below). It remains a major longstanding open problem to determine whether the bound of [140] could be improved to

$$\mathsf{e}(\mathbf{X}) = o\big(\dim(\mathbf{X})\big).$$

The new content of Theorem 13 is the lower bound on $\mathsf{e}(\mathbf{X})$, which improves over the previously known bound $\mathsf{e}(\mathbf{X}) \gtrsim \exp(c\sqrt{\log\dim(\mathbf{X})})$; see Remark 98 for the history of this question. It is a very interesting open problem to determine the supremum over those c for which Theorem 13 holds.[7] More generally, it is natural to aim to evaluate the precise power-type behavior of $\mathsf{e}(\mathbf{X})$ as $\dim(\mathbf{X}) \to \infty$ for specific (sequences of) finite dimensional normed spaces $\mathbf{X}$. However, prior to the present work and despite many efforts over the years, this was not achieved for *any* finite dimensional normed space whatsoever.

Theorem 14 (Restatement of Theorem 2). *For every $n \in \mathbb{N}$ we have $\mathsf{e}(\ell_\infty^n) \asymp \sqrt{n}$.*

The bound $\mathsf{e}(\ell_\infty^n) \gtrsim \sqrt{n}$ follows from a combination of [60, Theorem 4] and [62, Theorem 1.2]. The new content of Theorem 14 is the upper bound $\mathsf{e}(\ell_\infty^n) \lesssim \sqrt{n}$ (and, importantly, the extension procedure that leads to it; see below). The previously best-known upper bound on $\mathsf{e}(\ell_\infty^n)$ was the aforementioned $O(n)$ estimate of [140]. The question of evaluating the asymptotic behavior of $\mathsf{e}(\ell_p^n)$ as $n \to \infty$ for each $p \in [1,\infty]$ is natural and longstanding; it was stated in [60, Problem 2] and reiterated in [63, Section 4], [62, Problem 1.4] and [64, Problem 8.14]. Theorem 14 answers this question when $p = \infty$. The upper bound on $\mathsf{e}(\ell_\infty^n)$ of Theorem 14 is a special case of a general extension criterion that provides the best-known Lipschitz extension results in other settings (including for ℓ_p^n when $p > 2$), but we chose to state it separately because it yields the first (and currently essentially only) family of normed spaces for which the growth rate of their Lipschitz extension moduli has been determined.

Remark 15. It is also meaningful to study extension of θ-Hölder functions for any $0 < \theta \leqslant 1$. Namely, one can analogously define the θ-Hölder extension modulus of a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, denoted $\mathsf{e}^\theta(\mathfrak{M})$. Alternatively, this notion falls into the above

[7] Our proof of the lower bound on $\mathsf{e}(\mathbf{X})$ of Theorem 13 shows that this supremum is at least $\frac{1}{12}$; see equation (2.5).

Lipschitz-extension framework because one can define

$$\mathsf{e}^{\theta}(\mathcal{M}) \stackrel{\mathrm{def}}{=} \mathsf{e}\big(\mathcal{M}, d_{\mathcal{M}}^{\theta}\big). \tag{1.18}$$

The results that we obtain herein also yield improved estimates on θ-Hölder extension moduli; see Corollary 140. However, when $\theta < 1$ we never get a matching lower bound (the reason why we can do better in the Lipschitz regime $\theta = 1$ is essentially due to the fact that Lipschitz functions are differentiable almost everywhere). For example, in the setting of Theorem 14 we get the upper bound

$$\forall \theta \in (0,1], \quad \mathsf{e}^{\theta}\big(\ell_\infty^n\big) \lesssim n^{\frac{\theta}{2}}, \tag{1.19}$$

but the best lower bound on $\mathsf{e}^{\theta}(\ell_\infty^n)$ that we are at present able to prove is

$$\mathsf{e}^{\theta}\big(\ell_\infty^n\big) \gtrsim n^{\max\{\frac{\theta}{4}, \frac{\theta}{2}+\theta^2-1\}} = \begin{cases} n^{\frac{\theta}{4}} & \text{if } 0 \leqslant \theta \leqslant \frac{\sqrt{65}-1}{8}, \\ n^{\frac{\theta}{2}+\theta^2-1} & \text{if } \frac{\sqrt{65}-1}{8} \leqslant \theta \leqslant 1. \end{cases} \tag{1.20}$$

We conjecture that $\mathsf{e}^{\theta}(\ell_\infty^n) \asymp_\theta n^{\frac{\theta}{2}}$, but proving this for $\theta < 1$ would likely require a genuinely new idea.

Question 16. Despite its utility in many cases, the extension method that underlies Theorem 14 does not yield improved bounds for some important spaces, including notably ℓ_1^n and ℓ_2^n. Thus, determining the asymptotic behavior of $\mathsf{e}(\ell_1^n)$ and $\mathsf{e}(\ell_2^n)$ as $n \to \infty$ remains a tantalizing open question. Specifically, the currently best-known bounds on $\mathsf{e}(\ell_1^n)$ are

$$\sqrt{n} \lesssim \mathsf{e}\big(\ell_1^n\big) \lesssim n, \tag{1.21}$$

where the first inequality in (1.21) is due to Johnson and Lindenstrauss [138] and the second inequality in (1.21) is the aforementioned general upper bound of [140] on the Lipschitz extension modulus of *any* n-dimensional normed space. The currently best-known bounds in the Hilbertian setting are

$$\sqrt[4]{n} \lesssim \mathsf{e}\big(\ell_2^n\big) \lesssim \sqrt{n}, \tag{1.22}$$

where the first inequality in (1.22) is due to Mendel and the author [210] (a different proof of this lower bound on $\mathsf{e}(\ell_2^n)$ follows from [231]), and the second inequality in (1.22) is from [173].

By the bi-Lipschitz invariance of the Lipschitz extension modulus, the second inequality in (1.22) implies the following bound from [173], which holds for every finite dimensional normed space $\mathbf{X}$:

$$\mathsf{e}(\mathbf{X}) \lesssim d_{\mathrm{BM}}\big(\mathbf{X}, \ell_2^{\dim(\mathbf{X})}\big)\sqrt{\dim(\mathbf{X})}. \tag{1.23}$$

This refines the upper bound on $\mathsf{e}(\mathbf{X})$ in (1.17) because $d_{\mathrm{BM}}(\mathbf{X}, \ell_2^{\dim(\mathbf{X})}) \leqslant \sqrt{\dim(\mathbf{X})}$ by John's theorem [137].

Remark 17. In the context of the aforementioned question whether the bound $\mathsf{e}(\mathbf{X}) \lesssim \dim(\mathbf{X})$ of [140] is optimal, by (1.23) we see that $\mathsf{e}(\mathbf{X}) = o(\dim(\mathbf{X}))$ unless the Banach–Mazur distance between $\mathbf{X}$ and Euclidean space is of order $\sqrt{\dim(\mathbf{X})}$. Structural properties of such spaces of extremal distance to Euclidean space have been studied in [15, 43, 142, 221, 255]; see also [305, Chapters 6 and 7]. In particular, the Mil'man–Wolfson theorem [221] asserts that this holds if and only if $\mathbf{X}$ has a subspace of dimension $k = k(\dim(\mathbf{X}))$ whose Banach–Mazur distance to ℓ_1^k is $O(1)$, where $\lim_{n\to\infty} k(n) = \infty$.

As $d_{\mathrm{BM}}(\ell_p^n, \ell_2^n) \asymp n^{|p-2|/(2p)}$ for all $n \in \mathbb{N}$ and $p \in [1,\infty]$ (see [139, Section 8]), it follows from (1.23) that

$$\mathsf{e}(\ell_p^n) \lesssim \begin{cases} n^{\frac{1}{p}} & \text{if } p \in [1,2], \\ n^{1-\frac{1}{p}} & \text{if } p \in [2,\infty]. \end{cases} \tag{1.24}$$

(1.24) was the previously best-known upper bound on $\mathsf{e}(\ell_p^n)$, and here we improve it for every $p > 2$.

Theorem 18. *For every $n \in \mathbb{N}$ and every $p \in [1,\infty]$ we have $\mathsf{e}(\ell_p^n) \lesssim n^{\max\{\frac{1}{2},\frac{1}{p}\}}$.*

Theorem 14 is the case $p = \infty$ of Theorem 18. We do not know if Theorem 18 is optimal (perhaps up to lower order factors) as $n \to \infty$ for fixed $p \in [2,\infty)$, but we conjecture that this is indeed the case, which would resolve [60, Problem 2]. The currently best-known lower bound on $\mathsf{e}(\ell_p^n)$ for every $p \in [1,\infty]$ is

$$\mathsf{e}(\ell_p^n) \gtrsim \begin{cases} n^{\frac{1}{p}-\frac{1}{2}} & \text{if } 1 \leqslant p \leqslant \frac{4}{3}, \\ \sqrt[4]{n} & \text{if } \frac{4}{3} \leqslant p \leqslant 2, \\ n^{\frac{1}{2p}} & \text{if } 2 \leqslant p \leqslant 3, \\ n^{\frac{1}{2}-\frac{1}{p}} & \text{if } 3 \leqslant p \leqslant \infty. \end{cases} \tag{1.25}$$

A lower bound on $\mathsf{e}(\ell_p^n)$ that coincides with (1.25) when $p \in [1,4/3] \cup [3,\infty]$ is stated in [64, Corollary 8.12], but [64, Corollary 8.12] is weaker than (1.25) when $4/3 < p < 3$. The reason for this is that the lower bound of [210] on $\mathsf{e}(\ell_2^n)$ that appears in (1.22) was not available when [64] was written, but (1.25) for $4/3 < p < 3$ follows quickly by combining the first inequality in (1.22) with [99]; see Remark 2.4.

Remark 19. Theorem 18 resolves negatively a conjecture that A. Brudnyi and Y. Brudnyi posed as Conjecture 5 in [60]. They conducted a comprehensive study of the *linear* extension problem for real-valued Lipschitz functions, where one considers for a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ a quantity $\lambda(\mathfrak{M})$ which is defined the same as $\mathsf{e}(\mathfrak{M})$, but with the further requirements that the function f is real-valued and that the extended function F depends linearly on f. Namely, $\lambda(\mathfrak{M})$ is the infimum over those $K \in [1,\infty]$ such that for every $\mathcal{C} \subseteq \mathfrak{M}$ there is a linear operator $\mathsf{Ext}_{\mathcal{C}} : \mathrm{Lip}(\mathcal{C}) \to \mathrm{Lip}(\mathfrak{M})$ that assigns to every Lipschitz function $f : \mathcal{C} \to \mathbb{R}$ a function $\mathsf{Ext}_{\mathcal{C}} f : \mathfrak{M} \to \mathbb{R}$ satisfying $\mathsf{Ext}_{\mathcal{C}} f(s) = f(s)$ for every $s \in \mathcal{C}$, and

$$\|\mathsf{Ext}_{\mathcal{C}} f\|_{\mathrm{Lip}(\mathfrak{M})} \leqslant K\|f\|_{\mathrm{Lip}(\mathcal{C})}.$$

They also considered a natural variant of this quantity when $\mathfrak{M} = \mathbf{X}$ is a Banach space, denoted $\lambda_{\mathrm{conv}}(\mathbf{X})$, which is defined almost identically to $\lambda(\mathbf{X})$ except that now the subset $\mathcal{C}$ is only allowed to be any *convex* subset of $\mathbf{X}$ rather than a subset of $\mathbf{X}$ without any additional restriction. Conjecture 5 in [60] states that

$$\forall (p,n) \in [1,\infty] \times \mathbb{N}, \quad \lambda\big(\ell_p^n\big) \asymp_p \lambda_{\mathrm{conv}}\big(\ell_p^n\big)\sqrt{n}. \tag{1.26}$$

Theorem 18 implies that this conjecture is *false* for every $p \in (2,\infty]$. Indeed, the asymptotic behavior of $\lambda_{\mathrm{conv}}(\ell_p^n)$ was evaluated in [63, Theorem 2.19], where it was shown that

$$\forall p \in [1,\infty], \quad \lambda_{\mathrm{conv}}\big(\ell_p^n\big) \asymp n^{\left|\frac12-\frac1p\right|}.$$

Consequently, $\lambda_{\mathrm{conv}}(\ell_p^n)\sqrt{n} \asymp n^{1-\frac1p}$ when $p > 2$. Next, in [62] a quantity $\nu(\mathfrak{M})$ was associated to a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ by defining it almost identically to the definition of $\mathsf{e}(\mathfrak{M})$, except that the target Banach space $\mathbf{Z}$ is allowed to be any *finite dimensional* Banach space rather than any Banach space whatsoever. By definition $\nu(\mathfrak{M}) \leqslant \mathsf{e}(\mathfrak{M})$, but actually $\lambda(\mathfrak{M}) = \nu(\mathfrak{M})$ thanks to [62, Theorem 1.2] (see the work [11] of Ambrosio and Puglisi for more on this "linearization phenomenon"). Using these results in combination with Theorem 18, we see that for every $p \in (2,\infty]$, as $n \to \infty$ we have

$$\lambda\big(\ell_p^n\big) = \nu\big(\ell_p^n\big) \leqslant \mathsf{e}\big(\ell_p^n\big) \lesssim \sqrt{n} = o\big(n^{1-\frac1p}\big).$$

Thus, $\lambda(\ell_p^n) = o\big(\lambda_{\mathrm{conv}}(\ell_p^n)\sqrt{n}\big)$ as $n \to \infty$ for any $p > 2$, in contrast to the conjecture (1.26) of [60].

Prior to passing to the general Lipschitz extension theorem that underlies the new results that were described above, we will further illustrate its utility by stating one more concrete application. For each $p \in [1,\infty]$ and $n \in \mathbb{N}$, if $k \in \{1,\ldots,n\}$, then let $(\ell_p^n)_{\leqslant k}$ denote the subset of $\mathbb{R}^n$ consisting of those vectors with at most k nonzero coordinates, equipped with the metric that is inherited from ℓ_p^n.

Theorem 20. *For every $p \in [1,\infty]$, every $n \in \mathbb{N}$ and every $k \in \{1,\ldots,n\}$ we have*

$$\mathsf{e}\big((\ell_p^n)_{\leqslant k}\big) \lesssim k^{\max\{\frac1p,\frac12\}}.$$

Theorem 18 is the special case $k = n$ and $p \geqslant 2$ of Theorem 20. If $1 \leqslant p \leqslant 2$ and $k = n$, then Theorem 20 is the estimate (1.24), which is the best-known upper bound on $\mathsf{e}(\ell_p^n)$ for p in this range. However, for general $k \in \{1,\ldots,n\}$ Theorem 20 yields a refinement of (1.24) in the entire range $p \in [1,\infty]$ which does not seem to follow from previously known results. In particular, the case $p = 2$ of Theorem 20 becomes

$$\mathsf{e}\big((\ell_2^n)_{\leqslant k}\big) \lesssim \sqrt{k}. \tag{1.27}$$

Even though (1.27) concerns a Euclidean setting, its proof relies on a construction that employs a multi-scale partitioning scheme using balls of an auxiliary metric on $\mathbb{R}^n$ that differs from the ambient Euclidean metric. The utility of such a non-Euclidean geometric reasoning despite the Euclidean nature of the question being studied is discussed further in Section 1.4.

1.4 A volumetric upper bound on the Lipschitz extension modulus

We will prove that Theorem 20 (hence also its special cases Theorem 14 and Theorem 18) is a consequence of Theorem 21 below, which is a Lipschitz extension theorem for subsets of finite dimensional normed spaces in terms of volumes of hyperplane projections of their unit balls. Throughout what follows, for dealing with volumetric notions we will adhere to the following conventions. Given $n \in \mathbb{N}$, when we say that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space we mean that the underlying vector space is $\mathbb{R}^n$ and that $\|\cdot\|_{\mathbf{X}} : \mathbb{R}^n \to [0,\infty)$ is a norm on $\mathbb{R}^n$. This is, of course, always achievable by fixing any scalar product on an n-dimensional normed space. While the ensuing statements hold in this setting, i.e., for an arbitrary identification of $\mathbf{X}$ with $\mathbb{R}^n$, a judicious choice of such an identification is beneficial; the discussion of this important matter is postponed to Section 1.6.2 because it is not needed for the initial description of the main results. We will continue using the notation

$$B_{\mathbf{X}} = \{x \in \mathbb{R}^n : \|x\|_{\mathbf{X}} \leqslant 1\}$$

for the unit ball of $\mathbf{X}$. Also, given $\mathcal{C} \subseteq \mathbb{R}^n$ we denote by $\mathcal{C}_{\mathbf{X}}$ the metric space consisting of the set $\mathcal{C}$ equipped with the metric that is inherited from $\|\cdot\|_{\mathbf{X}}$. This notation is important for us because we will crucially need to simultaneously consider more than one norm on $\mathbb{R}^n$.

Theorem 21. *Suppose that $n \in \mathbb{N}$ and that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ are two normed spaces. Then, for every $\mathcal{C} \subseteq \mathbb{R}^n$ we have the following upper bound on the Lipschitz extension modulus of $\mathcal{C}_{\mathbf{X}}$:*

$$\mathsf{e}(\mathcal{C}_{\mathbf{X}}) \lesssim \Bigg(\sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \frac{\|x-y\|_{\mathbf{X}}}{\|x-y\|_{\mathbf{Y}}}\Bigg) \sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \Bigg(\frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^{\perp}} B_{\mathbf{Y}}\big)}{\mathrm{vol}_n(B_{\mathbf{Y}})} \cdot \frac{\|x-y\|_{\ell_2^n}}{\|x-y\|_{\mathbf{X}}}\Bigg). \tag{1.28}$$

We will next discuss the geometric meaning of Theorem 21 and derive some of its consequences, including Theorem 20. Firstly, by homogeneity the case $\mathcal{C} = \mathbb{R}^n$

of (1.28) becomes

$$\mathsf{e}(\mathbf{X}) \lesssim \Big(\sup_{y\in\partial B_{\mathbf{Y}}} \|y\|_{\mathbf{X}} \Big) \sup_{x\in\partial B_{\mathbf{X}}} \left(\frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp} B_{\mathbf{Y}}\big)}{\mathrm{vol}_n(B_{\mathbf{Y}})} \|x\|_{\ell_2^n} \right). \tag{1.29}$$

The quantity $\sup_{y\in\partial B_{\mathbf{Y}}} \|y\|_{\mathbf{X}}$ in (1.29) is the norm $\|\mathsf{Id}_n\|_{\mathbf{Y}\to\mathbf{X}}$ of the identity matrix $\mathsf{Id}_n \in \mathsf{M}_n(\mathbb{R})$ as an operator from $\mathbf{Y}$ to $\mathbf{X}$. Alternatively,

$$\sup_{y\in\partial B_{\mathbf{Y}}} \|y\|_{\mathbf{X}} = \frac{1}{2}\,\mathrm{diam}_{\mathbf{X}}(B_{\mathbf{Y}}),$$

where for each $\mathcal{C} \subseteq \mathbb{R}^n$ we denote its diameter with respect to the metric that $\mathbf{X}$ induces by

$$\mathrm{diam}_{\mathbf{X}}(\mathcal{C}) = \sup_{x,y\in\mathcal{C}} \|x-y\|_{\mathbf{X}}.$$

Given a convex body $K \subseteq \mathbb{R}^n$, let $\Pi^*K \subseteq \mathbb{R}^n$ be the polar of the *projection body* of K, which is defined to be the unit ball of the norm $\|\cdot\|_{\Pi^*K}$ on $\mathbb{R}^n$ that is given by setting for every $x \in \mathbb{R}^n \smallsetminus \{0\}$,

$$\|x\|_{\Pi^*K} \stackrel{\mathrm{def}}{=} \frac{1}{2}\int_{\partial K} \big|\langle x, N_K(y)\rangle\big|\,\mathrm{d}y = \mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp} K\big)\|x\|_{\ell_2^n}, \tag{1.30}$$

where $N_K(y) \in S^{n-1}$ denotes the unit outer normal to ∂K at $y \in \partial K$ (which is uniquely defined almost everywhere with respect to the surface-area measure on ∂K), and the final equality in (1.30) is the Cauchy projection formula (see, e.g., [102, Appendix A]). The projection body ΠK of K is the polar of Π^*K. These important notions were introduced by Petty [251]. When $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space let $\Pi^*\mathbf{X}$ be the normed space whose unit ball is $\Pi^*B_{\mathbf{X}}$. Let $\Pi\mathbf{X} = (\Pi^*\mathbf{X})^*$ be the normed space whose unit ball is $\Pi B_{\mathbf{X}}$.

By substituting (1.30) into (1.29) we get the following interpretation of our bound on $\mathsf{e}(\mathbf{X})$ in terms of analytic and geometric properties of projection bodies; it is worthwhile to state it as a separate corollary even though it is only a matter of notation because of its intrinsic interest and also because these alternative viewpoints were useful for guiding some of the subsequent considerations.

Corollary 22. *Any two normed spaces* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$, $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ *satisfy*

$$\begin{aligned}\mathsf{e}(\mathbf{X}) &\lesssim \frac{\mathrm{diam}_{\mathbf{X}}(B_{\mathbf{Y}})\,\mathrm{diam}_{\Pi^*\mathbf{Y}}(B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{Y}})} \asymp \frac{\|\mathsf{Id}_n\|_{\mathbf{Y}\to\mathbf{X}}\|\mathsf{Id}_n\|_{\mathbf{X}\to\Pi^*\mathbf{Y}}}{\mathrm{vol}_n(B_{\mathbf{Y}})}\\ &= \frac{\|\mathsf{Id}_n\|_{\mathbf{X}\to\mathbf{Y}}\|\mathsf{Id}_n\|_{\Pi\mathbf{Y}\to\mathbf{X}^*}}{\mathrm{vol}_n(B_{\mathbf{Y}})} \asymp \frac{\mathrm{diam}_{\mathbf{X}}(B_{\mathbf{Y}})\,\mathrm{diam}_{\mathbf{X}^*}(\Pi B_{\mathbf{Y}})}{\mathrm{vol}(B_{\mathbf{Y}})}.\end{aligned} \tag{1.31}$$

The penultimate step in (1.31) is duality (the norm of an operator equals the norm of its adjoint) and the final quantity in (1.31) relates Theorem 21 to the second estimate in Theorem 3.

Remark 23. It is worthwhile to note that Corollary 22 has the right *affine invariance*. For $S \in \mathsf{SL}_n(\mathbb{R})$ let $S\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{S\mathbf{X}})$ be the normed space whose unit ball is $SB_\mathbf{X}$. Equivalently, $\|x\|_{S\mathbf{X}} = \|S^{-1}x\|_\mathbf{X}$ for every $x \in \mathbb{R}^n$. Then $\mathbf{X}$ and $S\mathbf{X}$ are isometric, so $\mathsf{e}(S\mathbf{X}) = \mathsf{e}(\mathbf{X})$. We have $(S\mathbf{X})^* = (S^*)^{-1}\mathbf{X}^*$ (by definition), and $\Pi(SB_\mathbf{Y}) = (S^*)^{-1}\Pi B_\mathbf{Y}$ by [251]. From this we see that $\operatorname{diam}_{(S\mathbf{X})^*}(\Pi B_{S\mathbf{Y}}) = \operatorname{diam}_{\mathbf{X}^*}(\Pi B_\mathbf{Y})$. Thus, the minimum of the right-hand side of (1.31) over all normed spaces $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_\mathbf{Y})$ is also invariant under the action of $\mathsf{SL}_n(\mathbb{R})$.

The special case of Theorem 21 in which the normed space $\mathbf{Y}$ coincides with the given normed space $\mathbf{X}$ is in itself a nontrivial bound on the Lipschitz extension modulus. Examining this special case first will help elucidate how the idea arose to introduce an auxiliary space $\mathbf{Y}$ that may differ from $\mathbf{X}$, and why this can yield stronger estimates. If $\mathbf{X} = \mathbf{Y}$, then the bound (1.28) becomes

$$\mathsf{e}(\mathcal{C}_\mathbf{X}) \lesssim \sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \left(\frac{\operatorname{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^\perp} B_\mathbf{X}\big)}{\operatorname{vol}_n(B_\mathbf{X})} \cdot \frac{\|x-y\|_{\ell_2^n}}{\|x-y\|_\mathbf{X}} \right). \tag{1.32}$$

Correspondingly, the bound (1.29) becomes

$$\mathsf{e}(\mathbf{X}) \lesssim \sup_{z\in\partial B_\mathbf{X}} \left(\frac{\operatorname{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} B_\mathbf{X}\big)}{\operatorname{vol}_n(B_\mathbf{X})} \|z\|_{\ell_2^n} \right) = \frac{\operatorname{diam}_{\Pi^*\mathbf{X}}(B_\mathbf{X})}{\operatorname{vol}_n(B_\mathbf{X})}. \tag{1.33}$$

Even these weaker estimates suffice to obtain new results, e.g., we will see that this is so if $2 \leqslant p = O(1)$ and $\mathbf{X} = \ell_p^n$. However, as we will soon explain, (1.33) does not imply an upper bound on ℓ_∞^n that is better than the aforementioned general bound of [140]. Despite this shortcoming of (1.32) and (1.33) relative to (1.28), it is worthwhile to state these special cases of Theorem 21 separately because they are simpler than (1.28) and hence perhaps somewhat easier to remember. Moreover, a naïve way to enhance the applicability of (1.32) is to leverage the fact that the Lipschitz extension modulus is a bi-Lipschitz invariant, so that

$$\mathsf{e}(\mathcal{C}_\mathbf{X}) \leqslant \|\mathsf{Id}_n\|_{\operatorname{Lip}(\mathcal{C}_\mathbf{Y},\mathcal{C}_\mathbf{X})} \|\mathsf{Id}_n\|_{\operatorname{Lip}(\mathcal{C}_\mathbf{X},\mathcal{C}_\mathbf{Y})} \mathsf{e}(\mathcal{C}_\mathbf{Y}).$$

Consequently, by estimating $\mathsf{e}(\mathcal{C}_\mathbf{Y})$ through (1.32) we formally deduce from (1.32) that

$$\begin{aligned} \mathsf{e}(\mathcal{C}_\mathbf{X}) \lesssim &\left(\sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \frac{\|x-y\|_\mathbf{X}}{\|x-y\|_\mathbf{Y}} \right) \left(\sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \frac{\|x-y\|_\mathbf{Y}}{\|x-y\|_\mathbf{X}} \right) \\ &\cdot \sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}} \left(\frac{\operatorname{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^\perp} B_\mathbf{Y}\big)}{\operatorname{vol}_n(B_\mathbf{Y})} \cdot \frac{\|x-y\|_{\ell_2^n}}{\|x-y\|_\mathbf{Y}} \right). \end{aligned} \tag{1.34}$$

We do not see how to deduce Theorem 18 and Theorem 20 from (1.34). However, we will show that (1.34) suffices for proving Theorem 14 (as well as some other results that will be presented later). In summary, even the case of Theorem 21 in which the auxiliary space $\mathbf{Y}$ coincides with $\mathbf{X}$ is valuable, but Theorem 21 does not follow from merely combining its special case $\mathbf{Y} = \mathbf{X}$ with bi-Lipschitz invariance.

Given a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $z \in \mathbb{R}^n \setminus \{0\}$, the quantity

$$\frac{1}{n}\,\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} B_{\mathbf{X}}\big)\|z\|_{\ell_2^n} \tag{1.35}$$

is equal to the volume of the cone

$$\mathsf{Cone}_z(B_{\mathbf{X}}) \stackrel{\mathrm{def}}{=} \mathrm{conv}\big(\{z\} \cup \mathsf{Proj}_{z^\perp} B_{\mathbf{X}}\big) \subseteq \mathbb{R}^n \tag{1.36}$$

whose base is the $(n-1)$-dimensional convex set $\mathsf{Proj}_{z^\perp} B_{\mathbf{X}} \subseteq z^\perp$ and whose apex is z. In (1.36) and throughout what follows, $\mathrm{conv}(\cdot)$ denotes the convex hull. Thus, the estimate (1.33) can be restated as follows:

$$\mathsf{e}(\mathbf{X}) \lesssim n \sup_{z \in \partial B_{\mathbf{X}}} \frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(B_{\mathbf{X}})\big)}{\mathrm{vol}_n(B_{\mathbf{X}})}. \tag{1.37}$$

Through (1.37) we see that the geometric interpretation of the "bad spaces" $\mathbf{X}$ for (1.33) is that these are the spaces that have a "pointy direction" $z \in \partial B_{\mathbf{X}}$ for which the volume of the cone $\mathsf{Cone}_z(B_{\mathbf{X}})$ is a significant fraction of the volume of $B_{\mathbf{X}}$. Examples will be presented next, but note first that a short geometric argument (see the proof of [109, Lemma 5.1]) shows that $\mathrm{vol}_n(\mathsf{Cone}_z(B_{\mathbf{X}})) \leqslant \mathrm{vol}_n(B_{\mathbf{X}})/2$, so the right-hand side of (1.37) is at most $n/2$. Hence, (1.33) is a refinement of the classical bound $\mathsf{e}(\mathbf{X}) \lesssim n$ of [140].

Nevertheless, a "vanilla" application of (1.33) does not yield an asymptotically better estimate than that of [140] even when $\mathbf{X} = \ell_\infty^n$. Indeed, $B_{\ell_\infty^n} = [-1,1]^n$ and a simple argument (see [75]) shows that

$$\forall z \in \mathbb{R}^n \setminus \{0\}, \qquad \frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp}[-1,1]^n\big)}{\mathrm{vol}_n([-1,1]^n)} = \frac{\|z\|_{\ell_1^n}}{2\|z\|_{\ell_2^n}}. \tag{1.38}$$

So, by considering the all 1's vector $z = \mathbf{1}_{\{1,\ldots,n\}} \in \partial B_{\ell_\infty^n}$ we see that for $\mathbf{X} = \ell_\infty^n$ the right-hand side of (1.33) is at least $n/2$. The right-hand side of (1.33) is at least $n/2$ when $\mathbf{X} = \ell_1^n$, as seen by taking $z = (1,0,\ldots,0) \in \partial B_{\ell_1^n}$. Such "problematic" directions $z \in \partial B_{\mathbf{X}}$ can sometimes be the overwhelming majority of $\partial B_{\mathbf{X}}$. Consider Ball's counterexample [21] to the Shepard Problem [287], which states that for any $n \in \mathbb{N}$ there is a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ such that $\mathrm{vol}_n(B_{\mathbf{X}}) = 1$ yet $\mathrm{vol}_{n-1}(\mathsf{Proj}_{z^\perp} B_{\mathbf{X}}) \gtrsim \sqrt{n}$ for *every* $z \in S^{n-1}$. Since $\mathrm{vol}_n(B_{\ell_2^n}) \leqslant (3/\sqrt{n})^n$ while $\mathrm{vol}_n(B_{\mathbf{X}}) = 1$, the proportion of those $z \in \partial B_{\mathbf{X}}$ for which $\|z\|_{\ell_2^n} \geqslant \sqrt{n}/4$ tends to 1 as $n \to \infty$ (exponentially fast). Any such z satisfies

$$\frac{\mathrm{vol}_{n-1}(\mathsf{Proj}_{z^\perp} B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})}\|z\|_{\ell_2^n} \gtrsim n.$$

These obstacles can sometimes be overcome by perturbing the given normed space $\mathbf{X}$ prior to invoking (1.33), i.e., by using of Theorem 21 with a suitably chosen auxiliary normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$. In particular, $\|\cdot\|_{\ell_2^n} \leqslant n^{1/2-1/p}\|\cdot\|_{\ell_p^n}$ when $p \geqslant 2$ by Hölder's inequality, so Theorem 18 follows from a substitution of the space $\mathbf{Y}_p^n$ of Theorem 24 below into Theorem 21 (with $\mathbf{X} = \ell_p^n$), or even into (1.34).

Theorem 24. *For any $n \in \mathbb{N}$ and $p \in [1, \infty]$ there is a normed space*

$$\mathbf{Y}_p^n = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}_p^n})$$

that satisfies

$$\forall x \in \mathbb{R}^n \smallsetminus \{0\}, \quad \|x\|_{\mathbf{Y}_p^n} \asymp \|x\|_{\ell_p^n}, \quad \textit{and} \quad \frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp} B_{\mathbf{Y}_p^n}\big)}{\mathrm{vol}_n\big(B_{\mathbf{Y}_p^n}\big)} \lesssim n^{\frac{1}{p}}. \tag{1.39}$$

The case $p = \infty$ of Theorem 24 implies Theorem 20 by an application of Theorem 21. Indeed, fix $p \geqslant 1$ and $n \in \mathbb{N}$. Suppose that $x, y \in (\ell_p^n)_{\leqslant k}$ for some $k \in \{1, \ldots, n\}$. Then $x - y$ has at most $2k$ nonzero coordinates. Therefore, if $\mathbf{Y}_\infty^n$ is as in Theorem 24, then by Hölder's inequality we have

$$(2k)^{-\max\{\frac{1}{2}-\frac{1}{p},0\}}\|x-y\|_{\ell_2^n} \leqslant \|x-y\|_{\ell_p^n} \leqslant (2k)^{\frac{1}{p}}\|x-y\|_{\ell_\infty^n} \asymp k^{\frac{1}{p}}\|x-y\|_{\mathbf{Y}_\infty^n}. \tag{1.40}$$

Theorem 20 follows by substituting these bounds and the case $p = \infty$ of (1.39) into (1.28). Observe that we would have obtained the weaker bound $\mathsf{e}((\ell_p^n)_{\leqslant k}) \lesssim k^{1/p+1/2}$ if we used (1.34) instead of (1.28).

If $p = O(1)$, then one can take $\mathbf{Y}_p^n = \ell_p^n$ in Theorem 24. In fact, the direction $z \in S^{n-1}$ at which

$$\max_{z \in S^{n-1}} \mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} B_{\ell_p^n}\big) \tag{1.41}$$

is attained was determined by Barthe and the author in [32]. This result implies that

$$\forall p \geqslant 1, \quad \max_{z \in S^{n-1}} \frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} B_{\ell_p^n}\big)}{\mathrm{vol}_n(B_{\ell_p^n})} \asymp n^{\frac{1}{p}}\sqrt{\min\{p,n\}}. \tag{1.42}$$

As [32] computes (1.41) exactly, the implicit constant factors in (1.42) can be evaluated, but in the present context such precision is of secondary importance. While (1.42) follows from [32] (see the deduction in [227]), we will give a self-contained proof of (1.42) in Section 6 as a special case of a more general result that we will use for other purposes as well. In the range $q \in (2, \infty)$, a different approach to computing (1.41) was found in [157]. Earlier methods for estimating (1.41) with worse lower order factors are due to [223, 286]; the latter is an adaptation of an idea (used for related purposes) in [45].

For each $k \in \{1, \ldots, n\}$, by applying (1.28) with $\mathbf{Y} = \ell_q^n$ for some $q \geqslant p$, using (1.42) with p replaced by q, and optimizing the resulting bound over q, one obtains a result that matches Theorem 20 up to unbounded lower order factors. More precisely, the best that one can get with this approach is when

$$q = \max\left\{2\log\left(\frac{n}{k}\right), p\right\}$$

if $p \leqslant \log(2k)$. If $p \geqslant \log(2k)$, then use (1.28) with $\mathbf{Y} = \ell^n_{\log(2k)}$.

Theorem 24 provides an auxiliary space $\mathbf{Y}$ for which a use of (1.28) removes the above lower order factors, and yields a sharp result when $p = \infty$ (we conjecture that it is sharp for any $p \geqslant 2$). Regardless of whether we apply (1.28) with the space $\mathbf{Y} = \mathbf{Y}^n_\infty$ of Theorem 24 or with $\mathbf{Y} = \ell^n_q$ for a suitable choice of $q \geqslant p$, we have seen that without using an auxiliary space $\mathbf{Y} \neq \ell^n_p$ in (1.28) we do not come close to such results.

Even though in Theorem 21 we are interested in extending functions that are Lipschitz in the metric that is induced by the given norm $\|\cdot\|_{\mathbf{X}}$, the underlying reason for the bounds of Theorem 21 is a partitioning scheme (to be described below) that iteratively carves out balls in the metric that is induced by the auxiliary norm $\|\cdot\|_{\mathbf{Y}}$. So, the perturbation of $\mathbf{X}$ into $\mathbf{Y}$ amounts to exhibiting a Lipschitz extension operator through the use of a multi-scale construction that utilizes geometric shapes that differ from balls in the ambient metric. This strategy is feasible because the quantity $\mathsf{e}(\mathcal{C}_{\mathbf{X}})$ in the left-hand side of (1.32) is a bi-Lipschitz invariant, while the volumes that appear in the right-hand side of (1.32) scale exponentially in n. Hence, by passing to an equivalent norm one could hope to reduce the right-hand side of (1.32) significantly, while not changing the left-hand side of (1.32) by too much.

This perturbative approach is decisively useful for $\mathbf{X} = \ell^n_\infty$. When one unravels the ensuing proofs, the upper bound on $\mathsf{e}(\ell^n_\infty)$ of Theorem 14 arises from a multi-scale construction of an extension operator (using a *gentle partition of unity* [173]) that utilizes a partition of space that is obtained by iteratively removing sets of the form $x + rB_{\mathbf{Y}^n_\infty}$, where $\mathbf{Y}^n_\infty$ is as in Theorem 24. If one carries out the same procedure while using balls of the intrinsic metric of ℓ^n_∞ (namely, hypercubes $x + r[-1,1]^n$ in place of $x + rB_{\mathbf{Y}^n_\infty}$, which look like hypercubes with "rounded corners"), then only the weaker bound $\mathsf{e}(\ell^n_\infty) \lesssim n$ is obtained. We already mentioned that such a phenomenon even occurs in the proof of the Euclidean estimate (1.27).

The following two examples describe further uses of Theorem 21; we will work out several more later.

Example 25. In the forthcoming work [234], the author and Schechtman prove (for an application to metric embedding theory) the following asymptotic evaluation of the maximal volumes of hyperplane projections of the unit balls of the Schatten–von Neumann trace classes:

$$\forall q \geqslant 1, \qquad \max_{A \in \mathsf{M}_n(\mathbb{R}) \smallsetminus \{0\}} \frac{\mathrm{vol}_{n^2-1}\left(\mathsf{Proj}_{A^\perp} B_{\mathsf{S}^n_q}\right)}{\mathrm{vol}_{n^2}\left(B_{\mathsf{S}^n_q}\right)} \asymp n^{\frac{1}{2}+\frac{1}{q}} \sqrt{\min\{q,n\}}. \tag{1.43}$$

Upon substitution into Theorem 21, this yields the following new estimates on the Lipschitz extension moduli of Schatten–von Neumann trace classes, which holds for every $p \geqslant 1$ and every integer $n \geqslant 2$:

$$\mathsf{e}\big(\mathsf{S}_p^n\big) \lesssim \begin{cases} n^{\frac{1}{2}+\frac{1}{p}} & \text{if } p \in [1,2], \\ n\sqrt{\min\{p, \log n\}} & \text{if } p \in [2,\infty]. \end{cases} \tag{1.44}$$

Indeed, by Hölder's inequality

$$\|\cdot\|_{\mathsf{S}_2^n} \leqslant n^{\max\{0,\frac{1}{2}-\frac{1}{p}\}}\|\cdot\|_{\mathsf{S}_p^n},$$

so (1.44) for $p \leqslant \log n$ follows from a substitution of these point-wise bounds and (1.43) when $q = p$ into the case $\mathbf{X} = \mathbf{Y} = \mathsf{S}_p^n$ of Theorem 21. The case $p \geqslant \log n$ of (1.44) follows from the same reasoning using (1.43) when $q = \log n$ and Theorem 21 for $\mathbf{X} = \mathsf{S}_p^n$ and $\mathbf{Y} = \mathsf{S}_q^n$, since in this case $d_{\mathrm{BM}}(\mathsf{S}_p^n, \mathsf{S}_q^n) \lesssim 1$. Note that, since $\dim(\mathsf{S}_p^n) = n^2$, for every $p \in [1,\infty]$ the bound on $\mathsf{e}(\mathsf{S}_p^n)$ in (1.44) is $o(\dim(\mathsf{S}_p^n))$, i.e., it is asymptotically better than what follows from [140].

More generally, given $p \geqslant 1$, an integer $n \geqslant 2$ and $r \in \{3,\ldots,n\}$, let $(\mathsf{S}_p^n)_{\leqslant r}$ be the set of n by n matrices of rank at most r, equipped with the metric inherited from S_p^n. Then, (1.44) has the following strengthening:

$$\mathsf{e}\big((\mathsf{S}_p^n)_{\leqslant r}\big) \lesssim r^{\max\{\frac{1}{p},\frac{1}{2}\}}\sqrt{n}\cdot \begin{cases} \sqrt{\max\{\log(\frac{n}{r}), p\}} & \text{if } p \leqslant \log r, \\ \sqrt{\log n} & \text{if } p \geqslant \log r. \end{cases} \tag{1.45}$$

To justify (1.45), apply Theorem 21 with $\mathbf{X} = \mathsf{S}_p^n$ and $\mathbf{Y} = \mathsf{S}_q^n$ for some $q \geqslant p$ while using (1.43), and optimize the resulting bound over q. Specifically, as for $A, B \in (\mathsf{S}_p^n)_{\leqslant r}$ the matrix $A - B$ has at most $2r$ nonzero singular values, by Hölder's inequality we have

$$\|A-B\|_{\mathsf{S}_2^n} \leqslant (2r)^{\max\{0,\frac{1}{2}-\frac{1}{p}\}}\|A-B\|_{\mathsf{S}_p^n}$$

and

$$\|A-B\|_{\mathsf{S}_p^n} \leqslant (2r)^{\frac{1}{p}-\frac{1}{q}}\|A-B\|_{\mathsf{S}_q^n}.$$

In combination with (1.43), we therefore get the following bound from (1.28):

$$\begin{aligned}\mathsf{e}\big((\mathsf{S}_p^n)_{\leqslant r}\big) &\lesssim \Bigg(\sup_{\substack{A,B\in(\mathsf{S}_p^n)_{\leqslant r}\\ A\neq B}} \frac{\|A-B\|_{\mathsf{S}_p^n}}{\|A-B\|_{\mathsf{S}_q^n}}\Bigg) \sup_{\substack{A,B\in(\mathsf{S}_p^n)_{\leqslant r}\\ A\neq B}} \Bigg(n^{\frac{1}{2}+\frac{1}{q}}\sqrt{q}\frac{\|A-B\|_{\mathsf{S}_2^n}}{\|A-B\|_{\mathsf{S}_p^n}}\Bigg)\\ &\lesssim r^{\frac{1}{p}-\frac{1}{q}}n^{\frac{1}{2}+\frac{1}{q}}\sqrt{q}\,r^{\max\{\frac{1}{2}-\frac{1}{p},0\}}.\end{aligned} \tag{1.46}$$

The $q \geqslant p$ that minimizes the right-hand side of (1.46) is $\max\{2\log(n/r), p\}$, yielding (1.45) when $p \leqslant \log r$. If $p \geqslant \log r$, then $\|A-B\|_{\mathsf{S}_p^n} \asymp \|A-B\|_{\mathsf{S}_{\log r}^n}$ for every $A, B \in (\mathsf{S}_p^n)_{\leqslant r}$, so (1.45) reduces to its special case $p = \log r$.

We conjecture that it is possible to replace the logarithmic factor in (1.45) by a universal constant, i.e.,

$$\mathsf{e}\big((\mathsf{S}_p^n)_{\leqslant r}\big) \lesssim r^{\max\{\frac{1}{p},\frac{1}{2}\}}\sqrt{n}. \tag{1.47}$$

As we will see in Section 1.6, Conjecture 26 below is equivalent to the symmetric isomorphic reverse isoperimetry conjecture (see Conjecture 47) for $\mathsf{M}_n(\mathbb{R})$ equipped with the operator norm, which is an especially interesting special case of this much more general conjectural phenomenon; by reasoning as we did in the above deduction of Theorem 20 from (the special case $p=\infty$ of) Theorem 24 (recall the discussion immediately following (1.40)), a positive answer to Conjecture 26 would imply (1.47).

Conjecture 26. For every $n \in \mathbb{N}$ there exists a normed space

$$\mathbf{Y} = (\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\mathbf{Y}})$$

such that for every nonzero n by n matrix $A \in \mathsf{M}_n(\mathbb{R}) \smallsetminus \{0\}$ we have $\|A\|_{\mathbf{Y}} \asymp \|A\|_{\mathsf{S}_\infty^n}$ and

$$\mathrm{vol}_{n^2-1}(\mathsf{Proj}_{A^\perp} B_{\mathbf{Y}}) \lesssim \mathrm{vol}_{n^2}(B_{\mathbf{Y}})\sqrt{n}.$$

Example 27. Since the $\ell_\infty^n(\ell_\infty^n)$ norm on $\mathsf{M}_n(\mathbb{R})$ is isometric to $\ell_\infty^{n^2}$, by Theorem 24 there is a normed space $\mathbf{Y} = (\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\mathbf{Y}})$ that satisfies

$$\|A\|_{\ell_\infty^n(\ell_\infty^n)} \leqslant \|A\|_{\mathbf{Y}} \lesssim \|A\|_{\ell_\infty^n(\ell_\infty^n)}$$

for every $A \in \mathsf{M}_n(\mathbb{R})$, and

$$\max_{A\in \mathsf{M}_n(\mathbb{R})\smallsetminus\{0\}} \frac{\mathrm{vol}_{n^2-1}\big(\mathsf{Proj}_{A^\perp} B_{\mathbf{Y}}\big)}{\mathrm{vol}_{n^2}(B_{\mathbf{Y}})} = O(1).$$

By Hölder's inequality, for every $p,q \in [1,\infty]$ and $A \in \mathsf{M}_n(\mathbb{R})$ we have

$$\|A\|_{\ell_p^n(\ell_q^n)} \leqslant n^{\frac{1}{p}+\frac{1}{q}}\|A\|_{\ell_\infty^n(\ell_\infty^n)} \leqslant n^{\frac{1}{p}+\frac{1}{q}}\|A\|_{\mathbf{Y}}$$

and

$$\|A\|_{\ell_2^n(\ell_2^n)} \leqslant n^{\max\{\frac{1}{2}-\frac{1}{p},0\}+\max\{\frac{1}{2}-\frac{1}{q},0\}}\|A\|_{\ell_p^n(\ell_q^n)}.$$

Therefore, Theorem 21 gives the Lipschitz extension bound

$$\mathsf{e}\big(\ell_p^n(\ell_q^n)\big) \lesssim n^{\frac{1}{p}+\frac{1}{q}+\max\{\frac{1}{2}-\frac{1}{p},0\}+\max\{\frac{1}{2}-\frac{1}{q},0\}} = n^{\max\{1,\frac{1}{p}+\frac{1}{q},\frac{1}{2}+\frac{1}{p},\frac{1}{2}+\frac{1}{q}\}}. \tag{1.48}$$

As in the case of ℓ_p^n, we get (1.48) if $p,q = O(1)$ by using Theorem 21 with $\mathbf{Y} = \mathbf{X} = \ell_p^n(\ell_p^n)$, but otherwise we need to work with an auxiliary space $\mathbf{Y} \neq \mathbf{X}$ as above. Specifically, in Section 6 we will prove the following asymptotic evaluation of the maximal volume of hyperplane projections of the unit ball of $\ell_p^n(\ell_q^n)$:

$$\max_{A\in \mathsf{M}_n(\mathbb{R})\smallsetminus\{0\}} \frac{\mathrm{vol}_{n^2-1}\left(\mathsf{Proj}_{A^\perp} B_{\ell_p^n(\ell_q^n)}\right)}{\mathrm{vol}_{n^2}\left(B_{\ell_p^n(\ell_q^n)}\right)} \asymp \begin{cases} n & \text{if } n\leqslant \min\{\sqrt{p},q\},\\ \sqrt{q}n^{\frac12+\frac1q} & \text{if } q\leqslant n\leqslant \sqrt{p},\\ \sqrt{p} & \text{if } \sqrt{p}\leqslant n\leqslant \min\{p,q\},\\ \sqrt{pq}n^{\frac1q-\frac12} & \text{if } \max\{\sqrt{p},q\}\leqslant n\leqslant p,\\ n^{\frac12+\frac1p} & \text{if } p\leqslant n\leqslant q,\\ \sqrt{q}n^{\frac1p+\frac1q} & \text{if } n\geqslant \max\{p,q\}.\end{cases} \tag{1.49}$$

The intricacy of (1.49) is perhaps unexpected, though it is nonetheless sharp in all of the six ranges (depending on the relative locations of p,q,n and, somewhat curiously, $\sqrt{p}$) that appear in (1.49). By reasoning analogously to the discussion following (1.42), one can prove a bound on $\mathsf{e}(\ell_p^n(\ell_q^n))$ that matches (1.48) up to lower order factors by applying Theorem 21 with $\mathbf{Y}=\ell_r^n(\ell_s^n)$ and then optimizing over $r,s\geqslant 1$. For the sole purpose of this application, only the range $n\geqslant \max\{p,q\}$ of (1.49) is needed. However, results such as (1.49) have geometric interest in their own right for all of the possible values of the relevant parameters. We will actually prove a version of (1.49) for $\ell_p^n(\ell_q^m)$ even when $n\neq m$; the case of rectangular matrices is independently interesting, but we will also use it elsewhere (see Remark 56 below).

Problem 28. Determine the exact maximizers of volumes of hyperplane projections of the unit balls of S_p^n and $\ell_p^n(\ell_q^n)$, i.e., for which $A\in \mathsf{M}_n(\mathbb{R})\smallsetminus\{0\}$ are the maxima in (1.43) and (1.49) attained.

1.5 A dimension-independent extension theorem

In the preceding sections we stated all of the extension theorems using the traditional setup that aims to extend a Lipschitz function to a function that is Lipschitz with respect to the given metric. However, all of our new (positive) extension theorems are a consequence of Theorem 29 below, which is a nonstandard Lipschitz extension theorem.

Theorem 29 asserts that if $\mathbf{X}=(\mathbb{R}^n,\|\cdot\|_{\mathbf{X}})$ is a normed space and f is a 1-Lipschitz function from a subset of $\mathbb{R}^n$ to a Banach space $\mathbf{Z}$, then f can be extended to a $\mathbf{Z}$-valued function that is defined on all of $\mathbb{R}^n$ and is $O(1)$-Lipschitz with respect to the metric that is induced on $\mathbb{R}^n$ by the norm $|||\cdot||| = 2\|\cdot\|_{\Pi^*\mathbf{X}}/\mathrm{vol}_n(B_{\mathbf{X}})$, i.e., a suitable rescaling of the norm whose unit ball is the polar projection body of $B_{\mathbf{X}}$. This rescaling ensures that $|||\cdot|||$ dominates $\|\cdot\|_{\mathbf{X}}$; indeed, by an elementary geometric argument (see Remark 112),

$$\forall x \in \mathbb{R}^n, \qquad \|x\|_{\mathbf{X}} \leqslant \frac{2\|x\|_{\Pi^*\mathbf{X}}}{\mathrm{vol}_n(B_{\mathbf{X}})} \leqslant n\|x\|_{\mathbf{X}}. \tag{1.50}$$

Thus, the conclusion of Theorem 29 that the extended function is Lipschitz with respect to $|||\cdot|||$ is less stringent than the traditional requirement that it should be Lipschitz with respect to $\|\cdot\|_{\mathbf{X}}$, but Theorem 29 has the feature that the upper bound on the Lipschitz constant is independent of the dimension.

Theorem 29. *Fix $n \in \mathbb{N}$ and a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. Fix also a Banach space $(\mathbf{Z}, \|\cdot\|_{\mathbf{Z}})$. Suppose that $\mathcal{C} \subseteq \mathbb{R}^n$ and $f : \mathcal{C} \to \mathbf{Z}$ is 1-Lipschitz with respect to the metric that is induced by $\|\cdot\|_{\mathbf{X}}$, i.e., $\|f(x) - f(y)\|_{\mathbf{Z}} \leqslant \|x - y\|_{\mathbf{X}}$ for every $x, y \in \mathcal{C}$. Then, there exists $F : \mathbb{R}^n \to \mathbf{Z}$ that coincides with f on $\mathcal{C}$ and satisfies*

$$\forall x, y \in \mathbb{R}^n, \qquad \|F(x) - F(y)\|_{\mathbf{Z}} \lesssim \frac{\|x - y\|_{\Pi^*\mathbf{X}}}{\mathrm{vol}_n(B_{\mathbf{X}})}.$$

To see how Theorem 29 implies Theorem 21, denote (in the setting of the statement of Theorem 21):

$$M = \sup_{\substack{x,y \in \mathcal{C} \\ x \neq y}} \left(\frac{\|x - y\|_{\mathbf{X}}}{\|x - y\|_{\mathbf{Y}}} \right) \tag{1.51}$$

and

$$M' = \sup_{\substack{x,y \in \mathcal{C} \\ x \neq y}} \left(\frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^\perp} B_{\mathbf{Y}}\big)}{\mathrm{vol}_n(B_{\mathbf{Y}})} \cdot \frac{\|x - y\|_{\ell_2^n}}{\|x - y\|_{\mathbf{X}}} \right). \tag{1.52}$$

Thus, every $x, y \in \mathcal{C}$ satisfy $\|x - y\|_{\mathbf{X}} \leqslant M\|x - y\|_{\mathbf{Y}}$ and, recalling (1.30), also

$$\frac{\|x - y\|_{\Pi^*\mathbf{Y}}}{\mathrm{vol}_n(B_{\mathbf{Y}})} \leqslant M'\|x - y\|_{\mathbf{X}}.$$

Let $(\mathbf{Z}, \|\cdot\|_{\mathbf{Z}})$ be a Banach space. Consider an arbitrary subset $\mathcal{C}' \subseteq \mathcal{C}$. If $f : \mathcal{C}' \to \mathbf{Z}$ is 1-Lipschitz with respect to the metric that is induced by $\|\cdot\|_{\mathbf{X}}$, then the function f/M is 1-Lipschitz with respect to the metric that is induced by $\mathbf{Y}$. By Theorem 29 (with $\mathbf{X}$ replaced by $\mathbf{Y}$, $\mathcal{C}$ replaced by $\mathcal{C}'$, f replaced by f/M) we therefore see that there exists $F : \mathbb{R}^n \to \mathbf{Z}$ (for Theorem 21 we only need F to be defined on $\mathcal{C}$) that extends F and satisfies $\|F(x) - F(y)\|_{\mathbf{Z}} \lesssim M\|x - y\|_{\Pi^*\mathbf{Y}}/\,\mathrm{vol}_n(B_{\mathbf{Y}}) \leqslant MM'\|x - y\|_{\mathbf{X}}$ for all $x, y \in \mathcal{C}$. This coincides with (1.28).

Remark 30. Given $p \geqslant 1$, consider what happens when we apply Theorem 29 to the space $\mathbf{Y}_p^n$ of Theorem 24. We get that for any $\mathcal{C} \subseteq \mathbb{R}^n$ and any Banach space $\mathbf{Z}$, if $f : \mathcal{C} \to \mathbf{Z}$ is 1-Lipschitz with respect to the ℓ_p^n metric, then f can be extended to $F : \mathbb{R}^n \to \mathbf{Z}$ that is $O(n^{1/p})$-Lipschitz with respect to the Euclidean metric. When

$p < 2$, the Lipschitz assumption on f is less stringent than requiring it to be $O(1)$-Lipschitz with respect to the Euclidean metric, but we then get an extension F that is $O(n^{1/p})$-Lipschitz with respect to the Euclidean metric; this upper bound on the Lipschitz constant of F is asymptotically larger than the $O(\sqrt{n})$ bound that we would get if f were assumed to be 1-Lipschitz with respect to the Euclidean metric and we applied the second inequality in (1.22), but we get it while requiring less from f. In particular, when $p = 1$ we see that any $\mathbf{Z}$-valued function on a subset of $\mathbb{R}^n$ that is 1-Lipschitz with respect to the ℓ_1^n metric can be extended to a $\mathbf{Z}$-valued function defined on all of $\mathbb{R}^n$ whose Lipschitz constant with respect to the Euclidean metric is $O(n)$, while an application of [140] will give an extension that is $O(n)$-Lipschitz with respect to the ℓ_1^n metric. On the other hand, if $p > 2$, then the Lipschitz assumption on f is more stringent than requiring it to be $O(1)$-Lipschitz with respect to the Euclidean metric, but we then get an extension F that is $O(n^{1/p})$-Lipschitz with respect to the Euclidean metric, which is asymptotically better than the $O(\sqrt{n})$ bound from (1.22). In particular, when $p = \infty$ we see that any $\mathbf{Z}$-valued function on a subset of $\mathbb{R}^n$ that is 1-Lipschitz with respect to the ℓ_∞^n metric can be extended to a $\mathbf{Z}$-valued function on all of $\mathbb{R}^n$ whose Lipschitz constant with respect to the Euclidean metric is $O(1)$.

1.6 Isomorphic reverse isoperimetry

All of the applications that we found for Theorem 21 proceed by bounding the volumes of hyperplane projections of $B_{\mathbf{Y}}$ that appear in right-hand side of (1.28) by

$$\mathrm{MaxProj}(B_{\mathbf{Y}}) \stackrel{\mathrm{def}}{=} \max_{z \in S^{n-1}} \mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} B_{\mathbf{Y}}\big). \tag{1.53}$$

Thus, it follows from (1.29) that for any two normed spaces $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ with $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ we have

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\mathrm{MaxProj}(B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})} \mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}}). \tag{1.54}$$

Even though there could conceivably be an application of (1.29) that is more refined than (1.54), in this section we will investigate the ramifications of bounding $\mathrm{MaxProj}(B_{\mathbf{X}})$ as a way to use Theorem 21. This will relate to the isomorphic reverse isoperimetric phenomena that we conjectured in Section 1.1.1.

Any origin-symmetric convex body $L \subseteq \mathbb{R}^n$ satisfies

$$\mathrm{MaxProj}(L) \gtrsim \frac{\mathrm{vol}_{n-1}(\partial L)}{\sqrt{n}}. \tag{1.55}$$

Indeed, this follows immediately from the following classical *Cauchy surface area formula* (see, e.g., [282, equation (5.73)]) by bounding the integrand by its maximum:

$$\mathrm{vol}_{n-1}(\partial L) = \frac{2\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \fint_{S^{n-1}} \mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} L\big)\,\mathrm{d}z$$
$$\asymp \sqrt{n} \fint_{S^{n-1}} \mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z^\perp} L\big)\,\mathrm{d}z.$$

Remark 31. Using (1.55), Theorem 24 implies that Conjecture 9 (isomorphic reverse isoperimetry) holds (with S the identity mapping) when $K = B_{\ell_p^n}$ for any $p \geqslant 1$ and $n \in \mathbb{N}$. Indeed, let $\mathbf{Y}_p^n$ be the normed space from Theorem 24. By the first inequality in (1.40) we have

$$\mathrm{vol}_n\big(B_{\mathbf{Y}_p^n}\big)^{\frac{1}{n}} \asymp \mathrm{vol}_n\big(B_{\ell_p^n}\big)^{\frac{1}{n}} \asymp n^{-\frac{1}{p}}, \tag{1.56}$$

where the last equivalence in (1.56) is a standard computation (e.g., [263, p. 11]). By (1.55) and (1.56), the second inequality in (1.40) implies that the isoperimetric quotient of $B_{\mathbf{Y}_p^n}$ is $O(\sqrt{n})$. So, Conjecture 9 holds for $K = B_{\ell_p^n}$ if we take L to be a rescaling by a universal constant factor of $B_{\mathbf{Y}_p^n}$ so that $L \subseteq K$.

Thanks to (1.55), if we set $K = B_{\mathbf{X}}$ and $L = B_{\mathbf{Y}}$ in (1.54), then the right-hand side of (1.54) satisfies

$$\begin{aligned}\frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_n(L)}\,\mathrm{diam}_{\ell_2^n}(K) &\gtrsim \frac{\mathrm{vol}_{n-1}(\partial L)}{\sqrt{n}\,\mathrm{vol}_n(L)}\,\mathrm{diam}_{\ell_2^n}(K)\\ &= \frac{\mathrm{iq}(L)}{\sqrt{n}} \cdot \frac{\mathrm{diam}_{\ell_2^n}(K)}{\mathrm{vol}_n(L)^{\frac{1}{n}}} \gtrsim \frac{\mathrm{diam}_{\ell_2^n}(K)}{\mathrm{vol}_n(K)^{\frac{1}{n}}},\end{aligned} \tag{1.57}$$

where we recall notation (1.11) for the isoperimetric quotient $\mathrm{iq}(\cdot)$ and the last step uses the isoperimetric theorem (1.12) and the assumption $L \subseteq K$. The following proposition explains what it would entail for one to be able to reverse (1.57) after an application of a suitable linear transformation; in particular, it shows that one can find $S \in \mathsf{SL}_n(\mathbb{R})$ and an origin-symmetric convex body $L \subseteq SK$ such that

$$\frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_n(L)}\,\mathrm{diam}_{\ell_2^n}(SK) \lesssim \frac{\mathrm{diam}_{\ell_2^n}(SK)}{\mathrm{vol}_n(K)^{\frac{1}{n}}}$$

if and only if Conjecture 10 on weak isomorphic reverse isoperimetry holds for K.

Proposition 32. *The following two statements are equivalent for every $n \in \mathbb{N}$, every origin-symmetric convex body $K \subseteq \mathbb{R}^n$ and every $\alpha > 0$.*

(1) *There exist a linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$ and an origin-symmetric convex body $L \subseteq SK$ with*

$$\frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_n(L)}\,\mathrm{vol}_n(K)^{\frac{1}{n}} \lesssim \alpha. \tag{1.58}$$

(2) *There exist a linear transformation* $S \in \mathsf{SL}_n(\mathbb{R})$ *and an origin-symmetric convex body* $L \subseteq SK$ *that satisfies* $\sqrt[n]{\mathrm{vol}_n(L)} \geqslant \beta \sqrt[n]{\mathrm{vol}_n(K)}$ *and* $\mathrm{iq}(L) \leqslant \gamma\sqrt{n}$ *for some* $\beta \gtrsim 1/\alpha$ *and* $\gamma \lesssim \alpha$ *with* $\gamma/\beta \lesssim \alpha$.

Proof. For the implication (1)⇒(2) we introduce the notations $\gamma = \mathrm{iq}(L)/\sqrt{n}$ and $\beta = \sqrt[n]{\mathrm{vol}_n(L)}/\sqrt[n]{\mathrm{vol}_n(K)}$. Then,

$$\alpha \overset{(1.58)}{\gtrsim} \frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_n(L)} \mathrm{vol}_n(K)^{\frac{1}{n}} \overset{(1.55)}{\gtrsim} \frac{\mathrm{vol}_{n-1}(\partial L)}{\mathrm{vol}_n(L)\sqrt{n}} \mathrm{vol}_n(K)^{\frac{1}{n}} = \frac{\gamma}{\beta}.$$

Since by the isoperimetric theorem (1.12) we have $\gamma \gtrsim 1$, it follows from this that $\beta \gtrsim 1/\alpha$, and since $L \subseteq SK$ and $S \in SL_n(\mathbb{R})$, we have $\mathrm{vol}_n(L) \leqslant \mathrm{vol}_n(K)$, so $\beta \leqslant 1$ and it also follows from this that $\gamma \lesssim \alpha$.

For the implication (2)⇒(1), fix $T \in \mathsf{SL}_n(\mathbb{R})$ that satisfies

$$\mathrm{vol}_{n-1}(\partial TL) = \min\big\{\mathrm{vol}_{n-1}(\partial T'L) : T' \in \mathsf{SL}_n(\mathbb{R})\big\},$$

i.e., TL is in its *minimum surface area position* [250]. So, $\mathrm{vol}_{n-1}(\partial TL) \leqslant \mathrm{vol}_{n-1}(\partial L)$ by the definition of T, and by Proposition 3.1 in the work [104] of Giannopoulos and Papadimitrakis combined with (1.55) we have

$$\mathrm{MaxProj}(TL) \asymp \frac{\mathrm{vol}_{n-1}(\partial TL)}{\sqrt{n}}.$$

Consequently, if L satisfies part (2) of Proposition 32, then

$$\begin{aligned}\frac{\mathrm{MaxProj}(TL)}{\mathrm{vol}_n(TL)} \mathrm{vol}_n(K)^{\frac{1}{n}} &\asymp \frac{\mathrm{vol}_{n-1}(\partial TL)}{\mathrm{vol}_n(TL)\sqrt{n}} \mathrm{vol}_n(K)^{\frac{1}{n}} \\ &\leqslant \frac{\mathrm{vol}_{n-1}(\partial L)}{\mathrm{vol}_n(TL)\sqrt{n}} \mathrm{vol}_n(K)^{\frac{1}{n}} \\ &= \frac{\mathrm{iq}(L)}{\sqrt{n}} \left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)^{\frac{1}{n}} \leqslant \frac{\gamma}{\beta} \lesssim \alpha.\end{aligned}$$

Hence, (1) holds with S replaced by $TS \in \mathsf{SL}_n(\mathbb{R})$ and L replaced by $TL \subseteq TSK$. ∎

Since when $\alpha \lesssim 1$ in Proposition 32 the assertion of its part (2) coincides with Conjecture 10, it follows that Conjecture 10, and a fortiori Conjecture 9, imply that for any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ there is $S \in \mathsf{SL}_n(\mathbb{R})$ such that $\mathsf{e}(\mathbf{X})$ is at most a universal constant multiple of $\mathrm{diam}_{\ell_2^n}(SB_{\mathbf{X}})/\sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})}$. Indeed, this follows by applying Theorem 21 to the normed spaces $\mathbf{X}' = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}'})$ and $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ whose unit balls are $SB_{\mathbf{X}}$ and L, respectively, where S and L are as in part (1) of Proposition 32 for $K = B_{\mathbf{X}}$, while noting that $\mathsf{e}(\mathbf{X}') = \mathsf{e}(\mathbf{X})$ since $\mathbf{X}'$ is isometric to $\mathbf{X}$. We record this conclusion as the following corollary.

Corollary 33. *If Conjecture 10 holds for a normed space* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$*, then there is* $S \in \mathsf{SL}_n(\mathbb{R})$ *such that*

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\operatorname{diam}_{\ell_2^n}(SB_{\mathbf{X}})}{\operatorname{vol}_n(B_{\mathbf{X}})^{\frac{1}{n}}}. \tag{1.59}$$

The upshot of Corollary 33 is that the right-hand side of (1.59) involves only Euclidean diameters and nth roots of volumes, which are typically much easier to estimate than extremal volumes of hyperplane projections. This comes at the cost of having to find the auxiliary linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$, but we expect that in concrete settings it will be simple to determine S. Moreover, in all of the specific examples of spaces for which we are interested (at least initially) in estimating their Lipschitz extension modulus, S should be the identity mapping. We will discuss this matter and its consequences in Section 1.6.2.

Remark 34. There is a degree of freedom that the above discussion does not exploit. Let $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ be a normed space. By (1.31), we know that $\mathsf{e}(\mathbf{X})$ is bounded from above by a constant multiple of the minimum of $\operatorname{diam}_{\Pi^*\mathbf{Y}}(B_{\mathbf{X}})/\operatorname{vol}_n(B_{\mathbf{Y}})$ over all the normed spaces $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ for which $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$. By (1.54), to control this minimum it suffices to estimate the minimum of $\operatorname{MaxProj}(\mathrm{B}_{\mathbf{Y}})/\operatorname{vol}_n(B_{\mathbf{Y}})$ over all such $\mathbf{Y}$, which relates to isomorphic reverse isoperimetric phenomena. But, we could also take a normed space $\mathbf{W} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{W}})$ for $m \geqslant n$ such that $B_{\mathbf{W}} \cap \mathbb{R}^n = B_{\mathbf{X}}$ (we need that $\mathbf{W}$ contains an isometric copy of $\mathbf{X}$), estimate either of the two minima above for the super-space $\mathbf{W}$, and then use $\mathsf{e}(\mathbf{X}) \leqslant \mathsf{e}(\mathbf{W})$. Thus, it would suffice to embed $\mathbf{X}$ into a larger normed space that exhibits good isomorphic reverse isoperimetry. Our conjectures imply that such an embedding step is not needed, namely we expect that the desired isomorphic reverse isoperimetric property holds for $\mathbf{X}$. Nevertheless, it could be that by finding a suitable super-space $\mathbf{W}$ one could bound $\mathsf{e}(\mathbf{X})$ while circumventing the difficulty of proving Conjecture 10 for $\mathbf{X}$. For example, if $\mathbf{X}$ is a subspace of ℓ_∞^m for some $m = O(n)$, then by Theorem 14 we know that $\mathsf{e}(\mathbf{X}) \lesssim \sqrt{n}$, but this is because we know that ℓ_∞^m has the desired isomorphic reverse isoperimetric property, and it is not clear how to prove it for $\mathbf{X}$ itself. It is also unclear how to construct for a given normed $\mathbf{X}$ a super-space $\mathbf{W}$ that could be used as above. We leave the exploration of this possibility for future research.

1.6.1 A spectral interpretation, reverse Faber–Krahn and the Cheeger space of a normed space

We will henceforth quantify the extent to which Conjecture 10 holds through the following condition:

$$\frac{\operatorname{iq}(L)}{\sqrt{n}}\left(\frac{\operatorname{vol}_n(K)}{\operatorname{vol}_n(L)}\right)^{\frac{1}{n}} = \frac{\operatorname{vol}_n(K)^{\frac{1}{n}}}{\sqrt{n}}\left(\frac{\operatorname{vol}_{n-1}(\partial L)}{\operatorname{vol}_n(L)}\right) \leqslant \alpha. \tag{1.60}$$

The factors $\mathrm{iq}(L)/\sqrt{n}$ and $(\mathrm{vol}_n(K)/\mathrm{vol}_n(L))^{1/n}$ that appear in the left-hand side of (1.60) are at least a positive universal constant (by, respectively, the isoperimetric theorem and the assumed inclusion $L \subseteq K$), so (1.60) implies that

$$\mathrm{iq}(L) \leqslant \alpha\sqrt{n} \quad \text{and} \quad \sqrt[n]{\mathrm{vol}_n(L)} \gtrsim \alpha^{-1}\sqrt[n]{\mathrm{vol}_n(K)}.$$

Thus, if $\alpha = O(1)$, then (1.60) is equivalent to the conclusion of Conjecture 10. However, even though Conjecture 10 expresses our expectation that (1.60) is always achievable with $\alpha = O(1)$ upon a judicious choice of the Euclidean structure on $\mathbb{R}^n$, in lieu of Conjecture 10 it would still be valuable to obtain (1.60) with α unbounded but slowly growing. In such a situation, the bi-parameter quantification that we used in part (2) of Proposition 32 contains more geometric information than (1.60), but below we will work with (1.60) in order to simplify the ensuing discussion; this suffices for our purposes because (1.60) is what shows up in all of the applications herein (per the proof Proposition 32) since they all proceed by bounding the right-hand side of (1.54) from above.

Alter and Caselles proved [7] that for every convex body $K \subseteq \mathbb{R}^n$ there is a *unique* measurable set $A \subseteq K$, which we call the *Cheeger body* of K and denote $\mathrm{Ch}\,K$, satisfying $\mathrm{Per}(A)/\mathrm{vol}_n(A) \leqslant \mathrm{Per}(B)/\mathrm{vol}_n(B)$ for every measurable $B \subseteq K$ with $\mathrm{vol}_n(B) > 0$, where $\mathrm{Per}(\cdot)$ denotes perimeter in the sense of Caccioppoli and de Giorgi; this notion is covered in [9] but we do not need to recall its definition here since the perimeter of a convex body coincides with the $(n-1)$-dimensional Hausdorff measure of its boundary. It was proved in [7] that $\mathrm{Ch}\,K$ is convex and its boundary is $C^{1,1}$. Further information on this remarkable theorem can be found in [7], where $\mathrm{Ch}\,K$ is characterized in terms of the mean curvature of its boundary through the work [8] of Alter, Caselles and Chambolle (see also the precursor [74] by Caselles, Chambolle and Novaga which obtained these statements under stronger assumptions on K).

Beyond the fact that it allows us to use the notation $\mathrm{Ch}\,K$ and call it *the* Cheeger body of K, the aforementioned uniqueness statement will be used substantially in the ensuing reasoning. It implies in particular that if K is origin-symmetric, then so is $\mathrm{Ch}\,K$. Consequently, if $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space, then $\mathrm{Ch}\,B_{\mathbf{X}}$ is the unit ball of a normed space that we denote by $\mathrm{Ch}\,\mathbf{X}$ and call the *Cheeger space* of $\mathbf{X}$.

For a convex body $K \subseteq \mathbb{R}^n$, let $\lambda(K)$ be the smallest Dirichlet eigenvalue of the Laplacian on K, namely it is the smallest $\lambda > 0$ for which there is a nonzero function

$$\varphi : K \to \mathbb{R}$$

that is smooth on the interior of K, vanishes on the boundary of K, and satisfies $\Delta\varphi = -\lambda\varphi$ on the interior of K; see, e.g., [77,81,265] for background on this classical topic. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space, then we denote

$$\lambda(\mathbf{X}) = \lambda(B_{\mathbf{X}}).$$

The quantity $h(K) = \mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, K)/\, \mathrm{vol}_n(\mathrm{Ch}\, K)$ is called the Cheeger constant of K; it relates to $\lambda(K)$ by

$$\frac{2}{\pi}\sqrt{\lambda(K)} \leqslant h(K) = \frac{\mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, K)}{\mathrm{vol}_n(\mathrm{Ch}\, K)} \leqslant 2\sqrt{\lambda(K)}. \tag{1.61}$$

It is important for our purposes that the constants appearing in (1.61) are independent of the dimension n. The second inequality in (1.61) is the Cheeger inequality for the Dirichlet Laplacian on Euclidean domains. Cheeger's proof of it for compact Riemannian manifolds without boundary appears in [78] and that proof works mutatis mutandis in the present setting; see its derivation in, e.g., the appendix of [174]. The first inequality in (1.61) can be called the Buser inequality for the Dirichlet Laplacian on convex Euclidean domains, since Buser proved [69] its analogue for compact Riemannian manifolds without boundary that have a lower bound on their Ricci curvature. In our setting, this reverse Cheeger inequality is more recent, namely it was noted for planar convex sets by Parini [246] and in any dimension by Brasco [53]. It can be justified quickly using the convexity of K and its Cheeger body $\mathrm{Ch}\, K$ as follows. By a classical theorem of Pólya we have $\lambda(K) \leqslant \pi^2(\mathrm{vol}_{n-1}(\partial K)/\, \mathrm{vol}_n(K))^2/4$ (Pólya proved this for planar convex sets, but in [144] Joó and Stachó carried out Pólya's approach for convex bodies in $\mathbb{R}^n$ for any $n \in \mathbb{N}$). Therefore,

$$\lambda(K) \leqslant \lambda(\mathrm{Ch}\, K) \leqslant \left(\frac{\pi\, \mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, K)}{2\, \mathrm{vol}_n(\mathrm{Ch}\, K)}\right)^2 = \frac{\pi^2}{4} h(K)^2,$$

since $\mathrm{Ch}\, K$ is convex.

Let $j_{n/2-1,1}$ be the smallest positive zero of the Bessel function $J_{n/2-1}$; see [14, Chapter 4] for a treatment of Bessel functions and their zeros, though here we will only need to know that $j_{n/2-1,1} \asymp n$ (see [306] for more precise asymptotics). By classical computations (see, e.g., [129, equation (1.29)]),

$$\lambda\big(B_{\ell_2^n}\big) = j^2_{\frac{n}{2}-1,1}.$$

The Faber–Krahn inequality [95, 159] (see also, e.g., [77, 265]) asserts that $\lambda(K)$ is at least the first Dirichlet eigenvalue of a Euclidean ball whose volume is the same as the volume of K. Thus,

$$\lambda(K)\, \mathrm{vol}_n(K)^{\frac{2}{n}} \geqslant \lambda\big(B_{\ell_2^n}\big)\, \mathrm{vol}_n\big(B_{\ell_2^n}\big)^{\frac{2}{n}} = j^2_{\frac{n}{2}-1,1}\, \mathrm{vol}_n\big(B_{\ell_2^n}\big)^{\frac{2}{n}} \asymp n,$$

where we used the straightforward fact that $\lambda(rK) = \lambda(K)/r^2$ for every $r > 0$.

Observe that (1.61) can be rewritten as follows for every convex body $K \subseteq \mathbb{R}^n$:

$$\frac{2}{\pi}\left(\frac{\lambda(K)\, \mathrm{vol}_n(K)^{\frac{2}{n}}}{n}\right)^{\frac{1}{2}} \leqslant \frac{\mathrm{iq}(\mathrm{Ch}\, K)}{\sqrt{n}}\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(\mathrm{Ch}\, K)}\right)^{\frac{1}{n}} \leqslant 2\left(\frac{\lambda(K)\, \mathrm{vol}_n(K)^{\frac{2}{n}}}{n}\right)^{\frac{1}{2}}.$$

Hence, for every $\alpha > 0$ we have

$$\frac{\mathrm{iq}(\mathrm{Ch}\,K)}{\sqrt{n}}\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(\mathrm{Ch}\,K)}\right)^{\frac{1}{n}} \lesssim \alpha \iff \lambda(K)\,\mathrm{vol}_n(K)^{\frac{2}{n}} \lesssim \alpha^2 n. \tag{1.62}$$

Since Ch K is convex, the convex body $L \subseteq K$ that minimizes the left-hand side of (1.60) is equal to Ch K. We therefore see that Conjecture 35 below is equivalent to Conjecture 10. Furthermore, if one of these two conjectures hold for a matrix $S \in \mathsf{SL}_n(\mathbb{R})$, then the same matrix would work for the other conjecture.

Conjecture 35 (Reverse Faber–Krahn). For any origin-symmetric convex body $K \subseteq \mathbb{R}^n$ there exists a volume-preserving linear transformation $S \in \mathsf{SL}_n(\mathbb{R})$ such that

$$\lambda(SK)\,\mathrm{vol}(K)^{\frac{2}{n}} \asymp n.$$

Remark 36. One can also wonder about exact maximizers in the context of Conjecture 35. Specifically, Bucur and Fragalà stated in [67, p. 389] that they expect that for any origin-symmetric convex body $K \subseteq \mathbb{R}^n$ with $\mathrm{vol}_n(K) = 1$ there exists $S \in \mathsf{SL}_n(\mathbb{R})$ such that $\lambda(SK) \leqslant \lambda([0,1]^n) = \pi^2 n$. If true, then this would be a beautiful statement even though it does not have substantial impact on Conjecture 10 and its implications herein (it would only influence the value of the implicit constant factors in our statements, which incur further losses that are most likely not sharp in other steps of their derivations). The only available evidence for the aforementioned (speculative) exact statement is the partial result of [67] in the planar case $n = 2$, which proves that it indeed holds when $K \subseteq \mathbb{R}^2$ is a convex axisymmetric octagon that has four of its vertices lying on the axes at the same distance from the origin; see specifically [67, Proposition 10], whose proof involves delicate reasoning that incorporate computer-assisted steps. A complete result for $n = 2$ has been subsequently obtained by the same authors in [68] for the analogous question in which one replaces the Dirichlet eigenvalue of the Laplacian by the Cheeger constant. Namely, [68, Theorem 1.1] states that for every origin-symmetric convex body $K \subseteq \mathbb{R}^2$ with $\mathrm{vol}_2(K) = 1$ there exists $S \in \mathsf{SL}_2(\mathbb{R})$ such that $h(SK) \leqslant h([0,1]^2) = 2 + \sqrt{\pi}$ (furthermore, in this case S can be taken to be the matrix that puts K in John position, i.e., the ellipse of maximal area that is contained in SK is a circle).

This above spectral interpretation of Conjecture 10 is useful for multiple purposes, including the following lemma whose proof appears in Section 6.1. For its statement, as well as throughout the ensuing discussion, recall that a basis $x_1, \ldots, x_n$ of an n-dimensional normed space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ is a 1-unconditional basis of $\mathbf{X}$ if

$$\|\varepsilon_1 a_1 x_1 + \cdots + \varepsilon_n a_n x_n\|_{\mathbf{X}} = \|a_1 x_1 + \cdots + a_n x_n\|_{\mathbf{X}}$$

for every choice of scalars $a_1, \ldots, a_n \in \mathbb{R}$ and signs $\varepsilon_1, \ldots, \varepsilon_n \in \{-1, 1\}$. When we say that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is an unconditional normed space, we mean that the standard (coordinate) basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ is a 1-unconditional basis of $\mathbf{X}$.

Lemma 37 (Closure of Conjecture 10 under unconditional composition). *Fix $n \in \mathbb{N}$ and $m_1, \ldots, m_n \in \mathbb{N}$. Let $\mathbf{X}_1 = (\mathbb{R}^{m_1}, \|\cdot\|_{\mathbf{X}_1}), \ldots, \mathbf{X}_n = (\mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}_n})$ be normed spaces. Also, let $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ be an unconditional normed space. Define a normed space $\mathbf{X} = (\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}})$ by*

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}, \qquad \|x\|_{\mathbf{X}} \stackrel{\mathrm{def}}{=} \big\|\big(\|x_1\|_{\mathbf{X}_1}, \ldots, \|x_n\|_{\mathbf{X}_n}\big)\big\|_{\mathbf{E}}.$$

Suppose that there exist $\alpha > 0$, linear transformations $S_1 \in \mathsf{SL}_{m_1}(\mathbb{R}), \ldots, S_n \in \mathsf{SL}_{m_n}(\mathbb{R})$, and normed spaces $\mathbf{Y}_1 = (\mathbb{R}^{m_1}, \|\cdot\|_{\mathbf{Y}_1}), \ldots, \mathbf{Y}_n = (\mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{Y}_n})$ such that

$$B_{\mathbf{Y}_k} \subseteq S_k B_{\mathbf{X}_k} \quad \textit{and} \quad \frac{\mathrm{iq}\big(B_{\mathbf{Y}_k}\big)}{\sqrt{m_k}} \left(\frac{\mathrm{vol}_{m_k}\big(B_{\mathbf{X}_k}\big)}{\mathrm{vol}_{m_k}\big(B_{\mathbf{Y}_k}\big)}\right)^{\frac{1}{m_k}} \leqslant \alpha, \tag{1.63}$$

for every $k \in \{1, \ldots, n\}$. Then, there exist a normed space

$$\mathbf{Y} = (\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}})$$

and $S \in \mathsf{SL}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})$ such that

$$B_{\mathbf{Y}} \subseteq S B_{\mathbf{X}} \quad \textit{and} \quad \frac{\mathrm{iq}(B_{\mathbf{Y}})}{\sqrt{m_1 + \cdots + m_n}} \left(\frac{\mathrm{vol}_{m_1+\cdots+m_n}(B_{\mathbf{X}})}{\mathrm{vol}_{m_1+\cdots+m_n}(B_{\mathbf{Y}})}\right)^{\frac{1}{m_1+\cdots+m_n}} \lesssim \alpha. \tag{1.64}$$

As (1.63) with $\alpha = O(1)$ is immediate when $n_0 = 1$, Lemma 37 establishes Conjecture 10 for when K is the unit ball of an unconditional normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. This holds, in particular, for $\mathbf{X} = \ell_p^n$, though we will prove in Section 6.1 that the stronger conclusion of Conjecture 9 holds in this case (recall Remark 31). Lemma 37 also shows that Conjecture 10 holds for, say, $\mathbf{X} = \ell_p^n(\ell_q^m)$; we expect that the reasoning of Section 6.1 could be adapted to yield Conjecture 9 for these spaces as well, but we did not attempt to carry this out. Other spaces that satisfy (1.63) with α slowly growing will be presented in Section 1.6.2; upon their substitution into Lemma 37, more examples for which Conjecture 10 holds up to lower-order factors are obtained (of course, we are conjecturing here that it holds for *any* space).

Remark 38. Say that a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is in *Cheeger position* if

$$\forall S \in \mathsf{SL}_n(\mathbb{R}), \qquad \frac{\mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, B_{\mathbf{X}})}{\mathrm{vol}_n(\mathrm{Ch}\, B_{\mathbf{X}})} \leqslant \frac{\mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, SB_{\mathbf{X}})}{\mathrm{vol}_n(\mathrm{Ch}\, SB_{\mathbf{X}})}.$$

Observe that if $\mathbf{X}$ is in Cheeger position, then its Cheeger space $\mathrm{Ch}\,\mathbf{X}$ is in minimum surface area position, namely, $\mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, B_{\mathbf{X}}) \leqslant \mathrm{vol}_{n-1}(\partial S\, \mathrm{Ch}\, B_{\mathbf{X}})$ for every $S \in \mathsf{SL}_n(\mathbb{R})$. Indeed, $S\, \mathrm{Ch}\, B_{\mathbf{X}} \subseteq SB_{\mathbf{X}}$, so by the definition of the Cheeger body of $SB_{\mathbf{X}}$ we have $\mathrm{vol}_{n-1}(\partial S\, \mathrm{Ch}\, B_{\mathbf{X}})/\, \mathrm{vol}_n(\mathrm{Ch}\, B_{\mathbf{X}}) \geqslant \mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, SB_{\mathbf{X}})/\, \mathrm{vol}_n(\mathrm{Ch}\, SB_{\mathbf{X}})$. At the same time, $\mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, SB_{\mathbf{X}})/\, \mathrm{vol}_n(\mathrm{Ch}\, SB_{\mathbf{X}}) \geqslant \mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, B_{\mathbf{X}})/\, \mathrm{vol}_n(\mathrm{Ch}\, B_{\mathbf{X}})$ by the definition of the Cheeger position, so $\mathrm{vol}_{n-1}(\partial S\, \mathrm{Ch}\, B_{\mathbf{X}}) \geqslant \mathrm{vol}_{n-1}(\partial\, \mathrm{Ch}\, B_{\mathbf{X}})$. This shows that in the proof of the implication (2)⇒(1) of Proposition 32, if we worked with $L = \mathrm{Ch}\, SK$, then there would be no need to introduce the additional linear transformation $T \in \mathsf{SL}_n(\mathbb{R})$. It would be worthwhile to study the Cheeger position for its own sake even if it were not for its connection to reverse isoperimetry. In particular, we do not know if the converse of the above deduction holds, namely whether it is true that if $\mathrm{Ch}\, \mathbf{X}$ is in minimum surface area position, then $\mathbf{X}$ is in Cheeger position. We also do not know if the Cheeger position is unique up to orthogonal transformation (as is the case for the minimum surface area position [104]); we did not investigate these matters since they are not needed for the present purposes, but we expect that the characterisations of the Cheeger body in [7] would be relevant here. One could also define that a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is in *Dirichlet position* if $\lambda(\mathbf{X}) \leqslant \lambda(S\mathbf{X})$ for every $S \in \mathsf{SL}_n(\mathbb{R})$. It is unclear how the Cheeger position relates to the Dirichlet position and it would be also worthwhile to study the Dirichlet position for its own sake. By (1.61), working with either the Cheeger position or the Dirichlet position would be equally valuable for the reverse isoperimetric questions in which we are interested here.

1.6.2 Symmetries and positions

Thus far we considered an arbitrary scalar product on an n-dimensional normed space through which we identified its underlying vector space structure with $\mathbb{R}^n$. However, the Lipschitz extension modulus is insufficiently understood for "very nice" normed spaces (including even the Euclidean space ℓ_2^n) that belong to a natural class of normed spaces that have a canonical identification with $\mathbb{R}^n$. It therefore makes sense to first focus on this class.

For a finite dimensional normed space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, let $\mathsf{Isom}(\mathbf{X})$ be the group of all of the isometric automorphism of $\mathbf{X}$, i.e., all the linear operators $U : \mathbf{X} \to \mathbf{X}$ that satisfy $\|Ux\|_{\mathbf{X}} = \|x\|_{\mathbf{X}}$ for every $x \in \mathbf{X}$. We will denote the Haar probability measure on the compact group $\mathsf{Isom}(\mathbf{X})$ by $\hbar_{\mathbf{X}}$.

Definition 39. We say that a finite dimensional normed space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ is *canonically positioned* if any two $\mathsf{Isom}(\mathbf{X})$-invariant scalar products on $\mathbf{X}$ are proportional to each other. In other words, if $\langle\cdot,\cdot\rangle : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$ and $\langle\cdot,\cdot\rangle' : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$ are scalar products on $\mathbf{X}$ such that $\langle Ux, Uy\rangle = \langle x, y\rangle$ and $\langle Ux, Uy\rangle' = \langle x, y\rangle'$ for every $x, y \in \mathbf{X}$ and every $U \in \mathsf{Isom}(\mathbf{X})$, then there necessarily exists $\lambda \in \mathbb{R}$ such that $\langle\cdot,\cdot\rangle' = \lambda\langle\cdot,\cdot\rangle$.

On any finite dimensional normed space $\mathbf{X}$ there exists at least one scalar product $\langle\cdot,\cdot\rangle : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$ that is invariant under $\mathsf{Isom}(\mathbf{X})$, as seen, e.g., by averaging any given scalar product $\langle\cdot,\cdot\rangle_0$ on $\mathbf{X}$ with respect $\hbar_{\mathbf{X}}$, i.e., defining

$$\forall x, y \in \mathbf{X}, \quad \langle x, y\rangle \stackrel{\mathrm{def}}{=} \int_{\mathsf{Isom}(\mathbf{X})} \langle Sx, Sy\rangle_0 \,\mathrm{d}\hbar_{\mathbf{X}}(S).$$

Definition 39 concerns those spaces $\mathbf{X}$ for which such an invariant scalar product is unique up to rescaling, so there is (essentially, i.e., up to rescaling) no arbitrariness when we identify $\mathbf{X}$ with $\mathbb{R}^{\dim(\mathbf{X})}$.

Example 40. The class of n-dimensional canonically positioned spaces includes those normed spaces $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ that have a basis $e_1, \ldots, e_n$ such that for any distinct $i, j \in \{1, \ldots, n\}$ there are a permutation $\pi \in S_n$ with $\pi(i) = j$ and a sign vector $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, 1\}^n$ with $\varepsilon_i = -\varepsilon_j$ such that $T_\pi, S_\varepsilon \in \mathsf{Isom}(\mathbf{X})$, where we denote $T_\pi x = \sum_{i=1}^n a_{\pi(i)} e_i$ and $S_\varepsilon x = \sum_{i=1}^n \varepsilon_i a_i e_i$ for $x = \sum_{i=1}^n a_i e_i \in \mathbf{X}$ with $a_1, \ldots, a_n \in \mathbb{R}$. Indeed, let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathbf{X}$ that is $\mathsf{Isom}(\mathbf{X})$-invariant. For every distinct $i, j \in \{1, \ldots, n\}$, if $\pi \in S_n$ and $\varepsilon \in \{-1, 1\}^n$ are as above, then $\langle e_i, e_i \rangle = \langle e_{\pi(i)}, e_{\pi(i)} \rangle = \langle e_j, e_j \rangle$ while $\langle e_i, e_j \rangle = \langle \varepsilon_i e_i, \varepsilon_j e_j \rangle = -\langle e_i, e_j \rangle$, so $\langle e_i, e_j \rangle = 0$.

Example 40 covers all of the spaces for which we think that it is most pressing (given the current state of knowledge) to understand their Lipschitz extension modulus, including normed spaces $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ that have a 1-symmetric basis, i.e., a basis $e_1, \ldots, e_n \in \mathbf{E}$ such that $\| \sum_{i=1}^n \varepsilon_i a_{\pi(i)} e_i \|_{\mathbf{E}} = \| \sum_{i=1}^n a_i e_i \|_{\mathbf{E}}$ for every $(\varepsilon, \pi) \in \{-1, 1\}^n \times S_n$. In particular, ℓ_p^n, and more generally Orlicz and Lorentz spaces (see, e.g., [181]), are canonically positioned. We will use below the common convention that a normed space $(\mathbb{R}^n, \|\cdot\|)$ is said to be symmetric if it is 1-symmetric with respect to the standard (coordinate) basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

Example 40 also includes matrix norms

$$\mathbf{X} = (\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\mathbf{X}})$$

that remain unchanged if one transposes a pair of rows or columns, or changes the sign of an entire row or a column, such as S_p^n. More generally, if $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ is a symmetric normed space, then its unitary ideal $\mathsf{S}_{\mathbf{E}} = (\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\mathsf{S}_{\mathbf{E}}})$ is canonically positioned (see, e.g., [37]), where for $T \in \mathsf{M}_n(\mathbb{R})$ one denotes its singular values by $s_1(T) \geqslant \cdots \geqslant s_n(T)$ and defines $\|T\|_{\mathsf{S}_{\mathbf{E}}} = \|(s_1(T), \ldots, s_n(T))\|_{\mathbf{E}}$. More examples of such matrix norms are projective and injective tensor products (see, e.g., [276]) of symmetric spaces, where if $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{Y}})$ are normed spaces, then their projective tensor product $\mathbf{X} \hat{\otimes} \mathbf{Y}$ is the norm on $M_{n\times m}(\mathbb{R}) = \mathbb{R}^n \otimes \mathbb{R}^m$ whose unit ball is the convex hull of $\{x \otimes y : (x, y) \in B_{\mathbf{X}} \times B_{\mathbf{Y}}\}$, and their injective tensor product $\mathbf{X} \check{\otimes} \mathbf{Y}$ is the dual of $\mathbf{X}^* \hat{\otimes} \mathbf{Y}^*$ (equivalently, $\mathbf{X} \check{\otimes} \mathbf{Y}$ is isometric to the operator norm from $\mathbf{X}^*$ to $\mathbf{Y}$; see, e.g., [87, Section 1.1]).

Henceforth, when we will say that a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is canonically positioned it will always be tacitly assumed that the standard scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is $\mathsf{Isom}(\mathbf{X})$-invariant, i.e., $\mathsf{Isom}(\mathbf{X})$ is a subgroup of the orthogonal group $\mathsf{O}_n \subseteq \mathsf{M}_n(\mathbb{R})$. This is equivalent to the requirement that for every symmetric positive definite matrix $T \in \mathsf{M}_n(\mathbb{R})$, if $TU = UT$ for every $U \in \mathsf{Isom}(\mathbf{X})$, then there is $\lambda \in (0, \infty)$

such that $T = \lambda \mathsf{Id}_n$. Indeed, any scalar product $\langle\cdot,\cdot\rangle' : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is of the form $\langle x, y\rangle' = \langle Tx, y\rangle$ for some symmetric positive definite $T \in \mathsf{M}_n(\mathbb{R})$ and all $x, y \in \mathbb{R}^n$, and using the $\mathsf{Isom}(\mathbf{X})$-invariance of $\langle\cdot,\cdot\rangle$ we see that $\langle\cdot,\cdot\rangle'$ is $\mathsf{Isom}(\mathbf{X})$-invariant if and only if T commutes with all of the elements of $\mathsf{Isom}(\mathbf{X})$.

Remark 41. A symmetry assumption that is common in the literature is *enough symmetries*. A normed space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$ is said [103] to have enough symmetries if any linear transformation $T : \mathbf{X} \to \mathbf{X}$ must be a scalar multiple of the identity if T commutes with every element of $\mathsf{Isom}(\mathbf{X})$. By the above discussion, if $\mathbf{X}$ has enough symmetries, then $\mathbf{X}$ is canonically positioned. The converse implication does not hold, i.e., there exist normed spaces that are canonically positioned but do not have enough symmetries. For example, let $\mathsf{Rot}_{\pi/2} \in \mathsf{O}_2$ be the rotation by 90 degrees and let G be the subgroup of O_2 that is generated by $\mathsf{Rot}_{\pi/2}$. Thus, G is cyclic of order 4. Suppose that

$$\mathbf{X} = (\mathbb{R}^2, \|\cdot\|_{\mathbf{X}})$$

is a normed space with $\mathsf{Isom}(\mathbf{X}) = G$; the fact that there is such a normed space follows from the general result [118, Theorem 3.1] of Gordon and Loewy on existence of norms with a specified group of isometries, though in this particular case it is simple to construct such an example (e.g., the unit ball of $\mathbf{X}$ can be taken to be a suitable non-regular octagon). Since $\mathsf{Isom}(\mathbf{X})$ is Abelian, the matrix $\mathsf{Rot}_{\pi/2}$ commutes with all of the elements of $\mathsf{Isom}(\mathbf{X})$ yet it is not a multiple of the identity matrix, so $\mathbf{X}$ does not have enough symmetries. Nevertheless, $\mathbf{X}$ is canonically positioned. Indeed, suppose that $T \in \mathsf{M}_2(\mathbb{R})$ is a symmetric matrix that commutes with $\mathsf{Rot}_{\pi/2}$. Then, $\mathsf{Rot}_{\pi/2}$ preserves any eigenspace of T, which means that any such eigenspace must be $\{0\}$ or $\mathbb{R}^2$. But T is diagonalizable over $\mathbb{R}$, so it follows that for some $\lambda \in \mathbb{R}$ we have $T = \lambda \mathsf{Id}_2$. If n is even, then one obtains such an n-dimensional example by considering $\ell_\infty^{n/2}(\mathbf{X})$. However, a representation-theoretic argument due to Emmanuel Breuillard (private communication; details omitted) shows that if n is odd, then any n-dimensional normed space has enough symmetries if and only if it is canonically positioned.

The following lemma is important for us even though it is an immediate consequence of the (major) theorem of [7] that the Cheeger body of a given convex body in $\mathbb{R}^n$ is unique (recall Section 1.6.1).

Lemma 42. *Let $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ be a normed space such that $\mathsf{Isom}(\mathbf{X}) \leqslant \mathsf{O}_n$ is a subgroup of the orthogonal group. Then the isometry group of its Cheeger space* $\mathrm{Ch}\,\mathbf{X}$ *satisfies*

$$\mathsf{Isom}(\mathrm{Ch}\,\mathbf{X}) \supseteq \mathsf{Isom}(\mathbf{X}).$$

Consequently, if $\mathbf{X}$ is canonically positioned, then also $\mathrm{Ch}\,\mathbf{X}$ *is canonically positioned.*

Proof. For any $U \in \mathsf{Isom}(\mathbf{X})$ we have

$$\frac{\mathrm{vol}_{n-1}(\partial U\,\mathrm{Ch}\,B_{\mathbf{X}})}{\mathrm{vol}_n(U\,\mathrm{Ch}\,B_{\mathbf{X}})} = \frac{\mathrm{vol}_{n-1}(\partial\,\mathrm{Ch}\,B_{\mathbf{X}})}{\mathrm{vol}_n(\mathrm{Ch}\,B_{\mathbf{X}})},$$

and also $U\,\mathrm{Ch}\,B_{\mathbf{X}} \subseteq UB_{\mathbf{X}} = B_{\mathbf{X}}$, since $U \in \mathsf{O}_n$. Consequently, (by definition), $U\,\mathrm{Ch}\,B_{\mathbf{X}}$ is a Cheeger body of $B_{\mathbf{X}}$. The uniqueness of the Cheeger body now implies that $U\,\mathrm{Ch}\,B_{\mathbf{X}} = \mathrm{Ch}\,B_{\mathbf{X}}$. Therefore, $U \in \mathsf{Isom}(\mathrm{Ch}\,\mathbf{X})$. ∎

The following corollary is a quick consequence of Lemma 42.

Corollary 43. *Let* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *be a symmetric normed space. Then, its Cheeger space* $\mathrm{Ch}\,\mathbf{E}$ *is also symmetric and there exists a (unique) symmetric normed space* $\chi\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\chi\mathbf{E}})$ *such that the Cheeger space of the unitary ideal* $\mathsf{S}_{\mathbf{E}}$ *is the unitary ideal of* $\chi\mathbf{E}$*, i.e.,* $\mathrm{Ch}\,\mathsf{S}_{\mathbf{E}} = \mathsf{S}_{\chi\mathbf{E}}$*.*

Proof. The assertion that $\mathrm{Ch}\,\mathbf{E}$ is symmetric coincides with requiring that $\mathsf{Isom}(\mathrm{Ch}\,\mathbf{E})$ contains the group $\{-1,1\}^n \rtimes S_n = \{T_\varepsilon S_\pi : (\varepsilon,\pi) \in \{-1,1\}^n \times S_n\} \leqslant \mathsf{O}_n$, where we recall the notation of Example 40. We are assuming that $\mathsf{Isom}(\mathbf{E}) \supseteq \{-1,1\}^n \rtimes S_n$, so this follows from Lemma 42. For $U, V \in \mathsf{O}_n$ define $R_{U,V} : \mathsf{M}_n(\mathbb{R}) \to \mathsf{M}_n(\mathbb{R})$ by $(A \in M_n(\mathbb{R})) \mapsto UAV$. Since $\mathsf{Isom}(\mathsf{S}_{\mathbf{E}}) \supseteq \{R_{U,V} : U, V \in \mathsf{O}_n\}$, by Lemma 42 so does $\mathsf{Isom}(\mathrm{Ch}\,S_{\mathbf{E}})$. A normed space $(\mathsf{M}_n(\mathbb{R}), \|\cdot\|)$ that is invariant under $R_{U,V}$ for all $U, V \in \mathsf{O}_n$ is the unitary ideal of a symmetric normed space $\mathbf{F} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{F}})$; see, e.g., [37, Theorem IV.2.1]. This $\mathbf{F}$ is unique (consider the values of $\|\cdot\|_{S_{\mathbf{F}}}$ on diagonal matrices), so we can introduce the notation $\mathbf{F} = \chi\mathbf{E}$. ∎

The same reasoning as in the proof of Corollary 43 shows that if

$$\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$$

is an unconditional normed space, then so is $\mathrm{Ch}\,\mathbf{E}$. Thus, the space $\mathbf{Y}$ in Lemma 37 when

$$\mathbf{X}_1 = \cdots = \mathbf{X}_n = \mathbb{R}$$

that satisfies (1.64) can be taken to unconditional, as seen by an inspection of the proof of Lemma 37 (specifically, the operator S in (1.64) that arises in this case is diagonal, so $S\mathbf{E}$ is also unconditional and we can take $\mathbf{Y} = \mathrm{Ch}\,S\mathbf{E}$).

Problem 44. We associated above to every symmetric normed space

$$\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$$

two symmetric normed spaces $\mathrm{Ch}\,\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathrm{Ch}\,\mathbf{E}})$ and $\chi\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\chi\mathbf{E}})$. It would be valuable to understand these auxiliary norms on $\mathbb{R}^n$, and in particular how they relate to each other. By the definition of the Cheeger body, its convexity and uniqueness, Ch $\mathbf{E}$ is the unique minimizer of the functional

$$\mathbf{F} \mapsto \frac{\mathrm{vol}_{n-1}\big(\partial B_{\mathbf{F}}\big)}{\mathrm{vol}_n\big(B_{\mathbf{F}}\big)} = \frac{\int_{\partial B_{\mathbf{F}}} 1 \, \mathrm{d}x}{\int_{B_{\mathbf{F}}} 1 \, \mathrm{d}x} \tag{1.65}$$

over all symmetric normed spaces $\mathbf{F} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{F}})$ with $B_{\mathbf{F}} \subseteq B_{\mathbf{E}}$; denote the set of all such $\mathbf{F}$ by $\mathcal{S}\mathfrak{ym}(\subseteq B_{\mathbf{E}})$. In contrast to (1.65), $\chi\mathbf{E}$ is the unique minimizer of the functional

$$\mathbf{F} \mapsto \frac{\int_{\partial B_{\mathbf{F}}} \prod_{1\leqslant i<j\leqslant n} |x_i^2 - x_j^2| \, \mathrm{d}x}{\int_{B_{\mathbf{F}}} \prod_{1\leqslant i<j\leqslant n} |x_i^2 - x_j^2| \, \mathrm{d}x} \tag{1.66}$$

over the same domain $\mathcal{S}\mathfrak{ym}(\subseteq B_{\mathbf{E}})$. To justify (1.66), observe first that by Corollary 43 we know that $\chi\mathbf{E}$ is the unique minimizer of the following functional over $\mathcal{S}\mathfrak{ym}(\subseteq B_{\mathbf{E}})$:

$$\mathbf{F} \mapsto \frac{\mathrm{vol}_{n^2-1}\big(\partial B_{\mathsf{S}_{\mathbf{F}}}\big)}{\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{F}}}\big)} = \lim_{\varepsilon\to 0+} \frac{\int_{\left(B_{\mathsf{S}_{\mathbf{F}}}+\varepsilon B_{\mathsf{S}_2^n}\right)\smallsetminus B_{\mathsf{S}_{\mathbf{F}}}} 1 \, \mathrm{d}x}{\varepsilon \int_{B_{\mathbf{F}}} 1 \, \mathrm{d}x}. \tag{1.67}$$

We claim that for every $\mathbf{F} \in \mathcal{S}\mathfrak{ym}(\subseteq B_{\mathbf{E}})$ and $\varepsilon > 0$,

$$\begin{aligned}&\big(B_{\mathsf{S}_{\mathbf{F}}} + \varepsilon B_{\mathsf{S}_2^n}\big) \smallsetminus B_{\mathsf{S}_{\mathbf{F}}} \\ &= \Big\{A \in M_n(\mathbb{R}) : s(A) \stackrel{\mathrm{def}}{=} \big(s_1(A), \dots, s_n(A)\big) \in \big(B_{\mathbf{F}} + \varepsilon B_{\ell_2^n}\big) \smallsetminus B_{\mathbf{F}}\Big\},\end{aligned} \tag{1.68}$$

where we denote the singular values of $A \in \mathsf{M}_n(\mathbb{R})$ by $s_1(A) \geqslant \cdots \geqslant s_n(A)$. Indeed, if A belongs to the right-hand side of (1.68), then $\|s(A)\|_{\mathbf{F}} > 1$ and $s(A) = x + y$ for $x, y \in \mathbb{R}^n$ that satisfy $\|x\|_{\mathbf{F}} \leqslant 1$ and $\|y\|_{\ell_2^n} \leqslant \varepsilon$. Write $A = UDV$, where $D \in \mathsf{M}_n(\mathbb{R})$ is the diagonal matrix whose diagonal is the vector $s(A) \in \mathbb{R}^n$, and $U, V \in \mathsf{O}_n$. Let $D(x), D(y) \in \mathsf{M}_n(\mathbb{R})$ be the diagonal matrices whose diagonals equal x, y, respectively. By noting that $\|A\|_{\mathsf{S}_{\mathbf{F}}} = \|s(A)\|_{\mathbf{F}} > 1$ and $A = UD_xV + UD_yV$, where $\|UD(x)V\|_{\mathsf{S}_{\mathbf{F}}} \leqslant 1$ and $\|UD(y)V\|_{\mathsf{S}_2^n} \leqslant \varepsilon$, we conclude that A belongs to the left-hand side of (1.68). The reverse inclusion is less straightforward. If A belongs to the left-hand side of (1.68), then $\|A\|_{\mathsf{S}_{\mathbf{F}}} > 1$ and $A = B + C$, where $B, C \in \mathsf{M}_n(\mathbb{R})$ satisfy $\|B\|_{\mathsf{S}_{\mathbf{F}}} = \|s(B)\|_{\mathbf{F}} \leqslant 1$ and $\|C\|_{\mathsf{S}_2^n} \leqslant \varepsilon$. By an inequality of Mirsky [222] we have $\|s(A) - s(B)\|_{\ell_2^n} \leqslant \|A - B\|_{\mathsf{S}_2^n} = \|C\|_{\mathsf{S}_2^n} \leqslant \varepsilon$. Hence $s(A) = s(B) + (s(A) - s(B)) \in (B_{\mathbf{F}} + \varepsilon B_{\ell_2^n}) \smallsetminus B_{\mathbf{F}}$, i.e., A belongs to the right-hand side of (1.68). With (1.68) established, since membership of a matrix A in either $B_{\mathbf{F}}$ or $(B_{\mathbf{F}} + \varepsilon B_{\ell_2^n}) \smallsetminus B_{\mathbf{F}}$ depends only on $s(A)$, by the Weyl integration formula [311] (see [12, Proposition 4.1.3] for the formulation that we are using),

$$\frac{\int_{(B_{\mathsf{S}_{\mathbf{F}}}+\varepsilon B_{\mathsf{S}_2^n})\smallsetminus B_{\mathsf{S}_{\mathbf{F}}}} 1 \, \mathrm{d}x}{\int_{B_{\mathbf{F}}} 1 \, \mathrm{d}x} = \frac{\int_{(B_{\mathbf{F}}+\varepsilon B_{\ell_2^n})\smallsetminus B_{\mathbf{F}}} \prod_{1\leqslant i<j\leqslant n} |x_i^2 - x_j^2| \, \mathrm{d}x}{\int_{B_{\mathbf{F}}} \prod_{1\leqslant i<j\leqslant n} |x_i^2 - x_j^2| \, \mathrm{d}x}.$$

Thus (1.66) follows from (1.67). Analysing the functional in (1.66) seems nontrivial but likely tractable using ideas from random matrix theory. It would be especially interesting to treat the case $\mathbf{E} = \ell_\infty^n$. While we have a reasonably good understanding of the (isomorphic) geometry space $\mathrm{Ch}\,\ell_\infty^n$, its noncommutative counterpart $\chi\ell_\infty^n$ is still mysterious and understanding its geometry is closely related to Conjecture 10 (and likely also Conjecture 9) in the important special case of the operator norm S_∞^n; see also Remark 172.

If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is canonically positioned and μ is a Borel measure on $\mathbb{R}^n$ that is $\mathsf{Isom}(\mathbf{X})$-invariant, i.e., $\mu(UA) = \mu(A)$ for every $U \in \mathsf{Isom}(\mathbf{X})$ and every Borel subset $A \subseteq \mathbb{R}^n$, then consider the scalar product

$$\forall x, y \in \mathbb{R}^n, \quad \langle x, y\rangle' \stackrel{\mathrm{def}}{=} \int_{\mathbb{R}^n} \langle x, z\rangle\langle y, z\rangle \,\mathrm{d}\mu(z).$$

For every $U \in \mathsf{Isom}(\mathbf{X})$ and $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\langle Ux, Uy\rangle' &= \int_{\mathbb{R}^n} \langle Ux, z\rangle\langle Uy, z\rangle \,\mathrm{d}\mu(z) = \int_{\mathbb{R}^n} \langle x, U^{-1}z\rangle\langle y, U^{-1}z\rangle \,\mathrm{d}\mu(z)\\ &= \int_{\mathbb{R}^n} \langle x, z\rangle\langle y, z\rangle \,\mathrm{d}\mu(z) = \langle x, y\rangle',\end{aligned}$$

where the second step uses the $\mathsf{Isom}(\mathbf{X})$-invariance of $\langle\cdot,\cdot\rangle$, and the third step uses the $\mathsf{Isom}(\mathbf{X})$-invariance of μ. Hence $\langle x, y\rangle' = \lambda\langle x, y\rangle$ for some $\lambda \in \mathbb{R}$ and every $x, y \in \mathbb{R}^n$. By considering the case $x = y$ of this identity and integrating over $x \in S^{n-1}$ one sees that necessarily $n\lambda = \int_{\mathbb{R}^n} \|z\|_{\ell_2^n}^2 \,\mathrm{d}\mu(z)$. Hence,

$$\forall x, y \in \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \langle x, z\rangle\langle y, z\rangle \,\mathrm{d}\mu(z) = \frac{\int_{\mathbb{R}^n} \|z\|_{\ell_2^n}^2 \,\mathrm{d}\mu(z)}{n}\langle x, y\rangle. \tag{1.69}$$

By establishing (1.69) we have shown that if

$$\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$$

is a canonically positioned normed space, then any $\mathsf{Isom}(\mathbf{X})$-invariant Borel measure on $\mathbb{R}^n$ is *isotropic* [55, 107] (the converse also holds, i.e., $\mathbf{X}$ is canonically positioned if and only if every $\mathsf{Isom}(\mathbf{X})$-invariant Borel measure on $\mathbb{R}^n$ is isotropic). In particular, let $\sigma_{\mathbf{X}}$ be the measure on S^{n-1} that is given by

$$\sigma_{\mathbf{X}}(A) = \mathrm{vol}_{n-1}(\{x \in \partial B_{\mathbf{X}} : N_{\mathbf{X}}(x) \in A\})$$

for every measurable $A \subseteq S^{n-1}$, where for $x \in \partial B_{\mathbf{X}}$ the vector $N_{\mathbf{X}}(x) \in S^{n-1}$ is the (almost-everywhere uniquely defined) unit outer normal to $\partial B_{\mathbf{X}}$ at x, i.e., recalling (1.30), we use the simpler notation $N_{B_{\mathbf{X}}} = N_{\mathbf{X}}$. In other words, $\sigma_{\mathbf{X}}$ is the image under the Gauss map of the $(n-1)$-dimensional Hausdorff measure on $\partial B_{\mathbf{X}}$. Then,

$\sigma_{\mathbf{X}}$ is $\mathsf{Isom}(\mathbf{X})$-invariant because every $U \in \mathsf{Isom}(\mathbf{X})$ is an orthogonal transformation and $N_{\mathbf{X}} \circ U = U \circ N_{\mathbf{X}}$ almost everywhere on $\partial B_{\mathbf{X}}$. By [250], this implies that $\mathbf{X}$ is in its minimum surface area position (recall the proof of Proposition 32), so $\mathrm{MaxProj}(B_{\mathbf{X}}) \asymp \mathrm{vol}_{n-1}(\partial B_{\mathbf{X}})/\sqrt{n}$ by [104, Proposition 3.1].

The following corollary follows by substituting the above conclusion into Theorem 21.

Corollary 45. *Suppose that $n \in \mathbb{N}$ and that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ are two n-dimensional normed spaces. Suppose also that $\mathbf{Y}$ is canonically positioned and $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$. Then,*

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Y}})\, \mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{Y}})\sqrt{n}}.$$

The assumption in Corollary 45 that $\mathbf{Y}$ is canonically positioned can be replaced by the requirement $\mathrm{MaxProj}(B_{\mathbf{Y}}) \lesssim \mathrm{vol}_{n-1}(\partial B_{\mathbf{Y}})/\sqrt{n}$, which is much less stringent. In particular, by [104, Proposition 3.1] it is enough to assume here that $B_{\mathbf{Y}}$ is in its minimum surface area position; see also Section 6.2.

We will denote the John and Löwner ellipsoids of a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ by $\mathcal{J}_{\mathbf{X}}$ and $\mathcal{L}_{\mathbf{X}}$, respectively; see [128]. Thus, $\mathcal{J}_{\mathbf{X}} \subseteq \mathbb{R}^n$ is the ellipsoid of maximum volume that is contained in $B_{\mathbf{X}}$ and $\mathcal{L}_{\mathbf{X}} \subseteq \mathbb{R}^n$ is the ellipsoid of minimum volume that contains $B_{\mathbf{X}}$. Both of these ellipsoids are unique [137]. The *volume ratio* $\mathrm{vr}(\mathbf{X})$ of $\mathbf{X}$ and *external volume ratio* $\mathrm{evr}(\mathbf{X})$ of $\mathbf{X}$ are defined by

$$\mathrm{vr}(\mathbf{X}) \stackrel{\mathrm{def}}{=} \left(\frac{\mathrm{vol}_n(B_{\mathbf{X}})}{\mathrm{vol}_n(\mathcal{J}_{\mathbf{X}})}\right)^{\frac{1}{n}} \quad \text{and} \quad \mathrm{evr}(\mathbf{X}) \stackrel{\mathrm{def}}{=} \left(\frac{\mathrm{vol}_n(\mathcal{L}_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})}\right)^{\frac{1}{n}}. \tag{1.70}$$

By the Blaschke–Santaló inequality [39, 278] and the Bourgain–Milman inequality [50],

$$\mathrm{evr}(\mathbf{X}) \asymp \mathrm{vr}(\mathbf{X}^*). \tag{1.71}$$

By the above discussion, we can quickly deduce the following theorem that relates the Lipschitz extension modulus of a canonically positioned space to volumetric and spectral properties of its unit ball.

Theorem 46. *Suppose that $n \in \mathbb{N}$ and that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a canonically positioned normed space. Then,*

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\sqrt{n}}\sqrt{\lambda(\mathbf{X})} \asymp \mathrm{evr}(\mathbf{X})\sqrt{\lambda(\mathbf{X})\, \mathrm{vol}_n(B_{\mathbf{X}})^{\frac{2}{n}}}$$
$$\asymp \mathrm{vr}(\mathbf{X}^*)\sqrt{\lambda(\mathbf{X})\, \mathrm{vol}_n(B_{\mathbf{X}})^{\frac{2}{n}}}. \tag{1.72}$$

In fact, the minimum of the right-hand side of (1.54) over all those normed spaces $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ for which $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ is bounded above and below by universal constant multiples of $\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})\sqrt{\lambda(\mathbf{X})/n}$.

Proof. By Lemma 42 the Cheeger space $\mathrm{Ch}\,\mathbf{X}$ is canonically positioned. So, by Corollary 45 with $\mathbf{Y} = \mathrm{Ch}\,\mathbf{X}$,

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\mathrm{vol}_{n-1}(\partial\,\mathrm{Ch}\,B_{\mathbf{Y}})\,\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\mathrm{vol}_n(\mathrm{Ch}\,B_{\mathbf{Y}})\sqrt{n}} \overset{(1.61)}{\lesssim} \frac{\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\sqrt{n}}\sqrt{\lambda(\mathbf{X})}.$$

This proves the first inequality in (1.72). The final equivalence in (1.72) is (1.71). To prove the rest of (1.72), let $r_{\min} = \min\{r > 0 : rB_{\ell_2^n} \supseteq B_{\mathbf{X}}\}$ denote the radius of the circumscribing Euclidean ball of $B_{\mathbf{X}}$. We claim that $r_{\min}B_{\ell_2^n} = \mathcal{L}_{\mathbf{X}}$. Indeed, for every $U \in \mathsf{Isom}(\mathbf{X}) \subseteq \mathsf{O}_n$ the ellipsoid $U\mathcal{L}_{\mathbf{X}}$ contains $B_{\mathbf{X}}$ and has the same volume as $\mathcal{L}_{\mathbf{X}}$, so because the minimum volume ellipsoid that contains $B_{\mathbf{X}}$ is unique [137], it follows that $U\mathcal{L}_{\mathbf{X}} = \mathcal{L}_{\mathbf{X}}$. Hence, the scalar product that corresponds to $\mathcal{L}_{\mathbf{X}}$ is $\mathsf{Isom}(\mathbf{X})$-invariant and since $\mathbf{X}$ is canonically positioned, this means that $\mathcal{L}_{\mathbf{X}}$ is a multiple of $B_{\ell_2^n}$. Now,

$$\mathrm{vol}_n(B_{\mathbf{X}})^{\frac{1}{n}}\,\mathrm{evr}(\mathbf{X}) \overset{(1.70)}{=} \mathrm{vol}_n\big(r_{\min}B_{\ell_2^n}\big)^{\frac{1}{n}} \asymp \frac{r_{\min}}{\sqrt{n}} = \frac{\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{2\sqrt{n}}.$$

The above reasoning shows that the minimum of the right-hand side of (1.54) over all the normed spaces $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ with $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ is at most a universal constant multiple of $\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})\sqrt{\lambda(\mathbf{X})/n}$ (take $\mathbf{Y} = \mathrm{Ch}\,\mathbf{X}$). In the reverse direction, for any such $\mathbf{Y}$ by (1.55) with $L = B_{\mathbf{Y}}$ we have

$$\frac{\mathrm{MaxProj}(B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})} \gtrsim \frac{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})\sqrt{n}} \geqslant \frac{\mathrm{vol}_{n-1}(\partial\,\mathrm{Ch}\,B_{\mathbf{X}})}{\mathrm{vol}_n(\mathrm{Ch}\,B_{\mathbf{X}})\sqrt{n}} \overset{(1.61)}{\geqslant} \frac{2\sqrt{\lambda(\mathbf{X})}}{\pi\sqrt{n}},$$

where the penultimate step follows from the definition of the Cheeger body $\mathrm{Ch}\,B_{\mathbf{X}}$. ■

It is natural to expect that if $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a canonically positioned normed space, then in Conjecture 9 for $K = B_{\mathbf{X}}$ holds with S the identity matrix and with L being the unit ball of a canonically positioned normed space. We formulate this refined special case of Conjecture 9 as the following conjecture.

Conjecture 47. Fix $n \in \mathbb{N}$ and a canonically positioned normed space

$$\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}}).$$

Then, there exists a canonically positioned normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ that satisfies $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\mathbf{X}}$ and $\mathrm{iq}(B_{\mathbf{Y}}) \lesssim \sqrt{n}$.

Theorem 48 below shows that Conjecture 47 holds if $\mathbf{X} = \ell_p^n$ for any $p \geqslant 1$ and infinitely many dimensions $n \in \mathbb{N}$; specifically, it holds if n satisfies the mild arithmetic (divisibility) requirement (1.73) below. An obvious question that this leaves is to prove Conjecture 47 for $\mathbf{X} = \ell_p^n$ and arbitrary $(p, n) \in [1, \infty] \times \mathbb{N}$. We expect that this question is tractable by (likely nontrivially) adapting the approach herein, but we did not make a major effort to do so since obtaining Conjecture 47 for such a dense set of dimensions n suffices for our purposes (the bi-Lipschitz invariants that we consider can be estimated from above for any $n \in \mathbb{N}$ since the requirement (1.73) holds for some $N \in \mathbb{N} \cap [n, O(n)]$ and ℓ_p^n embeds isometrically into ℓ_p^N). In Section 6 we will prove Theorem 48, and deduce Theorem 24 from it. Recall Remark 31, which explains that Conjecture 9 when K is the unit ball of ℓ_p^n follows (with S the identity matrix) from Theorem 24. Thus, we *do* know that a body L as in Conjecture 9 exists for all the possible choices of $p \geqslant 1$ and $n \in \mathbb{N}$, and (1.73) is only relevant to ensure that L is the unit ball of a canonically positioned normed space.

Theorem 48. *Fix $n \in \mathbb{N}$ and $p \geqslant 1$. Conjecture 47 holds for $\mathbf{X} = \ell_p^n$ if the following condition is satisfied:*

$$\exists m \in \mathbb{N}, \quad m \mid n \quad \textit{and} \quad \max\{p, 2\} \leqslant m \leqslant e^p. \tag{1.73}$$

The following conjecture is a variant of Conjecture 11.

Conjecture 49. Fix $n \in \mathbb{N}$ and suppose that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a canonically positioned normed space. Then, there exists a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ with $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ yet $\sqrt[n]{\mathrm{vol}_n(B_{\mathbf{Y}})} \gtrsim \sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})}$ such that $\mathrm{iq}(B_{\mathbf{Y}}) \lesssim \sqrt{n}$.

Conjecture 47 requires $\mathbf{Y}$ to be canonically positioned while Conjecture 49 does not. The reason for this is that if any normed space $\mathbf{Y}$ satisfies the conclusion of Conjecture 49, then also the Cheeger space $\mathrm{Ch}\,\mathbf{X}$ of $\mathbf{X}$ satisfies it (this is so because the convex body L that minimizes the second quantity in (1.60) is, by definition, the Cheeger body of $K = B_{\mathbf{X}}$), and by Lemma 42 the Cheeger space of $\mathbf{X}$ inherits from $\mathbf{X}$ the property of being canonically positioned. This use of the uniqueness of the Cheeger body will be important below. By (1.62), Conjecture 49 is equivalent to the following symmetric version of Conjecture 35.

Conjecture 50. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a canonically positioned normed space, then $\lambda(\mathbf{X})\,\mathrm{vol}(B_{\mathbf{X}})^{\frac{2}{n}} \asymp n$.

The following corollary is a substitution of Conjecture 50 into Theorem 46.

Corollary 51. *If Conjecture 49 (equivalently, Conjecture 50) holds for a canonically positioned normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$, then the right-hand side of (1.54) when $\mathbf{Y} = \mathrm{Ch}\,\mathbf{X}$ is $O(\mathrm{evr}(\mathbf{X})\sqrt{n})$. Consequently,*

$$\mathsf{e}(\mathbf{X}) \lesssim \mathrm{evr}(\mathbf{X})\sqrt{n} \asymp \mathrm{vr}(\mathbf{X}^*)\sqrt{n}. \tag{1.74}$$

It is worthwhile to note that by [19], the rightmost quantity in (1.74) is maximized (over all possible n-dimensional normed spaces) when $\mathbf{X} = \ell_1^n$, in which case we have $\mathrm{evr}(\ell_1^n)\sqrt{n} \asymp n$.

Remark 52. We currently do not have any example of a normed space

$$\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$$

for which (1.74) provably does not hold. If (1.74) were true in general, or even if it were true for a restricted class of normed spaces that is affine invariant and closed under direct sums, such as spaces that embed into ℓ_1 with distortion $O(1)$, then it would be an excellent result. When one leaves the realm of canonically positioned spaces, (1.74) acquires a self-improving property[8] as follows. Suppose that $\mathbf{X}$ is in Löwner position, i.e., $\mathcal{L}_{\mathbf{X}} = B_{\ell_2^n}$. Fix $m \in \mathbb{N}$ and consider the $(n+m)$-dimensional space $\mathbf{X}' = \mathbf{X} \oplus_\infty \ell_2^m$. If (1.74) holds for $\mathbf{X}'$, then

$$\begin{aligned}
\mathsf{e}(\mathbf{X}) &\leqslant \mathsf{e}(\mathbf{X}') \\
&\lesssim \mathrm{evr}(\mathbf{X}')\sqrt{\dim(\mathbf{X}')} \\
&\lesssim \left(\frac{\mathrm{vol}_{n+m}\big(B_{\ell_2^{n+m}}\big)}{\mathrm{vol}_n(B_{\mathbf{X}})\,\mathrm{vol}_m\big(B_{\ell_2^m}\big)}\right)^{\frac{1}{n+m}}\sqrt{n+m} \\
&= \left(\frac{\mathrm{vol}_n(\mathcal{L}_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})}\right)^{\frac{1}{n+m}}\left(\frac{\mathrm{vol}_{n+m}\big(B_{\ell_2^{n+m}}\big)}{\mathrm{vol}_n\big(\ell_2^n\big)\,\mathrm{vol}_m\big(B_{\ell_2^m}\big)}\right)^{\frac{1}{n+m}}\sqrt{n+m} \\
&\asymp \mathrm{evr}(\mathbf{X})^{\frac{n}{n+m}}\, n^{\frac{n}{2(n+m)}}\, m^{\frac{m}{2(n+m)}}. \qquad (1.75)
\end{aligned}$$

The value of m that minimizes the right-hand side of (1.75) is

$$m \asymp n\log(\mathrm{evr}(\mathbf{X})+1),$$

for which (1.75) becomes

$$\mathsf{e}(\mathbf{X}) \lesssim \sqrt{n\log\big(\mathrm{evr}(\mathbf{X})+1\big)}. \qquad (1.76)$$

As $\mathrm{evr}(\mathbf{X}) \leqslant \sqrt{n}$ by John's theorem, (1.76) gives $\mathsf{e}(\mathbf{X}) \lesssim \sqrt{n\log n}$, which would be an improvement of [140]. Also, by (1.9) the bound (1.76) gives

$$\mathsf{e}(\mathbf{X}) \lesssim \sqrt{n\log(C_2(\mathbf{X})+1)},$$

which is better than the conjectural bound (1.10). Here and throughout what follows, for $1 \leqslant p \leqslant 2 \leqslant q$ the (Gaussian) type-p and cotype-q constants [204] of a Banach space $(\mathbf{X}, \|\cdot\|_{\mathbf{X}})$, denoted $T_p(\mathbf{X})$ and $C_q(\mathbf{X})$, respectively, are the infimum over those

[8] We recommend checking that the analogous stabilization argument does not lead to a similar self-improvement phenomenon in Conjecture 9, Conjecture 10 and Corollary 33; the computations in Section 4 of [198] are relevant for this purpose.

$T \in [1,\infty]$ and $C \in [1,\infty]$, respectively, for which the following inequalities hold for every $m \in \mathbb{N}$ and every $x_1,\ldots,x_m \in \mathbf{X}$, where the expectation is with respect to i.i.d. standard Gaussian random variables $\mathsf{g}_1,\ldots,\mathsf{g}_m$:

$$\frac{1}{C}\bigg(\sum_{j=1}^{m}\|x_j\|_{\mathbf{X}}^{q}\bigg)^{\frac{1}{q}} \leqslant \bigg(\mathbb{E}\bigg[\bigg\|\sum_{j=1}^{m}\mathsf{g}_j x_j\bigg\|_{\mathbf{X}}^{2}\bigg]\bigg)^{\frac12} \leqslant T\bigg(\sum_{j=1}^{m}\|x_j\|_{\mathbf{X}}^{p}\bigg)^{\frac{1}{p}}. \qquad (1.77)$$

This observation indicates that it might be too optimistic to expect that (1.74) holds in full generality, but it would be very interesting to understand the extent to which it does. Obvious potential counterexamples are $\ell_1^n \oplus \ell_2^m$; if (1.74) holds for these spaces, then $\mathsf{e}(\ell_1^n) \lesssim \sqrt{n\log n}$ by the above reasoning (with $m \asymp n\log n$), which would be a big achievement because the best-known bound remains $\mathsf{e}(\ell_1^n) \lesssim n$ from [140].

Lemma 53 below, whose proof appears in Section 6.1, shows that Conjecture 49 holds for a class of normed space that includes any normed spaces with a 1-symmetric basis, as well as, say, $\ell_p^n(\ell_q^m)$ for any $n,m\in\mathbb{N}$ and $p,q \geqslant 1$. Other (related) examples of such spaces arise from Lemma 151 below.

Lemma 53. *Let $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ be an unconditional normed space. Suppose that for any $j,k \in \{1,\ldots,n\}$ there is a permutation $\pi \in S_n$ with $\pi(j) = k$ such that $\|\sum_{i=1}^n a_{\pi(i)}e_i\|_{\mathbf{X}} = \|\sum_{i=1}^n a_i e_i\|_{\mathbf{X}}$ for every $a_1,\ldots,a_n \in \mathbb{R}$. Then, Conjecture 49 holds for $\mathbf{X}$. Therefore, we have $\lambda(\mathbf{X})\,\mathrm{vol}_n(B_{\mathbf{X}})^{2/n} \asymp n$ and $\mathsf{e}(\mathbf{X}) \lesssim \mathrm{evr}(\mathbf{X})\sqrt{n}$.*

By [293, Theorem 2.1], any unconditional normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ satisfies $\mathrm{vr}(\mathbf{X}) \lesssim C_2(\mathbf{X})\sqrt{n}$, where $C_2(\mathbf{X})$ is the cotype-2 constant of $\mathbf{X}$ (this is an earlier special case of (1.9) in which the logarithmic term is known to be redundant). Hence, if $\mathbf{X}$ satisfies the assumptions of Lemma 53, then we know that

$$\mathsf{e}(\mathbf{X}) \lesssim C_2(\mathbf{X}^*)\sqrt{n}. \qquad (1.78)$$

By combining [22, Theorem 6] and (1.71), for any $p \in [1,\infty]$, if a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is isometric to a quotient of L_p (equivalently, the dual of $\mathbf{X}$ is isometric to a subspace of $L_{p/(p-1)}$), then

$$\mathrm{evr}(\mathbf{X}) \lesssim \mathrm{evr}\Big(\ell^n_{\frac{p}{p-1}}\Big) \asymp \min\Big\{n^{\frac1p-\frac12},1\Big\}.$$

Consequently, if $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ satisfies the assumptions of Lemma 53 and is also a quotient of L_p, then

$$\mathsf{e}(\mathbf{X}) \lesssim n^{\max\{\frac12,\frac1p\}}. \qquad (1.79)$$

Both (1.78) and (1.79) are generalizations of Theorem 18.

Lemma 54 below, whose proof appears in Section 6.3, shows that the unitary ideal of any n-dimensional normed space with a 1-symmetric basis (in particular, any Schatten–von Neumann trace class), satisfies Conjecture 49 up to a factor of $O(\sqrt{\log n})$. Upon its substitution into Lemma 151 below, more such examples are obtained.

Lemma 54. *Let* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *be a symmetric normed space. Conjecture 49 holds up to lower order factors for its unitary ideal* $\mathsf{S}_{\mathbf{E}}$. *More precisely, there is a normed space* $\mathbf{Y} = (\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\mathbf{Y}})$ *such that* $B_{\mathbf{Y}} \subseteq B_{\mathsf{S}_{\mathbf{E}}}$ *and*

$$\mathrm{vol}_{n^2}(B_{\mathbf{Y}})^{\frac{1}{n^2}} \asymp \mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{E}}}\big)^{\frac{1}{n^2}} \quad \textit{and} \quad n \lesssim \mathrm{iq}(B_{\mathbf{Y}}) \lesssim n\sqrt{\log n}. \tag{1.80}$$

Therefore, we have

$$n^2 \lesssim \lambda\big(\mathsf{S}_{\mathbf{E}}\big)\, \mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{E}}}\big)^{\frac{2}{n^2}} \lesssim n^2 \log n \quad \textit{and} \quad \mathsf{e}(\mathsf{S}_{\mathbf{E}}) \lesssim \mathrm{evr}(\mathsf{S}_{\mathbf{E}})n \asymp \mathrm{evr}(\mathbf{E})n.$$

For the final assertion of Lemma 54, the fact that $\mathrm{evr}(\mathsf{S}_{\mathbf{E}}) \asymp \mathrm{evr}(\mathbf{E})$ follows by combining Proposition 2.2 in [285], which states that $\mathrm{vr}(\mathsf{S}_{\mathbf{E}}) \asymp \mathrm{vr}(\mathbf{E})$, with (1.71) and the duality $\mathsf{S}_{\mathbf{E}}^* = \mathsf{S}_{\mathbf{E}^*}$ (e.g., [289, Theorem 1.17]).

The proof of Lemma 54 also shows (see Remark 172 below) that if we could prove Conjecture 49 for S_∞^n, then it would follow that $\mathsf{S}_{\mathbf{E}}$ satisfies Conjecture 49 for any symmetric normed space $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$, i.e., the logarithmic factor in (1.80) could be replaced by a universal constant.

By substituting Lemma 54 into Corollary 51 and using volume ratio computations of Schütt [285], we will derive in Section 6.3 the following proposition.

Proposition 55. *If* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *is a symmetric normed space, then*

$$\mathsf{e}(\mathbf{E}) \lesssim \mathrm{diam}_{\ell_2^n}\big(B_{\mathbf{E}}\big)\|e_1 + \cdots + e_n\|_{\mathbf{E}}$$

and

$$\mathsf{e}\big(\mathsf{S}_{\mathbf{E}}\big) \lesssim \mathrm{diam}_{\ell_2^n}\big(B_{\mathbf{E}}\big)\|e_1 + \cdots + e_n\|_{\mathbf{E}}\sqrt{n \log n}.$$

The following remark sketches an alternative approach towards Conjecture 9 when K is the hypercube $[-1, 1]^n$ that differs from how we will prove Theorem 24. It yields the desired result up to a lower order factor that grows extremely slowly. Specifically, it constructs an origin-symmetric convex body $L \subseteq [-1, 1]^n$ with

$$\mathrm{iq}(L) = e^{O(\log^* n)} \quad \text{and} \quad [-1, 1]^n \subseteq e^{O(\log^* n)} L.$$

Here, for each $x \geqslant 1$ the quantity $\log^* x$ is defined to be the $k \in \mathbb{N}$ such that

$$\mathsf{tower}(k-1) \leqslant x < \mathsf{tower}(k)$$

for the sequence $\{\mathsf{tower}(i)\}_{i=0}^\infty$ that is defined by $\mathsf{tower}(0) = 1$ and $\mathsf{tower}(i+1) = \exp(\mathsf{tower}(i))$. We think that this approach is worthwhile to describe despite the fact that it falls slightly short of fully establishing Conjecture 9 for $[-1, 1]^n$ due to its flexibility that could be used for other purposes, as well as due to its intrinsic interest.

Remark 56. Fix $n \in \mathbb{N}$ and $q \geqslant 1$. Since the nth root of the volume of the unit ball of ℓ_q^n is of order $n^{-1/q}$ and ℓ_q^n is in minimum surface area position, we can restate (1.42) as

$$\mathrm{iq}\big(B_{\ell_q^n}\big) \asymp \min\{\sqrt{qn}, n\}. \tag{1.81}$$

In particular, for $\mathbf{Y} = \ell_q^n$ with $q = \log n$, we have $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\ell_\infty^n}$ and

$$\mathrm{iq}(\mathbf{Y}) \lesssim \sqrt{n \log n},$$

which already comes close to the conclusion of Conjecture 9. We can do better using the following evaluation of the isoperimetric quotient of the unit ball of $\ell_p^n(\ell_q^m)$, which holds for every $n, m \in \mathbb{N}$ and $p, q \geqslant 1$:

$$\mathrm{iq}\big(B_{\ell_p^n(\ell_q^m)}\big) \asymp \begin{cases} nm & m \leqslant \min\{\frac{p}{n}, q\}, \\ n\sqrt{qm} & q \leqslant m \leqslant \frac{p}{n}, \\ \sqrt{pnm} & \frac{p}{n} \leqslant m \leqslant \min\{p, q\}, \\ \sqrt{pqn} & \max\{\frac{p}{n}, q\} \leqslant m \leqslant p, \\ m\sqrt{n} & p \leqslant m \leqslant q, \\ \sqrt{qnm} & m \geqslant \max\{p, q\}. \end{cases} \tag{1.82}$$

We will prove (1.82) in Section 6. Note that when $m = 1$ this yields (1.81). The case $n = m$ of (1.82) is equivalent to (1.49) since $\ell_p^n(\ell_q^m)$ is canonically positioned (it belongs to the class of spaces in Example 40) and using a simple evaluation of the volume of its unit ball (see (6.6) below). The range of (1.82) that is most pertinent for the present context is $m \geqslant \max\{p, q\}$, which has the feature that the factor that multiplies the quantity

$$\sqrt{nm} = \sqrt{\dim\big(\ell_p^n(\ell_q^m)\big)}$$

is $O(\sqrt{q})$ and there is no dependence on p. This can be used as follows. Suppose that $n = ab$ for $a, b \in \mathbb{N}$ satisfying $a \asymp n/\log n$ and $b \asymp \log n$. Identify ℓ_∞^n with $\ell_\infty^a(\ell_\infty^b)$. If we set $\mathbf{Y} = \ell_p^a(\ell_q^b)$ for $p = \log a \asymp \log n$ and $q = \log b \asymp \log\log n$, then $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\ell_\infty^n}$, while

$$\mathrm{iq}(B_{\mathbf{Y}}) \asymp \sqrt{n \log\log n}$$

by (1.82). By iterating we get that for infinitely many $n \in \mathbb{N}$ there is a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ for which

$$\|\cdot\|_{\mathbf{Y}} \leqslant \|\cdot\|_{\ell_\infty^n} \lesssim e^{O(\log^* n)}\|\cdot\|_{\mathbf{Y}} \quad \text{and} \quad \mathrm{iq}(B_{\mathbf{Y}}) = e^{O(\log^* n)}.$$

Even though the set of $n \in \mathbb{N}$ for which this works is not all of $\mathbb{N}$, it is quite dense in $\mathbb{N}$ per Lemma 163 below. This will allow us to deduce that a space $\mathbf{Y}$ with the above properties exists for every $n \in \mathbb{N}$; see Section 6.1 for the details.

Remark 57. Recalling Remark 38, Conjecture 10 is equivalent to the assertion that if a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_\mathbf{X})$ is in Cheeger position, then $\mathrm{iq}(\mathrm{Ch}\, B_\mathbf{X}) \lesssim \sqrt{n}$ and $\mathrm{vol}_n(\mathrm{Ch}\, B_\mathbf{X})^{1/n} \gtrsim \mathrm{vol}_n(B_\mathbf{X})^{1/n}$. Since $\mathrm{Ch}\,\mathbf{X}$ is in minimum surface area position when $\mathbf{X}$ is in Cheeger position (as explained in Remark 38), the proof of Proposition 32 shows that Conjecture 10 implies that if $\mathbf{X}$ is in Cheeger position, then

$$\mathsf{e}(\mathbf{X}) \lesssim \frac{\mathrm{diam}_{\ell_2^n}(B_\mathbf{X})}{\mathrm{vol}_n(B_\mathbf{X})^{\frac{1}{n}}}. \tag{1.83}$$

In fact, the right-hand side of (1.54) is at most the right-hand side of (1.83) for a suitable choice of normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_\mathbf{Y})$, specifically for $\mathbf{Y} = \mathrm{Ch}\,\mathbf{X}$. The discussion in Section 1.6.2 was about establishing (1.83) when $\mathbf{X}$ is canonically positioned (conceivably that assumption implies that $\mathbf{X}$ is in Cheeger position or close to it, which would be a worthwhile to prove, if true). Even though, as we explained earlier, given the current state of knowledge, understanding the Lipschitz extension problem for canonically positioned spaces is the most pressing issue for future research, it would be very interesting to study if (1.83) holds in other situations. For examples, we pose the following two natural questions.

Question 58. Does (1.83) hold if the normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_\mathbf{X})$ is in minimum surface area position?

The extent to which $\Pi\mathbf{X}$ is close to being in minimum surface area position when $\mathbf{X}$ is in minimum surface area position seems to be unknown. Therefore, the connection between Question 59 below and Question 58 is unclear, but even if there is no formal link between these two questions, both are natural next steps beyond the setting of canonically positioned normed spaces.

Question 59. Let $\mathbf{Z} = (\mathbb{R}^n, \|\cdot\|_\mathbf{Z})$ be a normed space in minimum surface area position. Does (1.83) hold for the normed space $\mathbf{X} = \Pi\mathbf{Z}$ whose unit ball is the projection body of $B_\mathbf{X}$?

If $\mathbf{Z} = (\mathbb{R}^n, \|\cdot\|_\mathbf{Z})$ is a normed space in minimum surface area position, then

$$\frac{\mathrm{diam}_{\ell_2^n}(\Pi B_\mathbf{Z})}{\mathrm{vol}_n(\Pi B_\mathbf{Z})^{\frac{1}{n}}} \asymp \sqrt{n}. \tag{1.84}$$

Indeed, because $\mathbf{Z}$ is in minimum surface area position, by [104, Corollary 3.4] we have

$$\mathrm{vol}_n(\Pi B_\mathbf{Z})^{\frac{1}{n}} \asymp \frac{\mathrm{vol}_{n-1}(\partial B_\mathbf{Z})}{n},$$

and also by combining [104, Proposition 3.1] and (1.55) we have

$$\mathrm{MaxProj}(B_\mathbf{Z}) \asymp \frac{\mathrm{vol}_{n-1}(\partial B_\mathbf{Z})}{\sqrt{n}}.$$

We can therefore justify (1.84) using these results from [104] and duality as follows:

$$\frac{\mathrm{diam}_{\ell_2^n}(\Pi B_{\mathbf{Z}})}{\mathrm{vol}_n(\Pi B_{\mathbf{Z}})^{\frac{1}{n}}} \asymp \frac{n\|\mathsf{Id}_n\|_{\Pi\mathbf{Z}\to\ell_2^n}}{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Z}})} = \frac{n\|\mathsf{Id}_n\|_{\ell_2^n\to\Pi^*\mathbf{Z}}}{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Z}})}$$
$$= \frac{n\max_{z\in S^{n-1}}\|z\|_{\Pi^*\mathbf{Z}}}{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Z}})} \overset{(1.30)}{=} \frac{n\mathrm{MaxProj}(B_{\mathbf{Z}})}{\mathrm{vol}_{n-1}(\partial B_{\mathbf{Z}})} \asymp \sqrt{n}.$$

By this observation, a positive answer to Question 59 would show that $\mathsf{e}(\Pi\mathbf{Z}) \lesssim \sqrt{n}$ for any normed space $\mathbf{Z} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Z}})$. Indeed, if we take $S \in \mathsf{SL}_n(\mathbb{R})$ such that $S\mathbf{Z}$ is in minimum surface area position, then by [251] we know that $\Pi\mathbf{Z}$ and $\Pi S\mathbf{Z}$ are isometric, so $\mathsf{e}(\Pi\mathbf{Z}) = \mathsf{e}(\Pi S\mathbf{Z})$. As the class of projection bodies coincides with the class of zonoids [41, 283], which coincides with the class of convex bodies whose polar is the unit ball of a subspace of L_1, we have thus shown that a positive answer to Question 59 would imply the following conjecture (which would simultaneously improve (1.23) and generalize Theorem 18).

Conjecture 60. For any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ we have

$$\mathsf{e}(\mathbf{X}) \lesssim \mathsf{c}_{L_1}(\mathbf{X}^*)\sqrt{n}.$$

Note that Conjecture 60 is consistent with the estimate $\mathsf{e}(\mathbf{X}) \lesssim \mathrm{evr}(\mathbf{X})\sqrt{n}$ that has been arising thus far. Indeed, if $\mathbf{X}^*$ is isometric to a subspace of L_1 (it suffices to consider only this case in Conjecture 60 by a well-known differentiation argument; see, e.g., [36, Corollary 7.10]), then we have the bound $\mathrm{evr}(\mathbf{X}) \lesssim 1$ which can be seen to hold by combining (1.71) with (1.9), since $C_2(\mathbf{X}^*) \leqslant C_2(L_1) \lesssim 1$.[9]

Relating $\mathsf{e}(\mathbf{X})$ to $\mathrm{evr}(\mathbf{X})$ is valuable since the Lipschitz extension modulus is for the most part shrouded in mystery, while the literature contains extensive knowledge on volume ratios (we have already seen several examples of such consequences above, and we will derive more later). Section 6.3 contains examples of volume ratio evaluations for various canonically positioned normed spaces. Through their substitution into Corollary 51, they illustrate how our work yields a range of new Lipschitz extension results, some of which are currently conjectural because they hold assuming Conjecture 49 for the respective spaces; specifically, consider the Lipschitz extension bounds that correspond to using (1.14) and (1.15) with [173].

[9] Alternatively, $\mathrm{evr}(\mathbf{X}) \lesssim 1$ can be justified by writing $\mathbf{X} = \Pi\mathbf{Z}$ for some normed space $\mathbf{Z} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Z}})$ (using [41, 283]), and then applying the bound (1.84) that we derived above (this even demonstrates that the external volume ratio of $\Pi\mathbf{Z}$ is $O(1)$ when $\mathbf{Z}$ is in minimum surface area position rather when $\mathbf{Z}$ is in Löwner position). Actually, the sharp bound $\mathrm{evr}(\mathbf{X}) \leqslant \mathrm{evr}(\ell_\infty^n)$ holds, as seen by combining [22, Theorem 6] with Reisner's theorem [271] that the Mahler conjecture [193] holds for zonoids.

1.6.3 Intersection with a Euclidean ball

Fix an integer $n \geqslant 2$ and a canonically positioned normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. A natural first attempt to prove Conjecture 49 for $\mathbf{X}$ is to consider the normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ such that $B_{\mathbf{Y}} = B_{\mathbf{X}} \cap rB_{\ell_2^n}$ for a suitably chosen $r > 0$ (equivalently, we have $\|x\|_{\mathbf{Y}} = \max\{\|x\|_{\mathbf{X}}, \|x\|_{\ell_2^n}/r\}$ for every $x \in \mathbb{R}^n$). However, we checked with G. Schechtman that this fails even when $\mathbf{X} = \ell_\infty^n$. Specifically, if the nth root of the volume of $B_{\ell_\infty^n} \cap (rB_{\ell_2^n})$ is at least a universal constant, then necessarily $r \gtrsim \sqrt{n}$, but

$$\forall s > 0, \quad \mathrm{iq}\big(B_{\ell_\infty^n} \cap (s\sqrt{n}B_{\ell_2^n})\big) \gtrsim_s n. \tag{1.85}$$

A justification of (1.85) appears in Section 7 below. In terms of the quantification (1.60) of Conjecture 49 that is pertinent to the applications that we study herein, we will also show in Section 7 that

$$\min_{r>0} \frac{\mathrm{iq}\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big)}{\sqrt{n}} \left(\frac{\mathrm{vol}_n(B_{\ell_\infty^n})}{\mathrm{vol}_n\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big)}\right)^{\frac{1}{n}} \asymp \sqrt{\log n}, \tag{1.86}$$

where the minimum in the right-hand side of (1.86) is attained at some $r > 0$ that satisfies $r \asymp \sqrt{n/\log n}$.

Even though the above bounds demonstrate that it is impossible to resolve Conjecture 49 by intersecting with a Euclidean ball, this approach cannot fail by more than a lower-order factor; the reasoning that proves this assertion was shown to us by B. Klartag and E. Milman in unpublished private communication that is explained with their permission in Section 7. Specifically, we have the following proposition.

Proposition 61. *For any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ there exist a matrix $S \in \mathsf{SL}_n(\mathbb{R})$ and a radius $r > 0$ such that for $L = (SB_{\mathbf{X}}) \cap (rB_{\ell_2^n}) \subseteq SB_{\mathbf{X}}$ we have $\mathrm{iq}(L) \lesssim \sqrt{n}$ and $\sqrt[n]{\mathrm{vol}_n(L)} \gtrsim \sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})}/K(\mathbf{X})$, where $K(\mathbf{X})$ is the K-convexity constant of $\mathbf{X}$. If $\mathbf{X}$ is canonically positioned, then this holds when S is the identity matrix.*

For Proposition 61, the K-convexity constant of $\mathbf{X}$ is an isomorphic invariant that was introduced by Maurey and Pisier [204]; we defer recalling its definition to Section 7 since for the discussion here it suffices to state the following bounds that relate $K(\mathbf{X})$ to quantities that we already encountered. Firstly,

$$K(\mathbf{X}) \lesssim \log\big(d_{\mathrm{BM}}(\ell_2^n, \mathbf{X}) + 1\big) \lesssim \log n, \tag{1.87}$$

The first inequality in (1.87) is a useful theorem of Pisier [256, 257]. The second inequality in (1.87) follows from John's theorem [137], though for this purpose it suffices to use the older Auberbach lemma (see [27, p. 209] and [83, 300]). By [257] (see also, e.g., [143, Lemma 17]) the rightmost quantity in (1.87) can be reduced if $\mathbf{X}$ is a subspace of L_1, namely we have

$$K(\mathbf{X}) \lesssim \mathsf{c}_{L_1}(\mathbf{X})\sqrt{\log n}. \tag{1.88}$$

Secondly, $K(\mathbf{X})$ relates to the notion of type that we recalled in (1.77) through the following bounds:

$$T_{1+\frac{c}{K(\mathbf{X})^2}}(\mathbf{X})^{\frac{1}{2}} \lesssim K(\mathbf{X}) \leqslant \inf_{p\in(1,2]} e^{(CT_p(\mathbf{X}))^{\frac{p}{p-1}}}, \tag{1.89}$$

where $c, C > 0$ are universal constants. The qualitative meaning of (1.89) is that the K-convexity constant of a Banach space is finite if and only if it has type p for some $p > 1$; this is a landmark theorem of Pisier (the 'if' direction is due to [259] and the 'only if' direction is due to [254]). Since in our setting $\mathbf{X}$ is finite dimensional ($\dim(\mathbf{X}) = n \geqslant 2$), such a qualitative statement is vacuous without its quantitative counterpart (1.89). The first inequality in (1.89) can be deduced from [260] (together with the computation of the implicit dependence on p in [260] that was carried out in [131, Lemma 32]). The second inequality in (1.89) follows from an examination of the proof in [259]. We omit the details of both deductions as they would result in a (quite lengthy and tedious) digression. It would be very interesting to determine the best bounds in the context of (1.89).

Proposition 61 combined with (1.87) implies that Conjecture 10 holds up to a logarithmic factor in the sense that for every integer $n \geqslant 2$, any origin-symmetric convex body $K \subseteq \mathbb{R}^n$ admits a matrix $S \in \mathsf{SL}_n(\mathbb{R})$ and an origin-symmetric convex body $L \subseteq SK$ such that

$$\frac{\mathrm{iq}(L)}{\sqrt{n}}\left(\frac{\mathrm{vol}_n(K)}{\mathrm{vol}_n(L)}\right)^{\frac{1}{n}} \lesssim \log n. \tag{1.90}$$

Furthermore, by (1.88) the $\log n$ in (1.90) can be replaced by $\sqrt{\log n}$ if K is the unit ball of a subspace of L_1 (equivalently, the polar of K is a zonoid), and by the second inequality in (1.89) if $p > 1$, then the $\log n$ in (1.90) can be replaced by a dimension-independent quantity that depends only on p and the type-p constant of the norm whose unit ball is K. Also, Corollary 33 holds with the right-hand side of (1.59) multiplied by $\log n$, and the reverse Faber–Krahn inequality of Conjecture 35 holds up to a factor of $(\log n)^2$, i.e., for any origin-symmetric convex body $K \subseteq \mathbb{R}^n$ there is $S \in \mathsf{SL}_n(\mathbb{R})$ such that $\lambda(SK)\,\mathrm{vol}(K)^{2/n} \lesssim n(\log n)^2$. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a canonically positioned normed space, then it follows that for a suitable choice of normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ the right-hand side of (1.28), and hence also $\mathsf{e}(\mathbf{X})$ by Theorem 21, is at most a universal constant multiple of $\mathrm{evr}(\mathbf{X})\sqrt{n}\log n$, and also $n \lesssim \lambda(\mathbf{X})\,\mathrm{vol}_n(B_{\mathbf{X}})^{2/n} \lesssim n(\log n)^2$.

1.7 Randomized clustering

All of the new upper bounds on Lipschitz extension moduli that we stated above rely on a geometric structural result for finite dimensional normed spaces (and subsets thereof). Beyond the application to Lipschitz extension, this result is of value in its own right because it yields an improvement of a basic randomized clustering method from the computer science literature.

The link between random partitions of metric spaces and Lipschitz extension was found in [173]. We will adapt the methodology of [173] to deduce the aforementioned Lipschitz extension theorems from our new bound on randomized partitions of normed spaces. In order to formulate the corresponding definitions and results, one must first set some groundwork for a notion of a random partition of a metric space, whose subsequent applications necessitate certain measurability requirements.

A framework for reasoning about random partitions of metric spaces was developed in [173], but we will formulate a different approach. The reason for this is that the definitions of [173] are in essence the minimal requirements that allow one to use at once several different types of random partitions for Lipschitz extension, which leads to definitions that are more cumbersome than the approach that we take below. Greater simplicity is not the only reason why we chose to formulate a foundation that differs from [173]. The approach that we take is easier to implement, and, importantly, it yields a bi-Lipschitz invariant, while we do not know if the corresponding notions in [173] are bi-Lipschitz invariants (we suspect that they are *not*, but we did not attempt to construct examples that demonstrate this). The Lipschitz extension theorem of [173] is adapted accordingly in Section 5, thus making the present article self-contained, and also yielding simplification and further applications. Nevertheless, the key geometric ideas that underly this use of random partitions are the same as in [173].

Obviously, there are no measurability issues when one considers finite metric spaces (in our setting, finite subsets of normed spaces). The ensuing measurability discussions can therefore be ignored in the finitary setting. In particular, the computer science literature on random partitions focuses exclusively on finite objects. So, for the purpose of algorithmic clustering, one does not need the more general treatment below, but it is needed for the purpose of Lipschitz extension.

1.7.1 Basic definitions related to random partitions

Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space. Suppose that $\mathcal{P} \subseteq 2^{\mathfrak{M}}$ is a partition of $\mathfrak{M}$. For $x \in \mathfrak{M}$, denote by $\mathcal{P}(x) \subseteq \mathfrak{M}$ the unique element of $\mathcal{P}$ to which x belongs. The sets $\{\mathcal{P}(x)\}_{x\in\mathfrak{M}}$ are often called the *clusters* of $\mathcal{P}$. Given $\Delta > 0$, one says that $\mathcal{P}$ is Δ-bounded if $\mathrm{diam}_{\mathfrak{M}}(\mathcal{P}(x)) \leqslant \Delta$ for every $x \in \mathfrak{M}$, where

$$\mathrm{diam}_{\mathfrak{M}}(S) = \sup\{d_{\mathfrak{M}}(x, y) : x, y \in S\}$$

denotes the diameter of $\varnothing \neq S \subseteq \mathfrak{M}$.

Suppose that $(\mathcal{Z}, \mathcal{F})$ is a measurable space, i.e., $\mathcal{Z}$ is a set and $\mathcal{F} \subseteq 2^{\mathcal{Z}}$ is a σ-algebra of subsets of $\mathcal{Z}$. Recall (see [133] or the convenient survey [309]) that if

$(\mathfrak{M}, d_{\mathfrak{M}})$ is a metric space, then a set-valued mapping

$$\Gamma : \mathcal{Z} \to 2^{\mathfrak{M}}$$

is said to be strongly measurable if for every closed subset $E \subseteq \mathfrak{M}$ we have

$$\Gamma^{-}(E) \stackrel{\text{def}}{=} \big\{z \in \mathcal{Z} : E \cap \Gamma(z) \neq \varnothing\big\} \in \mathcal{F}. \tag{1.91}$$

Throughout what follows, when we say that $\mathcal{P}$ is a *random partition* of a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, we mean the following (formally, the objects that we will be considering are random *ordered* partitions into countably many clusters). There is a probability space $(\Omega, \mathbf{Prob})$ and a sequence of set-valued mappings

$$\big\{\Gamma^k : \Omega \to 2^{\mathfrak{M}}\big\}_{k=1}^{\infty}.$$

We write $\mathcal{P}^{\omega} = \{\Gamma^k(\omega)\}_{k=1}^{\infty}$ for each $\omega \in \Omega$ and require that the mapping $\omega \mapsto \mathcal{P}^{\omega}$ takes values in partitions of $\mathfrak{M}$. We also require that for every fixed $k \in \mathbb{N}$, the set-valued mapping $\Gamma^k : \Omega \to 2^{\mathfrak{M}}$ is strongly measurable, where the σ-algebra on Ω is the $\mathbf{Prob}$-measurable sets. Given $\Delta > 0$, we say that $\mathcal{P}$ is a Δ-bounded random partition of $(\mathfrak{M}, d_{\mathfrak{M}})$ if $\mathcal{P}^{\omega}$ is a Δ-bounded partition of $(\mathfrak{M}, d_{\mathfrak{M}})$ for every $\omega \in \Omega$.

Remark 62. Recall that when we say that $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is a normed space we mean that the underlying vector space is $\mathbb{R}^n$, equipped with a norm $\|\cdot\|_{\mathbf{X}} : \mathbb{R}^n \to [0, \infty)$. By doing so, we introduce a second metric on $\mathbf{X}$, i.e., $\mathbb{R}^n$ is also endowed with the standard Euclidean structure that corresponds to the norm $\|\cdot\|_{\ell_2^n}$. This leads to ambiguity when we discuss Δ-bounded partitions of $\mathbf{X}$ for some $\Delta > 0$, as there are two possible metrics with respect to which one could bound the diameters of the clusters. In fact, a key aspect of our work is that it can be beneficial to consider another auxiliary norm $\|\cdot\|_{\mathbf{Y}}$ on $\mathbb{R}^n$, as in, e.g., Theorem 21, thus leading to three possible interpretations of Δ-boundedness of a partition of $\mathbb{R}^n$. To avoid any confusion, we will adhere throughout to the convention that when we say that a partition $\mathcal{P}$ of $\mathbf{X}$ is Δ-bounded we mean exclusively that all the clusters of $\mathcal{P}$ have diameter at most Δ with respect to the norm $\|\cdot\|_{\mathbf{X}}$.

1.7.2 Iterative ball partitioning

Fix $\Delta \in (0, \infty)$. Iterative ball partitioning is a common procedure to construct a Δ-bounded random partition of a metric probability space. We will next describe it to clarify at the outset the nature of the objects that we investigate, and because our new positive partitioning results are solely about this type of partition. Thus, our contribution to the theory of random partitions is a sharp understanding of the performance of iterative ball partitioning of normed spaces, and, importantly, the demonstration of the utility of its implementation using balls that are induced by a suitably chosen auxiliary norm rather than the given norm that we aim to study. On the other hand, our impossibility results rule out the existence of any random partition whatsoever with certain desirable properties.

The iterative ball partitioning method is a ubiquitous tool in metric geometry and algorithm design. To the best of our knowledge, it was first used by Karger, Motwani and Sudan [152] and the aforementioned work [76] in the context of normed spaces, and it has become very influential in the context of general metric spaces due to its use in that setting (with the important twist of randomizing the radii) by Calinescu, Karloff and Rabani [71]. To describe it, suppose that $(\mathfrak{M}, d_{\mathfrak{M}})$ is a metric space and that μ is a Borel probability measure on $\mathfrak{M}$. Let $\{\mathsf{X}_k\}_{k=1}^{\infty}$ be a sequence of i.i.d. points sampled μ. Define inductively a sequence $\{\Gamma^k\}_{k=1}^{\infty}$ of random subsets of $\mathfrak{M}$ by setting $\Gamma^1 = B_{\mathfrak{M}}(\mathsf{X}_1, \Delta/2)$ and

$$\forall k \in \{2, 3, \ldots, \}, \qquad \Gamma^k \stackrel{\mathrm{def}}{=} B_{\mathfrak{M}}\Big(\mathsf{X}_k, \frac{\Delta}{2}\Big) \smallsetminus \bigcup_{j=1}^{k-1} B_{\mathfrak{M}}\Big(\mathsf{X}_j, \frac{\Delta}{2}\Big).$$

By design, $\mathrm{diam}_{\mathfrak{M}}(\Gamma^k) \leqslant \Delta$. Under mild assumptions on $\mathfrak{M}$ and μ that are simple to check, Γ^k will have the measurability properties that we require below and $\mathcal{P} = \{\Gamma^k\}_{k=1}^{\infty}$ will be a partition of $\mathfrak{M}$ almost-surely. While initially the clusters of $\mathcal{P}$ are quite "tame," e.g., they start out as balls in $\mathfrak{M}$, as the iteration proceeds and we discard the balls that were used thus far, the resulting sets become increasingly "jagged." In particular, even when the underlying metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ is very "nice," the clusters of $\mathcal{P}$ need not be connected; see Figure 1.2. Nevertheless, we will see that such a simple procedure results in a random partition with probabilistically small boundaries in sense that will be described rigorously below.

In the present setting, the metric space that we wish to partition is a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$, so it is natural to want to use the Lebesgue measure on $\mathbb{R}^n$ in the above construction. Since this measure is not a probability measure, we cannot use the above framework directly. For this reason, we will in fact use a periodic variant of iterative ball partitioning of $\mathbf{X}$ by adapting a construction that was used in [173].

1.7.3 Separation and padding

Fix $\Delta > 0$. Let $\mathcal{P}$ be a Δ-bounded random partition of a metric space $\mathfrak{M}$. As a random "clustering" of $\mathfrak{M}$ into pieces of small diameter, $\mathcal{P}$ yields a certain "simplification" of $\mathfrak{M}$. For such a simplification to be useful, one must add a requirement that it "mimics" the geometry of $\mathfrak{M}$ in a meaningful way. The literature contains multiple definitions that achieve this goal, leading to applications in both algorithms and pure mathematics. We will not attempt to survey the literature on this topic, quoting only the definitions of *separating* and *padded* random partitions, which are the simplest

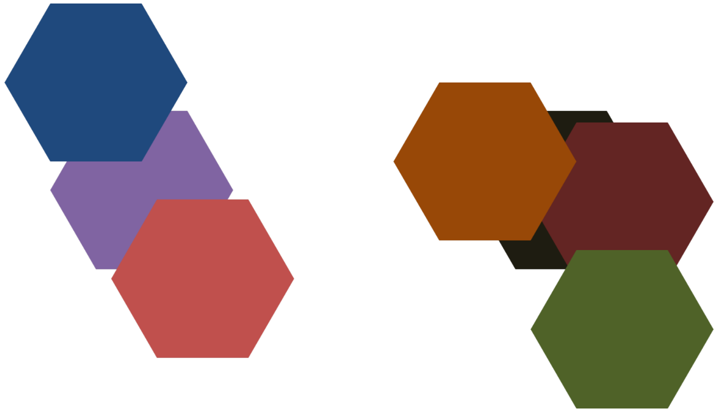

Figure 1.2. A schematic depiction of (randomized) iterative ball partitioning of a bounded subset of $\mathbb{R}^2$, where $\mathbb{R}^2$ is equipped with a norm whose unit ball is a regular hexagon. The centers of the above hexagons are chosen independently and uniformly at random from a large region that contains the given subset of $\mathbb{R}^2$. At each step of the iteration, a new hexagon appears, and it carves out a new cluster which consists of the part of the hexagon that does not intersect any of the clusters that have been formed in the previous stages of the iteration. The first few clusters that are formed by this procedure are typically hexagons, but at later stages the clusters become more complicated and less "round." In particular, they can eventually become disconnected, as exhibited by the region that is shaded black above.

and most popular notions of random partitions of metric spaces among those that have been introduced.

Definition 63 (Separating random partition and separation modulus). Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space. For $\sigma, \Delta > 0$, a Δ-bounded random partition $\mathcal{P}$ of $(\mathfrak{M}, d_{\mathfrak{M}})$ is *σ-separating* if

$$\forall x, y \in \mathfrak{M}, \quad \mathbf{Prob}\big[\mathcal{P}(x) \neq \mathcal{P}(y)\big] \leqslant \frac{\sigma}{\Delta} d_{\mathfrak{M}}(x, y). \tag{1.92}$$

The *separation modulus*[10] of $(\mathfrak{M}, d_{\mathfrak{M}})$, denoted $\mathsf{SEP}(\mathfrak{M}, d_{\mathfrak{M}})$ or simply $\mathsf{SEP}(\mathfrak{M})$ if the metric is clear from the context, is the infimum over those $\sigma > 0$ such that for every $\Delta > 0$ there exists a σ-separating Δ-bounded random partition of $(\mathfrak{M}, d_{\mathfrak{M}})$. If no such σ exists, then write $\mathsf{SEP}(\mathfrak{M}, d_{\mathfrak{M}}) = \infty$. Similarly, for $n \in \mathbb{N}$, the *size-n separation modulus* of $(\mathfrak{M}, d_{\mathfrak{M}})$, denoted $\mathsf{SEP}^n(\mathfrak{M}, d_{\mathfrak{M}})$ or simply $\mathsf{SEP}^n(\mathfrak{M})$ if the metric is clear from the context, is the infimum over those $\sigma > 0$ such that for every $S \subseteq \mathfrak{M}$ with $|S| \leqslant n$ and every $\Delta > 0$ there exists a σ-separating Δ-bounded random

[10]In [227] we called the same quantity the "modulus of separated decomposability."

partition of $(S, d_{\mathfrak{M}})$. In other words,

$$\mathsf{SEP}^n(\mathfrak{M}, d_{\mathfrak{M}}) \stackrel{\mathrm{def}}{=} \sup_{\substack{S\subseteq \mathfrak{M}\\ |S|\leqslant n}} \mathsf{SEP}(S, d_{\mathfrak{M}}).$$

While the notions that we presented in Definition 63 are standard (see below for the history), it will be beneficial for us (e.g., for proving Theorem 29) to introduce the following terminology.

Definition 64 (Separation profile). Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space. We say that a metric $\mathfrak{d} : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ on $\mathfrak{M}$ is a *separation profile* of $(\mathfrak{M}, d_{\mathfrak{M}})$ if for every $\Delta > 0$ there exists a Δ-bounded random partition $\mathcal{P}_\Delta$ of $(\mathfrak{M}, d_{\mathfrak{M}})$ that is defined on some probability space $(\Omega_\Delta, \mathbf{Prob}_\Delta)$ such that

$$\forall x, y \in \mathfrak{M}, \quad \mathfrak{d}(x, y) \geqslant \sup_{\Delta\in(0,\infty)} \Delta\mathbf{Prob}_\Delta\big[\mathcal{P}_\Delta(x) \neq \mathcal{P}_\Delta(y)\big]. \tag{1.93}$$

So, the separation modulus of $(\mathfrak{M}, d_{\mathfrak{M}})$ is the infimum over those $\sigma > 0$ for which $\sigma d_{\mathfrak{M}}$ is a separation profile of $(\mathfrak{M}, d_{\mathfrak{M}})$. Definition 64 would make sense for functions $\mathfrak{d} : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ that need not be metrics on $\mathfrak{M}$, but we prefer to deal only with separation profiles of $(\mathfrak{M}, d_{\mathfrak{M}})$ that are metrics on $\mathfrak{M}$ so as to be able to discuss the Lipschitz condition with respect to them; observe that the right-hand side of (1.93) is a metric on $\mathfrak{M}$, so any such function is always at least (point-wise) a metric that is a separation profile of $(\mathfrak{M}, d_{\mathfrak{M}})$. If $\mathfrak{d} : \mathfrak{M} \times \mathfrak{M} \to [0, \infty)$ is a separation profile of $(\mathfrak{M}, d_{\mathfrak{M}})$, then $\mathfrak{d}(x, y) \geqslant d_{\mathfrak{M}}(x, y)$ for all $x, y \in \mathfrak{M}$ because $\mathrm{diam}_{\mathfrak{M}}(\mathcal{P}_{d_{\mathfrak{M}}(x,y)-\varepsilon}(x)) \leqslant d_{\mathfrak{M}}(x, y) - \varepsilon < d_{\mathfrak{M}}(x, y)$ for any $0 < \varepsilon < d_{\mathfrak{M}}(x, y)$, so we necessarily have $y \notin \mathcal{P}_{d_{\mathfrak{M}}(x,y)-\varepsilon}(x)$ (deterministically) and therefore

$$\begin{aligned}\mathfrak{d}(x, y) &\geqslant (d_{\mathfrak{M}}(x, y) - \varepsilon)\mathbf{Prob}_{d_{\mathfrak{M}}(x,y)-\varepsilon}\big[\mathcal{P}_{d_{\mathfrak{M}}(x,y)-\varepsilon}(x) \neq \mathcal{P}_{d_{\mathfrak{M}}(x,y)-\varepsilon}(y)\big]\\ &= d_{\mathfrak{M}}(x, y) - \varepsilon.\end{aligned} \tag{1.94}$$

Definition 65 (Padded random partition and padding modulus). Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a metric space. For $\delta, \mathfrak{p}, \Delta > 0$, a Δ-bounded random partition $\mathcal{P}$ of $(\mathfrak{M}, d_{\mathfrak{M}})$ is $(\mathfrak{p}, \delta)$-*padded* if

$$\forall x \in \mathfrak{M}, \quad \mathbf{Prob}\Big[B_{\mathfrak{M}}\Big(x, \frac{\Delta}{\mathfrak{p}}\Big) \subseteq \mathcal{P}(x)\Big] \geqslant \delta. \tag{1.95}$$

Denote by $\mathsf{PAD}_\delta(\mathfrak{M}, d_{\mathfrak{M}})$, or simply $\mathsf{PAD}_\delta(\mathfrak{M})$ if the metric is clear from the context, the infimum over those $\mathfrak{p} > 0$ such that for every $\Delta > 0$ there exists a $(\mathfrak{p}, \delta)$-padded Δ-bounded random partition $\mathcal{P}$ of $(\mathfrak{M}, d_{\mathfrak{M}})$. If no such $\mathfrak{p}$ exists, then write $\mathsf{PAD}_\delta(\mathfrak{M}, d_{\mathfrak{M}}) = \infty$. For every $n \in \mathbb{N}$, denote

$$\mathsf{PAD}^n_\delta(\mathfrak{M}, d_{\mathfrak{M}}) \stackrel{\mathrm{def}}{=} \sup_{\substack{S\subseteq \mathfrak{M}\\ |S|\leqslant n}} \mathsf{PAD}_\delta(S, d_{\mathfrak{M}}).$$

See Section 3 for a quick justification why the above definition of random partition implies that the events that appear in (1.92) and (1.95) are indeed **Prob**-measurable.

Qualitatively, condition (1.92) says that despite the fact that $\mathcal{P}$ decomposes $\mathfrak{M}$ into clusters of small diameter, any two nearby points are likely to belong to the same cluster. Condition (1.95) says that every point in $\mathfrak{M}$ is likely to be "well within" its cluster (its distance to the complement of its cluster is at least a definite proportion of the assumed upper bound on the diameter of that cluster). Both of these requirements express the (often nonintuitive) property that the "boundaries" that the random partition induces are "thin" in a certain distributional sense, despite the fact that each realization of the partition consists only of small diameter clusters that can sometimes be very jagged. Neither of the above two definitions implies the other, but it follows from [170] that if $\mathcal{P}$ is a $(\mathfrak{p}, \delta)$-padded Δ-bounded random partition of $(\mathfrak{M}, d_{\mathfrak{M}})$, then there exists a random partition $\mathcal{P}'$ of $(\mathfrak{M}, d_{\mathfrak{M}})$ that is (2Δ)-bounded and $(4\mathfrak{p}/\delta)$-separating.

Separating and padded random partitions were introduced in the articles [29, 30] of Bartal, which contained decisive algorithmic applications and influenced a flurry of subsequent works that obtained many more applications in several directions. Other works considered such partitions implicitly, with a variety of applications; see the works of Leighton–Rao [175], Awerbuch–Peleg [18], Linial–Saks [184], Alon–Karp–Peleg–West [4], Klein–Plotkin–Rao [156] and Rao [269]. The nomenclature of Definition 63 and Definition 65 comes from [124, 160, 170, 171, 173].

By [29], for every metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ and every integer $n \geqslant 2$, we have the bound $\mathsf{SEP}^n(\mathfrak{M}) \lesssim \log n$. It was observed by Gupta, Krauthgamer and Lee [124] that [29] also implicitly yields the padding bound $\mathsf{PAD}^n_{0.5}(\mathfrak{M}) \lesssim \log n$. It was proved in [29] that both of these estimates are sharp.

Random partitions of normed spaces were first studied by Peleg and Reshef [248] for applications to network routing and distributed computing. The aforementioned work [76] improved and generalized the bounds of [248], and influenced later works; see, e.g., [173], and the work [13] of Andoni and Indyk. Similar partitioning schemes appeared implicitly in earlier work [152] on algorithms for graph colorings based on semidefinite programming.

1.7.4 From separation to Lipschitz extension

As we already explained, the connection between random partitions and Lipschitz extension was found in [173]. Here we will use the following theorem to deduce Theorem 29. It implies in particular the bound

$$\mathsf{e}(\mathfrak{M}) \lesssim \mathsf{SEP}(\mathfrak{M}) \tag{1.96}$$

of [173] and its proof is an adaptation of the ideas of [173] to both the present setup (extension to a function that is Lipschitz with respect to a different metric) and our different measurability requirements from the random partitions; we stress, however, that even though we cannot apply [173] directly as a "black box," the geometric ideas that underly the proof of Theorem 66 are the same as those of [173].

Theorem 66. *Suppose that $\mathfrak{d}$ is a separation profile of a locally compact metric space $(\mathfrak{M}, d_{\mathfrak{M}})$. For every Banach space $(\mathbf{Z}, \|\cdot\|_{\mathbf{Z}})$ and every subset $\mathcal{C} \subseteq \mathfrak{M}$, if $f : \mathcal{C} \to \mathbf{Z}$ is 1-Lipschitz with respect to the metric $d_{\mathfrak{M}}$, i.e., $\|f(x) - f(y)\|_{\mathbf{Z}} \leqslant d_{\mathfrak{M}}(x, y)$ for every $x, y \in \mathfrak{M}$, then there is $F : \mathfrak{M} \to \mathbf{Z}$ that extends f and is $O(1)$-Lipschitz with respect to the metric $\mathfrak{d}$, i.e., $\|F(x) - F(y)\|_{\mathbf{Z}} \lesssim \mathfrak{d}(x, y)$ for every $x, y \in \mathfrak{M}$.*

1.7.5 Bounds on the separation and padding moduli of normed spaces

To facilitate the ensuing discussion of upper and lower bounds on the separation and padding moduli of (subsets of) normed spaces, we will first record two of their rudimentary properties. Firstly, the following lemma formally expresses the aforementioned advantage of the definitions in Section 1.7.3 over those of [173], namely that the moduli $\mathsf{SEP}(\cdot)$ and $\mathsf{PAD}_\delta(\cdot)$ are bi-Lipschitz invariants; its straightforward proof appears in Section 3.

Lemma 67 (Bi-Lipschitz invariance of separation and padding moduli). *Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a complete metric space that admits a bi-Lipschitz embedding into a metric space $(\mathfrak{N}, d_{\mathfrak{N}})$. Then*

$$\mathsf{SEP}(\mathfrak{M}, d_{\mathfrak{M}}) \leqslant \mathsf{c}_{(\mathfrak{N}, d_{\mathfrak{N}})}(\mathfrak{M}, d_{\mathfrak{M}}) \mathsf{SEP}(\mathfrak{N}, d_{\mathfrak{N}}) \tag{1.97}$$

and

$$\forall \delta \in (0,1), \quad \mathsf{PAD}_\delta(\mathfrak{M}, d_{\mathfrak{M}}) \leqslant \mathsf{c}_{(\mathfrak{N}, d_{\mathfrak{N}})}(\mathfrak{M}, d_{\mathfrak{M}}) \mathsf{PAD}_\delta(\mathfrak{N}, d_{\mathfrak{N}}). \tag{1.98}$$

Secondly, we have the following tensorization property of the separation and padding moduli, whose simple proof appears in Section 3. For $s \in [1, \infty]$ and metric spaces $(\mathfrak{M}_1, d_{\mathfrak{M}_1}), (\mathfrak{M}, d_{\mathfrak{M}_2})$, the metric $d_{\mathfrak{M}_1 \oplus_s \mathfrak{M}_2} : \mathfrak{M}_1 \times \mathfrak{M}_2 \to [0, \infty)$ on the Cartesian product $\mathfrak{M}_1 \times \mathfrak{M}_2$ is defined by setting for every $(x_1, x_2), (y_1, y_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$,

$$d_{\mathfrak{M}_1 \oplus_s \mathfrak{M}_2}\big((x_1, x_2), (y_1, y_2)\big) \stackrel{\mathrm{def}}{=} \big(d_{\mathfrak{M}}(x_1, y_1)^s + d_{\mathfrak{N}}(x_2, y_2)^s\big)^{\frac{1}{s}}. \tag{1.99}$$

With the usual convention that when $s = \infty$ the right-hand side of (1.99) is equal to the maximum of $d_{\mathfrak{M}}(x_1,y_1)$ and $d_{\mathfrak{N}}(x_2,y_2)$. The metric space $(\mathfrak{M}_1 \times \mathfrak{M}_2, d_{\mathfrak{M}_1 \oplus_s \mathfrak{M}_2})$ is will be denoted $\mathfrak{M}_1 \oplus_s \mathfrak{M}_2$.

Lemma 68 (Tensorization of separation and padding moduli). *For any $s \in [1, \infty]$ and $\delta_1, \delta_2 \in (0, 1)$, any two metric spaces $(\mathfrak{M}_1, d_{\mathfrak{M}_1})$ and $(\mathfrak{M}_2, d_{\mathfrak{M}_2})$ satisfy*

$$\mathsf{SEP}(\mathfrak{M}_1 \oplus_s \mathfrak{M}_2) \leqslant \mathsf{SEP}(\mathfrak{M}_1) + \mathsf{SEP}(\mathfrak{M}_2), \tag{1.100}$$

and

$$\mathsf{PAD}_{\delta_1\delta_2}(\mathfrak{M}_1 \oplus_s \mathfrak{M}_2) \leqslant \big(\mathsf{PAD}_{\delta_1}(\mathfrak{M}_1)^s + \mathsf{PAD}_{\delta_2}(\mathfrak{M}_2)^s\big)^{\frac{1}{s}}. \tag{1.101}$$

The following theorem shows that the bi-Lipschitz invariant $\mathsf{PAD}_\delta(\cdot)$ is not sufficiently sensitive to distinguish substantially between normed spaces, as its value is essentially independent of the norm.

Theorem 69. *For every $n \in \mathbb{N}$, every normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ satisfies*

$$\forall \delta \in (0,1), \qquad \frac{1}{1-\sqrt[n]{\delta}} \leqslant \frac{1}{2}\mathsf{PAD}_\delta(\mathbf{X}) \leqslant \frac{1+\sqrt[n]{\delta}}{1-\sqrt[n]{\delta}}. \tag{1.102}$$

Therefore, $\mathsf{PAD}_\delta(\mathbf{X}) \asymp \max\{1, \frac{\dim(\mathbf{X})}{\log(1/\delta)}\}$ for every finite dimensional normed space $\mathbf{X}$ and $\delta \in (0,1)$.

As we explained above, in the setting of Theorem 69 the fact that

$$\mathsf{PAD}_{\frac{1}{2}}(\mathbf{X}) = O(n)$$

is well known. We will prove the upper bound on $\mathsf{PAD}_\delta(\mathbf{X})$ that appears in (1.102), i.e., with sharp dependence on both n and δ, in Section 4.1. The fact that $\mathsf{PAD}_{0.5}(\mathbf{X})$ is at least a universal constant multiple of n was proved in the manuscript [170]. Because [170] is not intended for publication, we will prove the lower bound on $\mathsf{PAD}_\delta(\mathbf{X})$ that appears in (1.102) in Section 2.6, by following the reasoning of [170] while taking more care than we did in [170] in order to obtain sharp dependence on δ in addition to sharp dependence on n.

In contrast to Theorem 69, the separation modulus of a finite dimensional normed space can have different asymptotic dependencies on its dimension. For example, we have $\mathsf{SEP}(\ell_2^n) \asymp \sqrt{n}$ and $\mathsf{SEP}(\ell_1^n) \asymp n$ by [76]. Using Lemma 67, we see from this that every normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ satisfies the a priori bounds

$$\frac{n}{d_{\mathrm{BM}}(\ell_1^n, \mathbf{X})} \lesssim \mathsf{SEP}(\mathbf{X}) \lesssim d_{\mathrm{BM}}(\ell_2^n, \mathbf{X})\sqrt{n}, \tag{1.103}$$

which we already quoted in the above overview as (1.2).

Giannopoulos proved [105] that every n-dimensional normed space $\mathbf{X}$ satisfies $d_{\mathrm{BM}}(\ell_1^n, \mathbf{X}) \lesssim n^{5/6}$, so the first inequality in (1.103) implies that $\mathsf{SEP}(\mathbf{X}) \gtrsim \sqrt[6]{n}$. Alternatively, the fact that $\mathsf{SEP}(\mathbf{X}) \geqslant n^c$ for some universal constant $c > 0$ follows from by combining Theorem 1 with (1.96). Actually, we always have

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \sqrt{n}, \tag{1.104}$$

which coincides with the first half of (1.7). Observe that (1.104) cannot follow from a "vanilla" application of the first inequality in (1.103) by Szarek's work [295]. In fact, the first inequality of (1.103) must sometimes yield a worse power type dependence on n than in (1.104), because Tikhomirov proved in [302] that there is a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ that satisfies $d_{\mathrm{BM}}(\ell_1^n, \mathbf{X}) \geqslant n^a$ for some universal constant $a > 1/2$.

Nevertheless, we can prove (1.104) by the following a "hereditary" application of (1.103). Bourgain–Szarek [51] and independently Ball (see [51, Remark 7], [296, Remark 7], [305, p. 138]) proved (relying on the Bourgain–Tzafriri restricted invertibility principle [52]) that there is $m \in \{1, \dots, n\}$ with $m \asymp n$ such that $\mathsf{c}_{\mathbf{X}}(\ell_1^m) \lesssim \sqrt{n}$ (in fact, by [51] any $2n$-dimensional normed space has Banach–Mazur distance $O(\sqrt{n})$ from $\ell_1^n \oplus \ell_2^n$). Hence, $\mathsf{SEP}(\mathbf{X}) \gtrsim \mathsf{SEP}(\ell_1^m)/\mathsf{c}_{\mathbf{X}}(\ell_1^m) \asymp m/\mathsf{c}_{\mathbf{X}}(\ell_1^m) \gtrsim \sqrt{n}$, by (1.97).

The second half of (1.7) is the following lower bound on $\mathsf{SEP}(\mathbf{X})$ in terms of the type 2 constant of $\mathbf{X}$:

$$\mathsf{SEP}(\mathbf{X}) \gtrsim T_2(\mathbf{X})^2. \tag{1.105}$$

We will prove (1.105) in Section 2.2 using Talagrand's refinement [298] of Elton's theorem [92], by the same hereditary use of (1.103), namely showing that there is $m \in \{1, \dots, n\}$ for which $m/\mathsf{c}_{\mathbf{X}}(\ell_1^m) \gtrsim T_2(\mathbf{X})^2$.

Remark 70. It is impossible to improve (1.7) for all the values of the relevant parameters, as seen by considering $\mathbf{X} = \ell_2^{n-m} \oplus_2 \ell_1^m$ for each $m \in \{1, \dots, n\}$. Indeed, since in this case $T_2(\mathbf{X}) \asymp \sqrt{m}$,

$$\begin{aligned}\mathsf{SEP}(\mathbf{X}) &\overset{(1.100)}{\leqslant} \mathsf{SEP}\big(\ell_2^{n-m}\big) + \mathsf{SEP}\big(\ell_1^m\big)\\ &\asymp \sqrt{n-m} + m \asymp \sqrt{n} + k \asymp \max\big\{\sqrt{\dim(\mathbf{X})}, T_2(\mathbf{X})^2\big\}.\end{aligned}$$

Thanks to (1.71), the following theorem is a restatement of the lower bound on $\mathsf{SEP}(\mathbf{X})$ in Theorem 3.

Theorem 71. *For every $n \in \mathbb{N}$, any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ satisfies*

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \mathrm{evr}(\mathbf{X})\sqrt{n}.$$

As $\mathrm{evr}(\mathbf{X}) \geqslant 1$ (by definition), Theorem 71 implies (1.104), via a proof that differs from the above reasoning. Also, Theorem 71 is stronger than the first inequality in (1.103) because $\mathrm{evr}(\ell_1^n) \asymp \sqrt{n}$, and hence

$$\mathrm{evr}(\mathbf{X})\sqrt{n} \geqslant \frac{\mathrm{evr}(\ell_1^n)}{d_{\mathrm{BM}}(\ell_1^n, \mathbf{X})}\sqrt{n} \asymp \frac{n}{d_{\mathrm{BM}}(\ell_1^n, \mathbf{X})}.$$

We will prove Theorem 71 in Section 2.5 by adapting to the setting of general normed spaces the strategy that was used in [76] to treat ℓ_1^n. The volumetric lower bound on

$\mathsf{SEP}(\mathbf{X})$ of Theorem 71 is typically quite easy to use and it often leads to estimates that are better than the first inequality in (1.103).

For example, by [285, Proposition 2.2] the Schatten–von Neumann trace class S_p^n satisfies

$$\forall p \geqslant 1, \quad \mathrm{evr}\big(\mathsf{S}_p^n\big) \asymp n^{\max\{\frac{1}{p}-\frac{1}{2},0\}}. \tag{1.106}$$

By substituting (1.106) into Theorem 71 we get that

$$\forall 1 \leqslant p \leqslant 2, \quad \mathsf{SEP}\big(\mathsf{S}_p^n\big) \gtrsim n^{\frac{1}{p}-\frac{1}{2}} \sqrt{\dim\big(\mathsf{S}_p^n\big)} \asymp n^{\frac{1}{p}+\frac{1}{2}}. \tag{1.107}$$

An upper bound that matches (1.107) is a consequence of the second inequality in (1.103) as follows

$$\mathsf{SEP}\big(\mathsf{S}_p^n\big) \lesssim d_{\mathrm{BM}}\big(\mathsf{S}_p^n, \ell_2^{n^2}\big)\sqrt{\dim\big(\mathsf{S}_p^n\big)} = d_{\mathrm{BM}}\big(\mathsf{S}_p^n, \mathsf{S}_2^n\big) n = n^{\frac{1}{p}+\frac{1}{2}}.$$

We therefore have

$$\forall 1 \leqslant p \leqslant 2, \quad \mathsf{SEP}\big(\mathsf{S}_p^n\big) \asymp n^{\frac{1}{p}+\frac{1}{2}}.$$

At the same time, the first inequality in (1.103) does not imply (1.107) since by a theorem of Davis (which was published only in the monograph [305]; see Theorem 41.10 there), for every $1 \leqslant p \leqslant 2$ we have

$$d_{\mathrm{BM}}\big(\ell_1^{n^2}, \mathsf{S}_p^n\big) \asymp n. \tag{1.108}$$

So, the first inequality in (1.103) only implies the weaker bound $\mathsf{SEP}(\mathsf{S}_p^n) \gtrsim n$. Of course, this rules out a "vanilla" use of (1.103) and a hereditary application of (1.103) as we did above could conceivably lead to (1.107), i.e., there could be $m \in \{1, \ldots, n\}$ such that $m/\mathsf{c}_{\mathsf{S}_p^n}(\ell_1^m)$ is at least the right-hand side of (1.107). However, this possibility seems to be unlikely, as it would mean that the following conjecture has a negative answer, which would entail finding a remarkable (and likely valuable elsewhere) subspace of S_p^n.

Conjecture 72. Fix $1 \leqslant p \leqslant 2$ and $0 < \delta \leqslant 1$. If $n, m \in \mathbb{N}$ satisfy $m \geqslant \delta n^2$, then

$$d_{\mathrm{BM}}(\ell_1^m, \mathbf{X}) \gtrsim_{p,\delta} n$$

for every m-dimensional subspace $\mathbf{X}$ of S_p^n.

Thus, (1.108) is the case $\delta = 1$ of Conjecture 72, which asserts that the same asymptotic lower bound persists if we consider subspaces of S_p^n of proportional dimension rather than S_p^n itself. Conjecture 72 is attractive in its own right, but it also implies that (1.107) does not follow from a hereditary application of the first inequality in (1.103). To see this, suppose for contradiction that there were $m \in \{1, \ldots, n\}$ such that

$$\frac{m}{\mathsf{c}_{\mathsf{S}_p^n}(\ell_1^m)} \gtrsim_p n^{\frac{1}{p}+\frac{1}{2}}. \tag{1.109}$$

By Rademacher's differentiation theorem [267] there is an m-dimensional subspace $\mathbf{X}$ of S_p^n satisfying

$$\mathsf{c}_{\mathsf{S}_p^n}(\ell_1^m) = d_{\mathrm{BM}}(\ell_1^m, \mathbf{X}) \gtrsim \frac{d_{\mathrm{BM}}(\ell_1^m, \ell_2^m)}{d_{\mathrm{BM}}(\mathsf{S}_p^n, \mathsf{S}_2^n)} = \frac{\sqrt{m}}{n^{\frac{1}{p}-\frac{1}{2}}}. \tag{1.110}$$

By contrasting (1.110) with (1.109) we deduce that necessarily $m \gtrsim_p n^2$, so an application of Conjecture 72 gives $m/\mathsf{c}_{\mathsf{S}_p^n}(\ell_1^m) \lesssim_p n$, which contradicts (1.109) since $p<2$.

Remark 73. The Löwner ellipsoid of $\ell_\infty^n(\ell_1^n)$ is $\sqrt{n}B_{\ell_2^n(\ell_2^n)}$, and $B_{\ell_\infty^n(\ell_1^n)} = (B_{\ell_1^n})^n$. Consequently,

$$\mathrm{evr}\big(\ell_\infty^n(\ell_1^n)\big)n = n\left(\frac{(\pi n)^{\frac{n^2}{2}}/\Gamma\big(\frac{n^2}{2}+1\big)}{2^{n^2}/(n!)^n}\right)^{\frac{1}{n^2}} \asymp n^{\frac{3}{2}}.$$

Therefore, Theorem 71 gives

$$\mathsf{SEP}\big(\ell_\infty^n(\ell_1^n)\big) \gtrsim n^{\frac{3}{2}}. \tag{1.111}$$

We will soon see that (1.111) is optimal, though unlike the above discussion for S_p^n when $1 \leqslant p \leqslant 2$, this does not follow from the second inequality in (1.103) because by [163],

$$d_{\mathrm{BM}}\big(\ell_2^{n^2}, \ell_\infty^n(\ell_1^n)\big) \asymp d_{\mathrm{BM}}\big(\ell_1^{n^2}, \ell_\infty^n(\ell_1^n)\big) \asymp n. \tag{1.112}$$

(1.112) also shows that (1.111) does not follow from the first inequality in (1.103). It seems that the method used in [163] to prove (1.112) is insufficient for proving that (1.111) does not follow from a hereditary application of the first inequality in (1.103). Analogously to Conjecture 72, we conjecture that this is impossible, which is a classical-sounding question about Banach–Mazur distances of independent interest.

Before passing to a description of our upper bounds on the separation modulus, we formulate the following corollary of Theorem 71 on the separation modulus of norms whose unit ball is a polytope; it restates the lower bound (1.6) and establishes its optimality.

Theorem 74. *Fix $n \in \mathbb{N}$ and a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. Suppose that $B_{\mathbf{X}}$ is a polytope that has exactly ρn vertices (note that necessarily $\rho \geqslant 2$, since $B_{\mathbf{X}}$ is origin-symmetric). Then*

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \frac{n}{\sqrt{\log \rho}}. \tag{1.113}$$

Moreover, this bound cannot be improved in general.

As an example of a consequence of Theorem 74, let

$$\mathbf{G} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{G}})$$

be a Gluskin space [111], i.e., it is a certain *random* norm on $\mathbb{R}^n$ whose unit ball has $O(n)$ vertices; see the survey [196] for extensive information about this important construction and its variants. The expected Banach–Mazur distance between two independent copies of $\mathbf{G}$ is at least cn for some universal constant $c > 0$, so the expected Banach–Mazur distance between $\mathbf{G}$ and ℓ_1^n is at least $\sqrt{cn}$. Thus, the first inequality in (1.103) only shows that $\mathsf{SEP}(\mathbf{G}) \gtrsim \sqrt{n}$ in expectation, while Theorem 74 shows that $\mathsf{SEP}(\mathbf{G}) \gtrsim n/\sqrt{\log n}$. It would be interesting to determine the growth rate of $\mathbb{E}[\mathsf{SEP}(\mathbf{G})]$. In particular, can it be that $\mathbb{E}[\mathsf{SEP}(\mathbf{G})] \gtrsim n$?

Proof of Theorem 74. By applying a linear isometry of $\mathbf{X}$ we may assume that $B_{\ell_2^n}$ is the Löwner ellipsoid of $B_{\mathbf{X}}$. Since $B_{\mathbf{X}}$ is a polytope with ρn vertices that is contained in $B_{\ell_2^n}$, we have

$$\sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})} \lesssim \frac{\sqrt{\log \rho}}{n}$$

by a result of Maurey [258] (see also [25, 28, 48, 72, 73, 112, 164] and the expository treatments in [24, 55]). Hence, $\mathrm{evr}(\mathbf{X}) \gtrsim \sqrt{n/\log \rho}$, so (1.113) follows from Theorem 71.

Consider the following (dual of an) example of Figiel and Johnson [98]. Fix $m \in \mathbb{N}$. Let $\mathbf{Z} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{Z}})$ be a normed space with $d_{\mathrm{BM}}(\ell_2^m, \mathbf{Z}) \lesssim 1$ such that $B_{\mathbf{Z}}$ is a polytope of $e^{O(m)}$ vertices; e.g., $B_{\mathbf{Z}}$ can be taken to be the convex hull of a net of S^{m-1}. For $k \in \mathbb{N}$, let $\mathbf{X} = \ell_1^k(\mathbf{Z})$. So, $\dim(\mathbf{X}) = km$ and $B_{\mathbf{X}}$ is a polytope of $2ke^{O(m)}$ vertices. Thus (1.113) becomes $\mathsf{SEP}(\mathbf{X}) \gtrsim k\sqrt{m}$. At the same time, since $d_{\mathrm{BM}}(\ell_2^m, \mathbf{Z}) \lesssim 1$ we have $d_{\mathrm{BM}}(\ell_2^{km}, \mathbf{X}) \lesssim \sqrt{k}$, so by (1.103) in fact $\mathsf{SEP}(\mathbf{X}) \lesssim \sqrt{k} \cdot \sqrt{km} = k\sqrt{m}$, i.e., (1.113) is sharp in this case. ∎

Theorem 29 follows from Theorem 66 thanks to the following randomized partitioning theorem.

Theorem 75. *For every $n \in \mathbb{N}$ and every normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$, the metric $\mathfrak{d}$ that is defined by*

$$\forall x, y \in \mathbb{R}^n, \qquad \mathfrak{d}(x, y) = \frac{4\|x - y\|_{\Pi^*\mathbf{X}}}{\mathrm{vol}_n(B_{\mathbf{X}})}$$

is a separation profile for $\mathbf{X}$.

To illustrate Theorem 75, fix $1 \leqslant p \leqslant \infty$ and apply it when $\mathbf{X}$ is the space $\mathbf{Y}_p^n$ of Theorem 24. By using Theorem 75 we see that for every $\Delta > 0$ there is a random partition $\mathcal{P}$ of $\mathbb{R}^n$ with the following properties.

(1) For every $x \in \mathbb{R}^n$ we have $\mathrm{diam}_{\ell_p^n}(\mathcal{P}(x)) \leqslant \Delta$.

(2) For every $x, y \in \mathbb{R}^n$ we have

$$\mathbf{Prob}\big[\mathcal{P}(x) \neq \mathcal{P}(y)\big] \lesssim \frac{\|x-y\|_{\Pi^*\mathbf{Y}_p^n}}{\mathrm{vol}_n(B_{\mathbf{Y}_p^n})} \overset{(1.30)\wedge(1.39)}{\lesssim} \frac{n^{\frac{1}{p}}}{\Delta}\|x-y\|_{\ell_2^n}. \quad (1.114)$$

In comparison to the $O(\sqrt{n})$-separating partition of ℓ_2^n from [76], when $p < 2$ the above random partition has smaller clusters in the sense that their diameter in the ℓ_p^n metric is at most Δ, which is more stringent than the requirement that their Euclidean diameter is at most Δ. This improved control on the size of the clusters comes at the cost that in the probabilistic separation requirement (1.114) the quantity that multiplies the Euclidean distance increases from $O(\sqrt{n})$ to $O(n^{1/p})$. When $p > 2$ this tradeoff is reversed, i.e., we get an asymptotic improvement in the separation guarantee (1.114) at the cost of requiring less from the cluster size, namely the diameter of each cluster is now guaranteed to be small in the ℓ_p^n metric rather than the more stringent requirement that it is small in the Euclidean metric.

Theorem 76 below follows from Theorem 75 the same way we deduced Theorem 21 from Theorem 29.

Theorem 76. *Fix $n \in \mathbb{N}$ and two normed spaces $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$, $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$. Every closed $\mathcal{C} \subseteq \mathbb{R}^n$ satisfies*

$$\begin{aligned}&\mathsf{SEP}(\mathcal{C}_{\mathbf{X}})\\ &\leqslant 4\Bigg(\sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}}\frac{\|x-y\|_{\mathbf{X}}}{\|x-y\|_{\mathbf{Y}}}\Bigg)\sup_{\substack{x,y\in\mathcal{C}\\ x\neq y}}\Bigg(\frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^\perp}(B_{\mathbf{Y}})\big)}{\mathrm{vol}_n(B_{\mathbf{Y}})}\cdot\frac{\|x-y\|_{\ell_2^n}}{\|x-y\|_{\mathbf{X}}}\Bigg). \quad (1.115)\end{aligned}$$

Proof of Theorem 76 assuming Theorem 75. Let M, M' be as in (1.51) and (1.52). By Theorem 75 applied to $\mathbf{Y}$, for every $\Delta > 0$ there is a random partition $\mathcal{P}$ of $\mathbb{R}^n$ that is (Δ/M)-bounded with respect to $\mathbf{Y}$, i.e.,

$$\frac{\mathrm{diam}_{\mathbf{X}}\big(\mathcal{P}(x)\big)}{M} \overset{(1.51)}{\leqslant} \mathrm{diam}_{\mathbf{Y}}\big(\mathcal{P}(x)\big) \leqslant \frac{\Delta}{M}$$

for every $x \in \mathbb{R}^n$, and also, recalling Definition 64, for every distinct $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}\frac{\Delta}{M}\mathbf{Prob}\big[\mathcal{P}(x) \neq \mathcal{P}(y)\big] &\leqslant \frac{4\|x-y\|_{\Pi^*\mathbf{Y}}}{\mathrm{vol}_n(B_{\mathbf{Y}})}\\ &\overset{(1.30)}{=} \frac{4\,\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{(x-y)^\perp}(B_{\mathbf{Y}})\big)\|x-y\|_{\ell_2^n}}{\mathrm{vol}_n(B_{\mathbf{Y}})}\\ &\overset{(1.52)}{\leqslant} 4M'\|x-y\|_{\mathbf{X}}.\end{aligned}$$

■

The special case $\mathcal{C} = \mathbb{R}^n$ of Theorem 76 coincides (with an explicitly stated constant factor) with the upper bound on $\mathsf{SEP}(\mathbf{X})$ in Theorem 3, since under the normalization $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ we have

$$\mathsf{SEP}(\mathbf{X}) \overset{(1.30)\wedge(1.115)}{\leqslant} 4\frac{\sup_{z\in\partial B_{\mathbf{X}}}\|z\|_{\Pi^*\mathbf{Y}}}{\mathrm{vol}_n(B_{\mathbf{Y}})} = 4\frac{\|\mathsf{Id}_n\|_{\mathbf{X}\to\Pi^*\mathbf{Y}}}{\mathrm{vol}_n(B_{\mathbf{Y}})} = 4\frac{\|\mathsf{Id}_n\|_{\Pi\mathbf{Y}\to\mathbf{X}^*}}{\mathrm{vol}_n(B_{\mathbf{Y}})} = 2\frac{\mathrm{diam}_{\mathbf{X}^*}(\Pi B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})}.$$

Also, Theorem 76 is stronger than the second inequality in (1.103) because by applying a linear isometry of $\mathbf{X}$ we may assume without loss of generality that $\|x\|_{\mathbf{X}} \leqslant \|x\|_{\ell_2^n} \leqslant d_{\mathrm{BM}}(\ell_2^n,\mathbf{X})\|x\|_{\mathbf{X}}$ for all $x\in\mathbb{R}^n$, in which case the special case $\mathcal{C}=\mathbb{R}^n$ and $\mathbf{Y}=\ell_2^n$ of (1.115) implies that

$$\mathsf{SEP}(\mathbf{X}) \leqslant \frac{4\,\mathrm{vol}_{n-1}\big(B_{\ell_2^{n-1}}\big)}{\mathrm{vol}_n\big(B_{\ell_2^n}\big)} d_{\mathrm{BM}}(\ell_2^n,\mathbf{X}) = \frac{4\pi^{\frac{n-1}{2}}\Gamma\big(\frac{n}{2}+1\big)}{\pi^{\frac{n}{2}}\Gamma\big(\frac{n-1}{2}+1\big)} d_{\mathrm{BM}}(\ell_2^n,\mathbf{X}) = \frac{2^{\frac{3}{2}}+o(1)}{\sqrt{\pi}} d_{\mathrm{BM}}(\ell_2^n,\mathbf{X})\sqrt{n}.$$

The right-hand side of (1.115) coincides (up to a universal constant factor) with the right-hand side of (1.28), so all of the upper bounds for the Lipschitz extension modulus that we derived in the previous sections from Theorem 21 hold for the separation modulus, by Theorem 76. For the separation modulus, we get several lower bounds from Theorem 71 that either provably match our upper bounds up to lower order factors, or match them assuming our conjectural isomorphic reverse isoperimetry. We will next spell out some of those consequences on randomized clustering of high dimensional norms.

Theorem 77. *For every $p\geqslant 1$, $n\in\mathbb{N}$ and $k,r\in\{1,\ldots,n\}$ we have*

$$\mathsf{SEP}\big((\ell_p^n)_{\leqslant k}\big) \asymp k^{\max\{\frac{1}{p},\frac{1}{2}\}} \tag{1.116}$$

and

$$r^{\max\{\frac{1}{p},\frac{1}{2}\}}\sqrt{n} \lesssim \mathsf{SEP}\big((\mathsf{S}_p^n)_{\leqslant r}\big) \lesssim r^{\max\{\frac{1}{p},\frac{1}{2}\}}\sqrt{n}\cdot\begin{cases}\sqrt{\max\{\log(\frac{n}{r}),p\}} & \text{if } p\leqslant \log r,\\ \sqrt{\log n} & \text{if } p\geqslant \log r.\end{cases} \tag{1.117}$$

Moreover, if Conjecture 49 holds for $\mathbf{X}=\mathsf{S}_p^n$, then in fact

$$\mathsf{SEP}\big((\mathsf{S}_p^n)_{\leqslant r}\big) \asymp r^{\max\{\frac{1}{p},\frac{1}{2}\}}\sqrt{n}.$$

Proof. The deduction of the upper bounds on the separation modulus that appear in (1.116) and (1.117) from Theorem 76 are identical, respectively, to the ways we deduced Theorem 20 and (1.45) from Theorem 21.

For the first inequality in (1.116), since $(\ell_p^n)_{\leqslant k}$ contains an isometric copy of ℓ_p^k, we have

$$\mathsf{SEP}\big((\ell_p^n)_{\leqslant k}\big) \geqslant \mathsf{SEP}\big(\ell_p^k\big) \gtrsim \frac{k}{d_{\mathrm{BM}}\big(\ell_p^k,\ell_1^k\big)} \overset{(1.103)}{\asymp} \frac{k}{k^{\max\{1-\frac{1}{p},\frac{1}{2}\}}} = k^{\min\{\frac{1}{p},\frac{1}{2}\}},$$

where the asymptotic evaluation of $d_{\mathrm{BM}}(\ell_p^k,\ell_q^k)$ for all $p,q \geqslant 1$ is due Gurariĭ, Kadec' and Macaev [125].

For the first inequality in (1.117), use the fact that $(\mathsf{S}_p^n)_{\leqslant r}$ contains an isometric copy of $\mathsf{S}_p^{r\times n}$, which is the Schatten–von Neumann trace class on the r-by-n real matrices $\mathsf{M}_{r\times n}(\mathbb{R})$, whose norm is given by

$$\forall A \in M_{r\times n}(\mathbb{R}), \quad \|A\|_{\mathsf{S}_p^{r\times n}} = \Big(\mathbf{Tr}\big((AA^*)^{\frac{p}{2}}\big)\Big)^{\frac{1}{p}}. \tag{1.118}$$

We then have the following rectangular version of (1.106) whose derivation is explained in Remark 171:

$$\mathrm{evr}\big(\mathsf{S}_p^{r\times n}\big) \asymp r^{\max\{\frac{1}{p}-\frac{1}{2},0\}}. \tag{1.119}$$

The desired lower bound on $\mathsf{SEP}((\mathsf{S}_p^n)_{\leqslant r})$ is now an application of Theorem 71. ∎

Remark 78. Theorem 3.3 in [76] asserts that $\mathsf{SEP}(\ell_p^n) \asymp n^{\max\{1/p,1-1/p\}}$ for every $p \geqslant 1$. Therefore, when $p > 2$ it was previously thought that $\mathsf{SEP}(\ell_p^n) \asymp n^{1-1/p}$, which contradicts the case $k = n$ of (1.116). While [76] provides a complete and correct proof that $\mathsf{SEP}(\ell_p^n) \asymp n^{1/p}$ when $1 \leqslant p \leqslant 2$, in the range $p > 2$ the assertion $\mathsf{SEP}(\ell_p^n) \asymp n^{1-1/p}$ in [76] is justified through the use of a result from reference [14] in [76], which is cited there as a “personal communication” with P. Indyk (dated April 1998). This reference was never published. After discovering Theorem 77, we confirmed with Indyk that his aforementioned personal communication with the authors of [76] contained a gap.

Corollary 79. *Conjecture 49 implies Conjecture 6. Namely, if Conjecture 49 holds for a canonically positioned normed space* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$*, then*

$$\mathsf{SEP}(\mathbf{X}) \asymp \mathrm{evr}(\mathbf{X})\sqrt{n} \asymp \mathrm{vr}(\mathbf{X}^*)\sqrt{n}. \tag{1.120}$$

In particular, if $\mathbf{X}$ *satisfies the assumptions of Lemma 53 (e.g., if* $\mathbf{X}$ *is symmetric), then (1.120) holds. Furthermore, if* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *is a symmetric normed space, then* $\mathsf{SEP}(\mathsf{S}_{\mathbf{E}}) = \mathrm{evr}(\mathbf{E})n^{1+o(1)}$*. More precisely,*

$$\mathrm{evr}(\mathbf{E})n \lesssim \mathsf{SEP}(\mathsf{S}_{\mathbf{E}}) \lesssim \mathrm{evr}(\mathbf{E})n\sqrt{\log n}.$$

Proof. The lower bound on $\mathsf{SEP}(\mathbf{X})$ in (1.120) is Theorem 71 (thus, it requires neither Conjecture 49 nor $\mathbf{X}$ being canonically positioned). The matching upper bound on

$\mathsf{SEP}(\mathbf{X})$ in (1.120) follows from Corollary 51 and the fact that by Theorem 76 the separation modulus of any (not necessarily canonically positioned) normed space

$$\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$$

is bounded from above by the right-hand side of (1.54). The rest of the assertions of Corollary 79 follow from Lemma 53 and Lemma 54. ∎

By incorporating Proposition 61 into the same reasoning as in the justification of Corollary 79, we also deduce the following stronger version of Theorem 12.

Theorem 80. *If* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ *is a canonically positioned normed space, then*

$$\mathrm{evr}(\mathbf{X})\sqrt{n} \lesssim \mathsf{SEP}(\mathbf{X}) \lesssim K(\mathbf{X})\,\mathrm{evr}(\mathbf{X})\sqrt{n} \overset{(1.87)}{\lesssim} \mathrm{evr}(\mathbf{X})\sqrt{n}\log n.$$

Section 6.3 contains volume ratio computations that show how Corollary 79 and Theorem 80 imply Corollary 4, as well as the conjectural (i.e., conditional on the validity of Conjecture 49 for the respective spaces) asymptotic evaluations (1.14) and (1.15), and several further results of this type. Most of the volume ratio computations in Section 6.3 rely on the available literature (notably Schütt's work [285]), with a few new twists that are perhaps of independent geometric/probabilisitic interest (e.g., Lemma 173).

1.7.6 Dimension reduction

Fix $n \in \mathbb{N}$ and a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$. Recall that in Definition 63 we denoted by $\mathsf{SEP}^n(\mathfrak{M}, d_{\mathfrak{M}})$ the supremum over all the separation moduli of subsets of $\mathfrak{M}$ of size at most n. In [76] it was shown that $\mathsf{SEP}^n(\ell_2) \lesssim \sqrt{\log n}$. Indeed, this follows from the Johnson–Lindenstrauss dimension reduction lemma [138], which asserts that any n-point subset of ℓ_2 can be embedded with $O(1)$ distortion into ℓ_2^m with $m \lesssim \log n$, combined with the proof in [76] that $\mathsf{SEP}(\ell_2^m) \lesssim \sqrt{m}$.

One might expect that the optimal bounds that we know for $\mathsf{SEP}(\ell_p^n)$ in the entire range $p \in (1, \infty)$ also translate to improved bounds on $\mathsf{SEP}^n(\ell_p)$. The term "improved" is used here to mean any upper bound of the form $o_p(\log n)$ as $n \to \infty$, since the benchmark general result is the aforementioned upper bound

$$\mathsf{SEP}^n(\mathfrak{M}, d_{\mathfrak{M}}) \lesssim \log n$$

from [29], which holds for any n-point metric space $(\mathfrak{M}, d_{\mathfrak{M}})$. This bound is sharp in general [29], so (because every n-point metric space embeds isometrically into ℓ_∞^n) we cannot hope to get a better bound on $\mathsf{SEP}^n(\ell_\infty)$ despite the fact that we obtained here an improved upper bound on $\mathsf{SEP}(\ell_\infty^n)$.

The obstacle is that when $p \in [1, \infty] \smallsetminus \{2\}$ no bi-Lipschitz dimension reduction result is known for finite subsets of ℓ_p, and poly-logarithmic bi-Lipschitz dimension reduction is impossible if $p \in \{1, \infty\}$; the case $p = \infty$ is due to Matoušek [200] (see also [228, 230]) and the case $p = 1$ is due to Brinkman and Charikar [58] (see also [172,232,240,270]). When $p \in [1, \infty] \setminus \{1, 2, \infty\}$ remarkably nothing is known, i.e., neither positive results nor impossibility results are available for bi-Lipschitz dimension reduction, and it is a major open problem to make any progress in this setting; see [229] for more on this area. Despite this obstacle, we have the following theorem that treats the range $p \in [1, 2]$.

Theorem 81. *For every $p \in (1, 2]$ and $n \in \mathbb{N}$ we have*

$$(\log n)^{\frac{1}{p}} \lesssim \mathsf{SEP}^n(\ell_p) \lesssim \frac{(\log n)^{\frac{1}{p}}}{p-1}.$$

The lower bound on $\mathsf{SEP}^n(\ell_p)$ of Theorem 81 can be deduced from [76]; see Section 4.2 for the details. An upper bound of $\mathsf{SEP}^n(\ell_p) \lesssim_p (\log n)^{1/p}$ was obtained when $p \in (1, 2]$ in the manuscript [170]. As [170] is not intended for publication, a proof of the upper bound on $\mathsf{SEP}^n(\ell_p)$ that is stated in Theorem 81 is included in Section 4.2, where we perform the argument with more care than the way we initially did it in [170], so as to obtain the best dependence on p that is achievable by this approach. Nevertheless, we conjecture that the dependence on p in Theorem 81 could be removed altogether, though this would likely require a substantially new idea.

Conjecture 82. The dependence on p in Theorem 81 can be improved to

$$\mathsf{SEP}^n(\ell_p) \lesssim (\log n)^{\frac{1}{p}}.$$

So, if $p \leqslant 1 + c(\log\log\log n)/\log\log n$ for some universal constant $c > 0$, then Theorem 81 does not improve asymptotically over $\mathsf{SEP}^n(\ell_p) \lesssim \log n$, while Conjecture 82 would imply that $\mathsf{SEP}^n(\ell_p) = o(\log n)$ if and only if

$$\lim_{n\to\infty} (p-1)\log\log n = \infty.$$

For fixed $p \in (2, \infty)$, at present we do not see how to obtain an upper bound on $\mathsf{SEP}^n(\ell_p)$ of the form $o_p(\log n)$ as $n \to \infty$. We state this separately as an interesting and challenging open question.

Question 83. Is it true that for every $n \in \mathbb{N}$ and $p \in (2, \infty)$ we have

$$\lim_{n\to\infty} \frac{\mathsf{SEP}^n(\ell_p)}{\log n} = 0?$$

More ambitiously, is it true that $\mathsf{SEP}^n(\ell_p) \lesssim_p \sqrt{\log n}$?

Note that $\mathsf{SEP}^n(\mathbf{X}) \gtrsim \sqrt{\log n}$ for any infinite-dimensional normed space $\mathbf{X}$, because by Dvoretzky's theorem [90] we have $\mathsf{c}_{\mathbf{X}}(\ell_2^m) = 1$ for every $m \in \mathbb{N}$, and therefore $\mathsf{SEP}^n(\mathbf{X}) \geqslant \mathsf{SEP}^n(\ell_2) \asymp \sqrt{\log n}$.

1.8 Consequences in the linear theory

Even though the purpose of the present article was to investigate the nonlinear invariants $\mathsf{e}(\cdot)$ and $\mathsf{SEP}(\cdot)$, by relating them to volumetric quantities and other linear invariants of Banach spaces (such as type and cotype), we arrive at consequences that have nothing to do with nonlinear issues. In this section, we will give a flavor of such consequences, though we will not be exhaustive since it would be more natural to pursue them separately for their own right in future work.

Denote the Minkowski functional of an origin-symmetric convex body $K \subseteq \mathbb{R}^n$ by $\|\cdot\|_K$, i.e., it is the norm on $\mathbb{R}^n$ whose unit ball is equal to K. The following theorem coincides with the second inequality in (1.1) upon a straightforward application of duality as we did in (1.31); this formulation is intended to highlight how we are bounding a convex-geometric quantity by a bi-Lipschitz invariant.

Theorem 84 (Nonsandwiching between a convex body and its polar projection body). *Fix $n \in \mathbb{N}$ and $\alpha, \beta \in (0, \infty)$. Let $K, L \subseteq \mathbb{R}^n$ be origin-symmetric convex bodies with* $\mathrm{vol}_n(L) = 1$. *Suppose that*

$$\alpha L \subseteq K \subseteq \beta \Pi^* L. \tag{1.121}$$

Then,

$$\frac{\beta}{\alpha} \gtrsim \mathsf{SEP}\big(\mathbb{R}^n, \|\cdot\|_K\big). \tag{1.122}$$

Since the separation modulus of a metric space is at least the separation modulus of any of its subsets, by combining (1.122) with the first inequality in (1.1) we see that the sandwiching hypothesis (1.121) implies the following purely volumetric consequence for every linear subspace $\mathbf{V} \subseteq \mathbb{R}^n$:

$$\frac{\beta}{\alpha} \gtrsim \mathrm{evr}\big(K \cap \mathbf{V}\big)\sqrt{n} \asymp \mathrm{vr}\big(\mathsf{Proj}_{\mathbf{V}} K^\circ\big)\sqrt{n}. \tag{1.123}$$

In particular, using $\mathrm{evr}(\ell_1^n) \asymp \sqrt{n}$, we record separately the following special case of (1.123).

Corollary 85 (Nonsandwiching of the cross-polytope). *Fix $n \in \mathbb{N}$ and $\alpha, \beta \in (0, \infty)$. If $L \subseteq \mathbb{R}^n$ is a convex body of volume 1 that satisfies $\alpha L \subseteq B_{\ell_1^n} \subseteq \beta \Pi^* L$, then necessarily $\beta/\alpha \gtrsim n$.*

The geometric meaning of Theorem 84 when $L = K$ is spelled out in the following corollary.

Corollary 86 (Every origin-symmetric convex body admits a large cone). *For every $n \in \mathbb{N}$, every origin-symmetric convex body $K \subseteq \mathbb{R}^n$ has a boundary point $z \in \partial K$ that satisfies*

$$\frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(K)\big)}{\mathrm{vol}_n(K)} \gtrsim \frac{1}{n}\mathsf{SEP}\big(\mathbb{R}^n, \|\cdot\|_K\big). \tag{1.124}$$

To see that Corollary 86 coincides with the case $L = K$ of Theorem 84, simply recall the definition of the polar projection body $\Pi^* K$ in (1.30), while also recalling that for $z \in \mathbb{R}^n \smallsetminus \{0\}$ we denote the cone whose base is $\mathsf{Proj}_{z^\perp}(K) \subseteq z^\perp$ and whose apex is z by $\mathsf{Cone}_z(K)$, and the volume of $\mathsf{Cone}_z(K)$ is given in (1.35).

A substitution of (1.104) into Corollary 86 shows that any origin-symmetric convex body $K \subseteq \mathbb{R}^n$ has a boundary point $z \in \partial K$ that satisfies

$$\frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(K)\big)}{\mathrm{vol}_n(K)} \gtrsim \frac{1}{\sqrt{n}}. \tag{1.125}$$

It seems (based on inquiring with experts in convex geometry) that the classical-looking geometric statement (1.125) did not previously appear in the literature. However, in response to our inquiry Lutwak found a different proof of (1.125) which in addition shows that the best possible constant in (1.125) is $1/\sqrt{2\pi}$. More precisely, we have the following proposition, whose proof (which relies on classical Brunn–Minkowski theory, unlike the indirect way by which we found (1.125)), is included in Section 2.7 (this proof is a restructuring of the proof that Lutwak found; we thank him for allowing us to include it here).

Proposition 87 (Lutwak). *For every $n \in \mathbb{N}$, any origin symmetric convex body $K \subseteq \mathbb{R}^n$ satisfies*

$$\max_{z \in \partial K} \frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(K)\big)}{\mathrm{vol}_n(K)} \geqslant \frac{\Gamma\left(\frac{n}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{n+1}{2}\right)} \geqslant \frac{1+\frac{1}{4n}}{\sqrt{2\pi n}}. \tag{1.126}$$

Moreover, the first inequality in (1.126) holds as equality if and only if K is an ellipsoid.

A substitution of (1.105) into Corollary 86 yields the following geometric inequality.

Corollary 88. *Fix $n \in \mathbb{N}$ and suppose that $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body. There is a boundary point $z \in \partial K$ such that the following inequality holds for every $x_1, \ldots, x_n \in K$:*

$$\frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(K)\big)}{\mathrm{vol}_n(K)} \gtrsim \frac{1}{n} \fint_{S^{n-1}} \Big\| \sum_{i=1}^n \theta_i x_i \Big\|_K^2 \,\mathrm{d}\theta. \tag{1.127}$$

By combining [303] with Lemma 102 below, the maximum of the right-hand side of (1.127) over all possible $x_1, \ldots, x_n \in K$ is bounded above and below by universal constant multiples of $T_2(\mathbb{R}^n, \|\cdot\|_K)^2/n$ (recall the definition (1.77) of the type-2 constant), so Corollary 88 is indeed a substitution of (1.105) into (1.124).

Returning to Corollary 86, recall that both the cross-polytope $B_{\ell_1^n}$ and the hypercube $[-1,1]^n$ are examples of extremal symmetric convex bodies $K \subseteq \mathbb{R}^n$ that have a boundary point $z \in \partial K$ for which the volume of $\mathsf{Cone}_z(K)$ is a universal constant proportion of the volume of K (the Euclidean ball is an example of a convex body that is not extremal in this regard). But, there is a difference between the cross-polytope and the hypercube in terms of the stability of this property. Specifically, there is an origin-symmetric convex body $K \subseteq [-1,1]^n \subseteq O(1)K$ such that for every $z \in \partial K$ the left-hand side of (1.124) is at most a universal constant multiple of $1/\sqrt{n}$. In contrast, the following proposition shows that the extremality of $\max_{z\in\partial B_{\ell_1^n}} \mathrm{vol}_n(\mathsf{Cone}_z(B_{\ell_1^n}))/\,\mathrm{vol}_n(B_{\ell_1^n})$ (up to constant factors) persists under $O(1)$ perturbations.

Proposition 89. *Fix $n \in \mathbb{N}$ and $\alpha, \beta \in (0,\infty)$. Suppose that $K \subseteq \mathbb{R}^n$ is an origin-symmetric convex body that satisfies $\alpha K \subseteq B_{\ell_1^n} \subseteq \beta K$. Then there exists a boundary point $z \in \partial K$ such that*

$$\frac{\mathrm{vol}_n\big(\mathsf{Cone}_z(K)\big)}{\mathrm{vol}_n(K)} \gtrsim \frac{\alpha}{\beta}.$$

Proposition 89 is a direct consequence of Corollary 86, the bi-Lipschitz invariance of the modulus of separated decomposability, and the lower bound $\mathsf{SEP}(\ell_1^n) \gtrsim n$ of [76].

The following proposition is an application in a different direction of the results that we described in the preceding sections.

Proposition 90. *If $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ is a finite dimensional normed space with a 1-symmetric basis, then every subspace $\mathbf{X}$ of $\mathbf{E}$ satisfies*

$$\mathrm{evr}(\mathbf{X})\sqrt{\dim(\mathbf{X})} \lesssim \mathrm{evr}(\mathbf{E})\sqrt{\dim(\mathbf{E})}. \tag{1.128}$$

Proposition 90 holds because $\mathsf{SEP}(\mathbf{E}) \lesssim \mathrm{evr}(\mathbf{E})\sqrt{\dim(\mathbf{E})}$ by Corollary 79, while

$$\mathsf{SEP}(\mathbf{X}) \gtrsim \mathrm{evr}(\mathbf{X})\sqrt{\dim(\mathbf{X})}$$

by Theorem 71, so (1.128) follows from $\mathsf{SEP}(\mathbf{X}) \leqslant \mathsf{SEP}(\mathbf{E})$. This justification shows that Proposition 90 holds for a class of spaces that is larger than those that have a 1-symmetric basis, and Conjecture 6 would imply that Proposition 90 holds when $\mathbf{E}$ is any canonically positioned normed space.

Nevertheless, Proposition 90 fails to hold true without any further assumption on the normed space $\mathbf{E}$. For example, the computation in Remark 52 shows that for any $n, m \in \mathbb{N}$ with $n \geqslant 2$ and $m \asymp n\log n$, the space $\mathbf{E} = \ell_1^n \oplus \ell_2^m$ satisfies

$$\mathrm{evr}(\mathbf{E})\sqrt{\dim(\mathbf{E})} \lesssim \sqrt{n\log n}$$

while its subspace $\mathbf{X} = \ell_1^n$ satisfies $\mathrm{evr}(\mathbf{X})\sqrt{\dim(\mathbf{X})} \asymp n$.

Proposition 90 shows that if $\mathbf{E}$ has a 1-symmetric basis, then among the linear subspaces $\mathbf{X}$ of $\mathbf{E}$ the invariant $\mathrm{evr}(\mathbf{X})\sqrt{\dim(\mathbf{X})}$ is maximized up to universal constant factors at $\mathbf{X} = \mathbf{E}$. The fact we are multiplying here the external volume ratio of $\mathbf{X}$ by the square root of its dimension is an artifact of our proof and it would be interesting to understand what correction factors allow for such a result to hold.

Question 91. Characterize (up to universal constant factors) those $A : [1, \infty) \to [1, \infty)$ with the property that for any $n \geqslant 1$ we have $\mathrm{evr}(\mathbf{X})A(k) \leqslant \mathrm{evr}(\mathbf{E})A(n)$ for every normed space $(\mathbf{E}, \|\cdot\|_{\mathbf{E}})$ of dimension at most n that has a 1-symmetric basis, every $k \in \{1, \ldots, n\}$, and every k-dimensional subspace $\mathbf{X}$ of $\mathbf{E}$.

Proposition 90 shows that if $A(n) \asymp \sqrt{n}$, then $A : [1, \infty) \to [1, \infty)$ has the properties that are described in Question 91. At the same time, no $A : [1, \infty) \to [1, \infty)$ with $A(n) = O(1)$ can be as in Question 91. Indeed, for any such A consider the symmetric normed space $\mathbf{E} = \ell_\infty^n$. There is a universal constant $\eta > 0$ such that any normed space $\mathbf{X}$ with $\dim(\mathbf{X}) \leqslant \eta \log n$ is at Banach–Mazur distance at most 2 from a subspace of ℓ_∞^n.[11] In particular, this holds for $\mathbf{X} = \ell_1^m$ when $m \in \mathbb{N}$ satisfies $m \leqslant \eta \log n$, so we get that

$$A(\eta \log n)\sqrt{\log n} \asymp \mathrm{evr}\big(\ell_1^m\big)A(\eta \log n) \leqslant 2\,\mathrm{evr}\big(\ell_\infty^n\big)A(n) \asymp A(n). \tag{1.129}$$

So, $A(n) \gtrsim \sqrt{\log n}$ and by iterating (1.129) one gets the slightly better lower bound $A(n) \gtrsim \sqrt{(\log n) \log\log n}$, as well as $A(n) \gtrsim \sqrt{(\log n)(\log\log n) \log\log\log n}$ and so forth, yielding in the end the estimate

$$A(n) \geqslant \frac{\left(\prod_{k=1}^{\log^* n} \log^{[k]} n\right)^{\frac{1}{2}}}{e^{O(\log^* n)}}, \tag{1.130}$$

where for $k \in \mathbb{N} \cup \{0\}$ we denote the kth iterant of the logarithm by $\log^{[k]}$, i.e., $\log^{[0]} x = x$ for $x > 0$, and

$$\log^{[k]} x > 0 \implies \log^{[k+1]} x = \log\big(\log^{[k]} x\big). \tag{1.131}$$

There is no reason to expect that the lower bound (1.130) is close to being optimal, but in combination with Proposition 90 it does show that the answer to Question 91 is likely nontrivial.

These considerations lead to the following open-ended question. The literature contains multiple results showing that ℓ_p^n maximizes certain geometric invariants (for examples, Banach–Mazur distance to ℓ_2^n [176], or volume ratio [22]) among all

[11]This assertion is standard, here is a quick sketch. Take a δ-net $\mathfrak{N}$ of the unit sphere of $\mathbf{X}^*$ for a sufficiently small universal constant $\delta > 0$ and consider the embedding $x \mapsto (x^*(x))_{x^* \in \mathfrak{N}}$ from $\mathbf{X}$ to $\ell_\infty(\mathfrak{N})$. Since $\log|\mathfrak{N}| \asymp \dim(\mathbf{X})$, this gives a distortion 2-embedding (say, for $\delta = 1/10$) of $\mathbf{X}$ into ℓ_∞^n provided $\log n$ is at least a sufficiently large universal constant multiple of $\dim(\mathbf{X})$.

the n-dimensional subspaces or quotients of L_p. Is there an analogous theory in the spirit of (1.128) in the much more general setting of spaces that have a 1-symmetric basis? This could be viewed as a symmetric space variant of the classical work of Lewis [176, 177]. An interesting step in this direction can be found in [304]; specifically, see [304, Theorem 1.2], which could be relevant to Question 91 through the approach of [22, Section 2].