Chapter 3

Preliminaries on random partitions

This section treats basic properties of random partitions, including measurability issues that we need for subsequent applications. As such, it is of a technical/foundational nature and it can be skipped on first reading if one is willing to accept the measurability requirements that are used in the proofs that appear in Section 4 and Section 5.

Recall that a random partition $\mathcal{P}$ of a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ was defined in the Introduction as follows. One is given a probability space $(\Omega, \mathbf{Prob})$ and a sequence of set-valued mappings $\{\Gamma^k : \Omega \to 2^{\mathfrak{M}}\}_{k=1}^{\infty}$ such that for each fixed $k \in \mathbb{N}$ the mapping $\Gamma^k : \Omega \to 2^{\mathfrak{M}}$ is strongly measurable relative to the σ-algebra of **Prob**-measurable subsets of Ω, i.e., the set $(\Gamma^k)^-(E) = \{\omega \in \Omega : E \cap \Gamma^k(\omega) \neq \varnothing\}$ is **Prob**-measurable for every closed $E \subseteq \mathfrak{M}$. We require that $\mathcal{P}^{\omega} = \{\Gamma^k(\omega)\}_{k=1}^{\infty}$ is a partition of $\mathfrak{M}$ for every $\omega \in \Omega$.

Definition 63 and Definition 65 (of separating and padded random partitions, respectively) assumed implicitly that the quantities that appear in the left-hand sides of equations (1.92) and (1.95) are well defined, i.e., that the events $\{\mathcal{P}(x) \neq \mathcal{P}(y)\}$ and $\{B_{\mathfrak{M}}(x, r) \subseteq \mathcal{P}(x)\}$ are **Prob**-measurable for every $x, y \in \mathfrak{M}$ and $r > 0$. This follows from the above definition, because for every closed subset $E \subseteq \mathfrak{M}$ we have

$$\begin{aligned}&\big\{\omega \in \Omega : \mathcal{P}^{\omega}(x) \neq \mathcal{P}^{\omega}(y)\big\}\\ &\quad= \bigcup_{\substack{k,\ell \in \mathbb{N}\\ k \neq \ell}} \Big(\big\{\omega \in \Omega : \{x\} \cap \Gamma^k(\omega) \neq \varnothing\big\} \cap \big\{\omega \in \Omega : \{y\} \cap \Gamma^{\ell}(\omega) \neq \varnothing\big\}\Big)\end{aligned}$$

and

$$\begin{aligned}&\big\{\omega \in \Omega : E \not\subseteq \mathcal{P}^{\omega}(x)\big\}\\ &\quad= \bigcup_{\substack{k,\ell \in \mathbb{N}\\ k \neq \ell}} \Big(\big\{\omega \in \Omega : \{x\} \cap \Gamma^k(\omega) \neq \varnothing\big\} \cap \big\{\omega \in \Omega : E \cap \Gamma^{\ell}(\omega) \neq \varnothing\big\}\Big).\end{aligned}$$

Another “leftover” from the Introduction is the proof of Lemma 67, which asserts that the moduli of Definition 63 and Definition 65 are bi-Lipschitz invariants. The proof of this simple but needed statement is the following direct use of the definition of a Δ-bounded random partition.

Proof of Lemma 67. Fix $D > \mathsf{c}_{(\mathfrak{N}, d_{\mathfrak{N}})}(\mathfrak{M}, d_{\mathfrak{M}})$. There is an embedding $\phi : \mathfrak{M} \to \mathfrak{N}$ and a scaling factor $\lambda > 0$ such that (1.16) holds. Fix $\Delta > 0$ and let $\mathcal{P}$ be a $\lambda\Delta$-bounded random partition of $\mathfrak{N}$. Suppose that $\mathcal{P}$ is induced by the probability space $(\Omega, \mathbf{Prob})$, i.e., there are strongly measurable mappings $\{\Gamma^k : \Omega \to 2^{\mathfrak{N}}\}_{k=1}^{\infty}$ such that $\mathcal{P}^{\omega} = \{\Gamma^k(\omega)\}_{k=1}^{\infty}$ for every $\omega \in \Omega$. For every $k \in \mathbb{N}$ the mapping $\omega \mapsto \phi^{-1}(\Gamma^k(\omega)) \in 2^{\mathfrak{M}}$ is strongly measurable. Indeed, if $E \subseteq \mathfrak{M}$ is closed then, because $\mathfrak{M}$ is complete and ϕ is a homeomorphism, also $\phi(E) \subseteq \mathfrak{N}$ is closed. So,

$$\{\omega \in \Omega : \phi(E) \cap \Gamma^k(\omega) \neq \varnothing\} = \{\omega \in \Omega : E \cap \phi^{-1}(\Gamma^k(\omega)) \neq \varnothing\}$$

is **Prob**-measurable, as required. Therefore, if we define $\mathcal{Q}^{\omega} = \{\phi^{-1}(\Gamma^k(\omega))\}_{k=1}^{\infty}$ for $\omega \in \Omega$, then $\mathcal{Q}$ is a random partition of $\mathfrak{M}$.

$\mathcal{Q}$ is Δ-bounded since for $x \in \mathfrak{M}$ and $u, v \in \mathcal{Q}(x)$ we have $\phi(u), \phi(v) \in \mathcal{P}(\phi(x))$, hence $d_{\mathfrak{M}}(u, v) \leqslant d_{\mathfrak{N}}(\phi(u), \phi(v))/\lambda \leqslant \mathrm{diam}_{\mathfrak{N}}(\mathcal{P}(\phi(x)))/\lambda \leqslant \Delta$, using (1.16) and that $\mathcal{P}$ is $\lambda\Delta$-bounded. For every $x, y \in \mathfrak{M}$ the events $\{\mathcal{Q}(x) \neq \mathcal{Q}(y)\}$ and $\{\mathcal{P}(\phi(x)) \neq \mathcal{P}(\phi(y))\}$ coincide. So, if $\mathcal{P}$ is σ-separating for some $\sigma > 0$,

$$\begin{aligned}\mathbf{Prob}\big[\mathcal{Q}(x) \neq \mathcal{Q}(y)\big] &= \mathbf{Prob}\big[\mathcal{P}\big(\phi(x)\big) \neq \mathcal{P}\big(\phi(y)\big)\big] \\ &\leqslant \frac{\sigma}{\lambda\Delta} d_{\mathfrak{N}}\big(\phi(x), \phi(y)\big) \overset{(1.16)}{\leqslant} \frac{D\sigma}{\Delta} d_{\mathfrak{M}}(x, y).\end{aligned}$$

This shows that $\mathcal{Q}$ is $(D\sigma)$-separating, thus establishing the first assertion (1.97) of Lemma 67.

Suppose that $\mathcal{P}$ is $(\mathfrak{p}, \delta)$-padded for some $\mathfrak{p} > 0$ and $0 < \delta < 1$. Fix $x \in \mathfrak{M}$. Assuming that the event $\{B_{\mathfrak{N}}(\phi(x), \lambda\Delta/\mathfrak{p}) \subseteq \mathcal{P}(\phi(x))\}$ occurs, if $z \in B_{\mathfrak{M}}(x, \Delta/(D\mathfrak{p}))$, then $d_{\mathfrak{N}}(\phi(z), \phi(x)) \leqslant \lambda D d_{\mathfrak{M}}(z, x) \leqslant \lambda\Delta/\mathfrak{p}$ by (1.16). Thus,

$$\phi(z) \in B_{\mathfrak{N}}\Big(\phi(x), \frac{\lambda\Delta}{\mathfrak{p}}\Big)$$

and therefore $\phi(z) \in \mathcal{P}(\phi(x))$, i.e., $z \in \mathcal{Q}(x)$. This shows the inclusion of events

$$\Big\{B_{\mathfrak{N}}\Big(\phi(x), \frac{\lambda\Delta}{\mathfrak{p}}\Big) \subseteq \mathcal{P}\big(\phi(x)\big)\Big\} \subseteq \Big\{B_{\mathfrak{M}}\Big(x, \frac{\Delta}{D\mathfrak{p}}\Big) \subseteq \mathcal{Q}(x)\Big\}.$$

Since $\mathcal{P}$ is $(\mathfrak{p}, \delta)$-padded, it follows from this that also $\mathcal{Q}$ is $(D\mathfrak{p}, \delta)$-padded, thus establishing the second assertion (1.98) of Lemma 67. ■

The final basic "leftover" from the Introduction is the following simple proof of Lemma 68.

Proof of Lemma 68. Fix $\Delta > 0$ and suppose that $\sigma_1 > \mathsf{SEP}(\mathfrak{M}_1)$ and $\sigma_2 > \mathsf{SEP}(\mathfrak{M}_2)$. Define

$$\Delta_1 = \Delta\Big(\frac{\sigma_1}{\sigma_1 + \sigma_2}\Big)^{\frac{1}{s}} \quad \text{and} \quad \Delta_2 = \Delta\Big(\frac{\sigma_2}{\sigma_1 + \sigma_2}\Big)^{\frac{1}{s}}. \tag{3.1}$$

Let $\mathcal{P}_{\Delta_1}$ be a σ_1-separating Δ_1-bounded random partition of $\mathfrak{M}_1$. Similarly, let $\mathcal{P}_{\Delta_2}$ be a σ_2-separating Δ_2-bounded random partition of $\mathfrak{M}_2$. Assume that $\mathcal{P}_{\Delta_1}$ and $\mathcal{P}_{\Delta_2}$

are independent random variables. Let $\mathcal{P}_\Delta$ be the corresponding product random partition of $\mathfrak{M}_1 \times \mathfrak{M}_2$, i.e., its clusters are give by

$$\forall (x_1, x_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2, \quad \mathcal{P}_\Delta(x_1, x_2) = \mathcal{P}_{\Delta_1}(x_1) \times \mathcal{P}_{\Delta_2}(x_2). \tag{3.2}$$

By (3.1) we have $\Delta_1^s + \Delta_2^s = \Delta^s$, so $\mathcal{P}_\Delta$ is a Δ-bounded random partition of the metric space $\mathfrak{M}_1 \oplus_s \mathfrak{M}_2$ (the required measurability is immediate). It therefore remains to observe that every $(x_1, x_2), (y_1, y_2) \in \mathfrak{M}_1 \times \mathfrak{M}_2$ satisfy

$$\begin{aligned}
&\mathbf{Prob}\big[\mathcal{P}_\Delta(x_1, x_2) \neq \mathcal{P}_\Delta(y_1, y_2)\big] \\
&= 1 - \mathbf{Prob}\big[\mathcal{P}_{\Delta_1}(x_1) = \mathcal{P}_{\Delta_1}(y_1)\big]\mathbf{Prob}\big[\mathcal{P}_{\Delta_2}(x_2) = \mathcal{P}_{\Delta_2}(y_2)\big] &(3.3)\\
&\leqslant 1 - \left(1 - \frac{\sigma_1 d_{\mathfrak{M}_1}(x_1, y_1)}{\Delta_1}\right)\left(1 - \frac{\sigma_2 d_{\mathfrak{M}_2}(x_2, y_2)}{\Delta_2}\right) &(3.4)\\
&= \frac{\sigma_1 d_{\mathfrak{M}_1}(x_1, y_1)}{\Delta_1} + \frac{\sigma_2 d_{\mathfrak{M}_2}(x_2, y_2)}{\Delta_2} - \frac{\sigma_1\sigma_2 d_{\mathfrak{M}_1}(x_1, y_1) d_{\mathfrak{M}_2}(x_2, y_2)}{\Delta_1\Delta_2} &(3.5)\\
&\leqslant \left(\left(\frac{\sigma_1}{\Delta_1}\right)^{\frac{s}{s-1}} + \left(\frac{\sigma_2}{\Delta_2}\right)^{\frac{s}{s-1}}\right)^{\frac{s-1}{s}} \left(d_{\mathfrak{M}_1}(x_1, y_1)^s + d_{\mathfrak{M}_2}(x_2, y_2)^s\right)^{\frac{1}{s}} &(3.6)\\
&= \frac{\sigma_1 + \sigma_2}{\Delta} d_{\mathfrak{M}_1 \oplus_s \mathfrak{M}_2}\big((x_1, x_2), (y_1, y_2)\big), &(3.7)
\end{aligned}$$

where (3.3) uses (3.2) and the independence of $\mathcal{P}_{\Delta_1}$ and $\mathcal{P}_{\Delta_2}$, the bound (3.4) is an application of the assumption that $\mathcal{P}_{\Delta_1}$ is σ_1-separating and $\mathcal{P}_{\Delta_2}$ is σ_2-separating, (3.6) is an application of Hölder's inequality, and (3.7) follows from (1.99) and (3.1). This proves (1.100). Note that even though we dropped the quadratic additive improvement in (3.5), this does not change the final bound in (1.100) due to the need to work with all possible scales $\Delta > 0$ and all possible values of $d_{\mathfrak{M}_1}(x_1, y_1)$ and $d_{\mathfrak{M}_2}(x_2, y_2)$.

To prove (1.101), fix $\mathfrak{p}_1 > \mathsf{PAD}_{\delta_1}(\mathfrak{M}_1)$ and $\mathfrak{p}_2 > \mathsf{PAD}_{\delta_2}(\mathfrak{M}_2)$ and replace (3.1) by

$$\Delta_1 = \frac{\Delta \mathfrak{p}_1}{\left(\mathfrak{p}_1^s + \mathfrak{p}_2^s\right)^{\frac{1}{s}}} \quad \text{and} \quad \Delta_2 = \frac{\Delta \mathfrak{p}_2}{\left(\mathfrak{p}_1^s + \mathfrak{p}_2^s\right)^{\frac{1}{s}}}.$$

This time, we choose $\mathcal{P}_{\Delta_1}$ to be a $(\mathfrak{p}_1, \delta_1)$-padded Δ_1-bounded random partition of $\mathfrak{M}_1$. Similarly, let $\mathcal{P}_{\Delta_2}$ be a $(\mathfrak{p}_2, \delta_2)$-padded Δ_2-bounded random partition of $\mathfrak{M}_2$, with $\mathcal{P}_{\Delta_1}$ and $\mathcal{P}_{\Delta_2}$ independent, and we again combine them as in (3.2) to give the product partition $\mathcal{P}_\Delta$ of $\mathfrak{M}_1 \times \mathfrak{M}_2$. By reasoning analogously, $\mathcal{P}_\Delta$ is a $((\mathfrak{p}_1^s + \mathfrak{p}_2^s)^{1/s}, \delta_1\delta_2)$-padded Δ-bounded random partition of $\mathfrak{M}_1 \oplus_s \mathfrak{M}_2$. ∎

3.1 Standard set-valued mappings

Recall that a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ is said to be Polish if it is separable and complete. Polish metric spaces are the appropriate setting for Lipschitz extension theorems that are based on the assumption that for every $\Delta > 0$ there is a probability distribution over Δ-bounded partitions of $\mathfrak{M}$ with certain properties. Indeed, a Banach space-valued Lipschitz function can always be extended to the completion of $\mathfrak{M}$ while preserving the Lipschitz constant, and the mere existence of countably many sets of diameter at most Δ that cover $\mathfrak{M}$ for every $\Delta > 0$ implies that $\mathfrak{M}$ is separable.

Theorem 66 assumes local compactness. Even though this assumption is more restrictive than being Polish, it suffices for the applications that we obtain herein because they deal with finite dimensional normed spaces. It is, however, possible to treat general Polish metric spaces by working with a notion of measurability of set-valued mappings that differs from the strong measurability that was assumed in Section 1.7. We call this notion *standard set-valued mappings*; see Definition 113.

The requirements for a set-valued mapping to be standard are quite innocuous and easy to check. In particular, the clusters of the specific random partitions that we will study are easily seen to be standard set-valued mappings. It is also simple to verify that the clusters of the random partitions that we construct are strongly measurable. So, we have two approaches, which are both easy to work with. We chose to work in the Introduction with the requirement that the clusters are strongly measurable because this directly makes the quantity $\mathsf{SEP}(\cdot)$ be bi-Lipschitz invariant, and it is also slightly simpler to describe. Nevertheless, in practice it is straightforward to check that the clusters are standard, and even though we do not know that this leads to a bi-Lipschitz invariant (we suspect that it *does not*), it does lead to an easily implementable Lipschitz extension criterion that holds in the maximal generality of Polish spaces.

Definition 113 (Standard set-valued mapping). Suppose that $(\mathcal{Z}, d_{\mathcal{Z}})$ is a Polish metric space and that $\Omega \subseteq \mathcal{Z}$ is a Borel subset of $\mathcal{Z}$. Given a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, a set-valued mapping $\Gamma : \Omega \to 2^{\mathfrak{M}}$ is said to be *standard* if the following three conditions hold.

- For every $x \in \mathfrak{M}$ the set $\{\omega \in \Omega : x \in \Gamma(\omega)\}$ is Borel.
- The set $\mathcal{G}_\Gamma = \Gamma^-(\mathfrak{M}) = \{\omega \in \Omega : \Gamma(\omega) \neq \varnothing\}$ is Borel.
- For every $x \in \mathfrak{M}$ the mapping $(\omega \in \mathcal{G}_\Gamma) \mapsto d_{\mathfrak{M}}(x, \Gamma(\omega))$ is Borel measurable on $\mathcal{G}_\Gamma$.

The following extension criterion is a counterpart to Theorem 66 that works in the maximal generality of Polish metric spaces; its proof, which is an adaptation of ideas of [173], appears in Section 5.

Theorem 114. *Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a Polish metric space and fix another metric $\mathfrak{d}$ on $\mathfrak{M}$. Suppose that for every $\Delta > 0$ there is a Polish metric space $\mathcal{Z}_\Delta$, a Borel subset $\Omega_\Delta \subseteq \mathcal{Z}_\Delta$, a Borel probability measure* $\mathbf{Prob}_\Delta$ *on Ω_Δ and a sequence of standard set-valued mappings $\{\Gamma^k_\Delta : \Omega_\Delta \to 2^{\mathfrak{M}}\}_{k=1}^\infty$ such that $\mathcal{P}^\omega_\Delta = \{\Gamma^k_\Delta(\omega)\}_{k=1}^\infty$ is a partition* *of* $\mathfrak{M}$ *for every* $\omega \in \Omega_\Delta$*, for every* $x \in \mathfrak{M}$ *and* $\omega \in \Omega_\Delta$ *we have* $\mathrm{diam}_{\mathfrak{M}}(\mathcal{P}_\Delta^\omega(x)) \leqslant \Delta$*, and*

$$\forall x, y \in \mathfrak{M}, \quad \Delta \mathbf{Prob}_\Delta\big[\omega \in \Omega_\Delta : \mathcal{P}_\Delta^\omega(x) \neq \mathcal{P}_\Delta^\omega(y)\big] \leqslant \mathfrak{d}(x, y).$$

Then, for every Banach space $(\mathbf{Z}, \|\cdot\|_{\mathbf{Z}})$*, every subset* $\mathcal{C} \subseteq \mathfrak{M}$ *and every* 1*-Lipschitz mapping* $f : \mathcal{C} \to \mathbf{Z}$*, there exists a mapping* $F : \mathfrak{M} \to \mathbf{Z}$ *that extends* f *and satisfies* $\|F(x) - F(y)\|_{\mathbf{Z}} \lesssim \mathfrak{d}(x, y)$ *for every* $x, y \in \mathfrak{M}$ *(namely,* F *is Lipschitz on* $\mathfrak{M}$ *with respect to the metric* $\mathfrak{d}$*). Moreover,* F *depends linearly on* f*.*

3.2 Proximal selectors

For later applications we need to know that set-valued mappings that are either strongly measurable or standard admit certain auxiliary measurable mappings that are (perhaps approximately) the closest point to a given (but arbitrary) nonempty closed subset of the metric space in question. We will justify this now using classical descriptive set theory.

Lemma 115. *Fix a measurable space* $(\Omega, \mathcal{F})$*. Suppose that* $(\mathfrak{M}, d_{\mathfrak{M}})$ *is a metric space and that* $S \subseteq \mathfrak{M}$ *is nonempty and locally compact. Let* $\Gamma : \Omega \to 2^{\mathfrak{M}}$ *be a strongly measurable set-valued mapping such that* $\Gamma(\omega)$ *is a bounded subset of* $\mathfrak{M}$ *for every* $\omega \in \Omega$*. Then there exists an* $\mathcal{F}$*-to-Borel measurable mapping* $\gamma : \Omega \to S$ *that satisfies* $d_{\mathfrak{M}}(\gamma(\omega), \Gamma(\omega)) = d_{\mathfrak{M}}(S, \Gamma(\omega))$ *for every* $\omega \in \Omega$ *for which* $\Gamma(\omega) \neq \varnothing$*.*

Proof. For every $\omega \in \Omega$ define a subset $\Phi(\omega) \subseteq S$ as follows:

$$\Phi(\omega) \stackrel{\mathrm{def}}{=} \begin{cases} \{s \in S : d_{\mathfrak{M}}(s, \Gamma(\omega)) = d_{\mathfrak{M}}(S, \Gamma(\omega))\} & \text{if } \Gamma(\omega) \neq \varnothing, \\ S & \text{if } \Gamma(\omega) = \varnothing. \end{cases}$$

The goal of Lemma 115 is to show the existence of an $\mathcal{F}$-to-Borel measurable mapping $\gamma : \Omega \to S$ that satisfies $\gamma(\omega) \in \Phi(\omega)$ for every $\omega \in \Omega$. Since $(S, d_{\mathfrak{M}})$ is locally compact, it is in particular Polish, so by the measurable selection theorem of Kuratowski and Ryll-Nardzewski [161] (see also [309] or [291, Chapter 5.2]) it suffices to check that $\Phi(\omega)$ is nonempty and closed for every $\omega \in \Omega$, and that we have $\{\omega \in \Omega : E \cap \Phi(\omega) = \varnothing\} \in \mathcal{F}$ for every closed $E \subseteq S$. Since S is locally compact, every closed subset of S is a countable union of compact subsets, so it suffices to check the latter requirement for compact subsets of S, i.e., to show that $\{\omega \in \Omega : K \cap \Phi(\omega) = \varnothing\} \in \mathcal{F}$ for every compact $K \subseteq S$.

Fix $\omega \in \Omega$. If $\Gamma(\omega) = \varnothing$ then $\Phi(\omega) = S$ is closed (since S is locally compact) and nonempty by assumption. If $\Gamma(\omega) \neq \varnothing$ then the continuity of $s \mapsto d_{\mathfrak{M}}(s, \Gamma(\omega))$ on S implies that $\Phi(\omega)$ is closed. Moreover, in this case since $\Gamma(\omega)$ is bounded and S is locally compact, the continuous mapping $s \mapsto d_{\mathfrak{M}}(s, \Gamma(\omega))$ attains its minimum on S, so that $\Phi(\omega) \neq \varnothing$.

It therefore remains to check that $\{\omega\in\Omega : K\cap\Phi(\omega)=\varnothing\}\in\mathcal{F}$ for every nonempty compact $K\subsetneq S$. Fixing such a K, since S is locally compact and hence separable, there exist $\{\kappa_i\}_{i=1}^{\infty}\subseteq K$ and $\{\sigma_j\}_{j=1}^{\infty}\subseteq S$ that are dense in K and S, respectively. Denote $\mathcal{G}_\Gamma=\{\omega\in\Omega:\Gamma(\omega)\neq\varnothing\}$. Then $\mathcal{G}_\Gamma\in\mathcal{F}$, because Γ is strongly measurable. Observe that the following identity holds:

$$\begin{aligned}&\big\{\omega\in\Omega : K\cap\Phi(\omega)=\varnothing\big\}\\ &\quad=\big\{\omega\in\mathcal{G}_\Gamma : \forall\kappa\in K,\ d_{\mathfrak{M}}(\kappa,\Gamma(\omega))>d_{\mathfrak{M}}(S,\Gamma(\omega))\big\}\\ &\quad=\bigcup_{m=1}^{\infty}\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{\infty}\left\{\omega\in\mathcal{G}_\Gamma : d_{\mathfrak{M}}(\kappa_i,\Gamma(\omega))>d_{\mathfrak{M}}(\sigma_j,\Gamma(\omega))+\frac{1}{m}\right\}. \end{aligned}\tag{3.8}$$

The verification of (3.8) proceeds as follows. Since $\Phi(\omega)\neq\varnothing$ for every $\omega\in\Omega$ and $K\neq\varnothing$, if $K\cap\Phi(\omega)=\varnothing$ then $\omega\in\mathcal{G}_\Gamma$ (otherwise $\Phi(\omega)=S$). This explains the first equality (3.8). For the second equality in (3.8), note that since $\Gamma(\omega)$ is bounded and K is compact, $\inf_{\kappa\in K} d_{\mathfrak{M}}(\kappa,\Gamma(\omega))$ is attained. Therefore, the second set in (3.8) is equal to $A=\{\omega\in\mathcal{G}_\Gamma : d_{\mathfrak{M}}(K,\Gamma(\omega))>d_{\mathfrak{M}}(S,\Gamma(\omega))\}$. If $\omega\in A$, then there is $m\in\mathbb{N}$ such that $d_{\mathfrak{M}}(K,\Gamma(\omega))>d_{\mathfrak{M}}(S,\Gamma(\omega))+2/m$, implying in particular that $d_{\mathfrak{M}}(\kappa_i,\Gamma(\omega))>d_{\mathfrak{M}}(S,\Gamma(\omega))+2/m$ for every $i\in\mathbb{N}$. As $\{\sigma_j\}_{j=1}^{\infty}$ is dense in S, for every $i\in\mathbb{N}$ there is $j\in\mathbb{N}$ such that $d_{\mathfrak{M}}(\kappa_i,\Gamma(\omega))>d_{\mathfrak{M}}(\sigma_j,\Gamma(\omega))+1/m$. Hence, the second set in (3.8) is contained in the third set in (3.8). For the reverse inclusion, if ω is in third set in (3.8) then

$$d_{\mathfrak{M}}(K,\Gamma(\omega))=\inf_{i\in\mathbb{N}} d_{\mathfrak{M}}(\kappa_i,\Gamma(\omega))>\inf_{j\in\mathbb{N}} d_{\mathfrak{M}}(\sigma_j,\Gamma(\omega))=d_{\mathfrak{M}}(S,\Gamma(\omega)).$$

By (3.8), it suffices to show that $\{\omega\in\mathcal{G}_\Gamma : d_{\mathfrak{M}}(x,\Gamma(\omega))>d_{\mathfrak{M}}(y,\Gamma(\omega))+r\}\in\mathcal{F}$ for every fixed $x,y\in S$ and $r>0$. For this, it suffices to show that for every $z\in\mathfrak{M}$ the mapping $\omega\mapsto d_{\mathfrak{M}}(z,\Gamma(\omega))$ is $\mathcal{F}$-to-Borel measurable on $\mathcal{G}_\Gamma$. Since $\mathcal{G}_\Gamma\in\mathcal{F}$, this is a consequence of the strong measurability of Γ, because for every $t\geqslant 0$ we have

$$\begin{aligned}&\{\omega\in\mathcal{G}_\Gamma : d_{\mathfrak{M}}(z,\Gamma(\omega))>t\}\\ &\quad=\bigcup_{k=1}^{\infty}\mathcal{G}_\Gamma\cap\left\{\omega\in\Omega : B_{\mathfrak{M}}\left(z,t+\frac{1}{k}\right)\cap\Gamma(\omega)=\varnothing\right\}.\end{aligned}$$ ■

Lemma 115 is a satisfactory treatment of measurable nearest point selectors for strongly measurable set-valued mappings, though under an assumption of local compactness. We did not investigate the minimal assumptions that are required for the conclusion of Lemma 115 to hold. We will next treat the setting of standard set-valued mappings without assuming local compactness.

Let $(\mathcal{Z},d_{\mathcal{Z}})$ be a Polish metric space. Recall that a subset A of $\mathcal{Z}$ is said to be *universally measurable* if it is measurable with respect to *every* complete σ-finite Borel measure μ on $\mathcal{Z}$ (see, e.g., [154, p. 155]). If $(\mathfrak{M},d_{\mathfrak{M}})$ is another metric space and

$\Omega \subseteq \mathcal{Z}$ is Borel, then a mapping $\psi : \Omega \to \mathfrak{M}$ is said to be universally measurable if $\psi^{-1}(E)$ is a universally measurable subset of Ω for every Borel subset E of $\mathfrak{M}$. Finally, recall that $A \subseteq \mathfrak{M}$ is said to be *analytic* if it is an image under a continuous mapping of a Borel subset of a Polish metric space (see, e.g., [154, Chapter 14] or [136, Chapter 11]). By Lusin's theorem [189, 192] (see also, e.g., [154, Theorem 21.10]), analytic subsets of Polish metric spaces are universally measurable.

Lemma 116. *Let $(\mathfrak{M}, d_{\mathfrak{M}})$ and $(\mathcal{Z}, d_{\mathcal{Z}})$ be Polish metric spaces and fix a Borel subset $\Omega \subseteq \mathcal{Z}$. Fix also $\Delta > 0$ such that* $\operatorname{diam}(\mathfrak{M}) \geqslant \Delta$. *Suppose that $\Gamma : \Omega \to 2^{\mathfrak{M}}$ satisfies the following two properties.*

(1) *For every $\omega \in \Omega$ such that $\Gamma(\omega) \neq \varnothing$ we have* $\operatorname{diam}_{\mathfrak{M}}(\Gamma(\omega)) < \Delta$.

(2) *For every $x \in \mathfrak{M}$ and $t \in \mathbb{R}$ the set $\{\omega \in \Omega : \Gamma(\omega) \neq \varnothing \wedge d_{\mathfrak{M}}(x, \Gamma(\omega)) > t\}$ is analytic.*

Then, for every closed subset $\varnothing \neq S \subseteq \mathfrak{M}$ there exists a universally measurable mapping $\gamma : \Omega \to S$ such that

$$\forall (\omega, x) \in \Omega \times \mathfrak{M}, \quad x \in \Gamma(\omega) \implies d_{\mathfrak{M}}(x, \gamma(\omega)) \leqslant d_{\mathfrak{M}}(x, S) + \Delta.$$

Proof. For every $\omega \in \Omega$, define a subset $\Psi(\omega) \subseteq S$ as follows:

$$\Psi(\omega) \stackrel{\mathrm{def}}{=} \begin{cases} \bigcap_{x\in\mathfrak{M}} \{s \in S : d_{\mathfrak{M}}(x, s) \leqslant 2 d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x, S) + \Delta\} & \text{if } \Gamma(\omega) \neq \varnothing, \\ S & \text{if } \Gamma(\omega) = \varnothing. \end{cases} \tag{3.9}$$

We will show that there exists a universally measurable mapping $\gamma : \Omega \to S$ such that $\gamma(\omega) \in \Psi(\omega)$ for every $\omega \in \Omega$. Since S is a closed subset of $\mathfrak{M}$, it is Polish. Hence, by the Kuratowski–Ryll-Nardzewski measurable selection theorem [161], it suffices to prove that $\Psi(\omega)$ is nonempty and closed for every $\omega \in \Omega$, and that $\Psi^{-}(E) = \{\omega \in \Omega : E \cap \Psi(\omega) \neq \varnothing\}$ is universally measurable for every closed $E \subseteq S$.

By design, $\Psi(\omega) = S$ is nonempty and closed if $\Gamma(\omega) = \varnothing$. So, fix $\omega \in \Omega$ such that $\Gamma(\omega) \neq \varnothing$. Then $\Psi(\omega)$ is closed because if $\{s_k\}_{k=1}^{\infty} \subseteq \Psi(\omega)$ and $s \in \mathfrak{M}$ satisfy $\lim_{k\to\infty} d_{\mathfrak{M}}(s_k, s) = 0$, then for every $k \in \mathbb{N}$ and $x \in \mathfrak{M}$, since $s_k \in \Psi(\omega)$ we have $d_{\mathfrak{M}}(s_k, x) \leqslant 2 d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x, S) + \Delta$. Hence, by continuity also

$$d_{\mathfrak{M}}(s, x) \leqslant 2 d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x, S) + \Delta$$

for every $x \in \mathfrak{M}$, i.e., $s \in \Psi(\omega)$.

We will next check that $\Psi(\omega) \neq \varnothing$ for every $\omega \in \Omega$ such that $\Gamma(\omega) \neq \varnothing$. Denote $\varepsilon_\omega = \Delta - \operatorname{diam}_{\mathfrak{M}}(\Gamma(\omega))$. By assumption (1) of Lemma 116 we have $\varepsilon_\omega > 0$, so we may choose $s_\omega \in S$ and $y_\omega \in \Gamma(\omega)$ that satisfy $d_{\mathfrak{M}}(y_\omega, s_\omega) \leqslant d_{\mathfrak{M}}(\Gamma(\omega), S) + \varepsilon_\omega$.

We claim that $s_\omega \in \Psi(\omega)$. Indeed, for every $x \in \mathfrak{M}$ and $z \in \Gamma(\omega)$ we have

$$\begin{aligned} d_{\mathfrak{M}}(x, s_\omega) &\leqslant d_{\mathfrak{M}}(x,z) + d_{\mathfrak{M}}(z, y_\omega) + d_{\mathfrak{M}}(y_\omega, s_\omega) \\ &\leqslant d_{\mathfrak{M}}(x,z) + \operatorname{diam}_{\mathfrak{M}}(\Gamma(\omega)) + d_{\mathfrak{M}}(\Gamma(\omega), S) + \varepsilon_\omega \\ &\leqslant d_{\mathfrak{M}}(x,z) + d_{\mathfrak{M}}(z,S) + \Delta \\ &\leqslant d_{\mathfrak{M}}(x,z) + d_{\mathfrak{M}}(x,S) + d_{\mathfrak{M}}(x,z) + \Delta, \end{aligned} \tag{3.10}$$

where in the penultimate step of (3.10) we used the fact that $d_{\mathfrak{M}}(\Gamma(\omega), S) \leqslant d_{\mathfrak{M}}(z, S)$, since $z \in \Gamma(\omega)$, and in the final step of (3.10) we used the fact that $p \mapsto d_{\mathfrak{M}}(p, S)$ is 1-Lipschitz on $\mathfrak{M}$. Since (3.10) holds for every $z \in \Gamma(\omega)$, it follows that

$$d_{\mathfrak{M}}(x, s_\omega) \leqslant 2 d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x, S) + \Delta.$$

Because this holds for every $x \in \mathfrak{M}$, it follows that $s_\omega \in \Psi(\omega)$.

Having checked that Ψ takes values in closed and nonempty subsets of S, it remains to show that $\Psi^-(E)$ is universally measurable for every closed $E \subseteq S$. To this end, since $\mathfrak{M}$ is separable, we may fix from now on a sequence $\{x_j\}_{j=1}^\infty$ that is dense in $\mathfrak{M}$. Note that by the case $t = 0$ of assumption (2) of Lemma 116, for every $j \in \mathbb{N}$ the following set is analytic:

$$\{\omega \in \Omega : \Gamma(\omega) \neq \emptyset \wedge d_{\mathfrak{M}}(x_j, \Gamma(\omega)) > 0\} = \big\{\omega \in \Omega : \Gamma(\omega) \neq \emptyset \wedge x_j \notin \overline{\Gamma(\omega)}\big\}.$$

Countable unions and intersections of analytic sets are analytic (see, e.g., [154, Proposition 14.4]), so we deduce that the following set is analytic:

$$\begin{aligned} &\bigcup_{j=1}^{\infty} \big\{\omega \in \Omega : \Gamma(\omega) \neq \emptyset \wedge x_j \notin \overline{\Gamma(\omega)}\big\} \\ &\quad = \big\{\omega \in \Omega : \Gamma(\omega) \neq \emptyset \wedge \{x_j\}_{j=1}^\infty \not\subseteq \overline{\Gamma(\omega)}\big\} = \big\{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\big\}, \end{aligned} \tag{3.11}$$

where for the final step of (3.11) observe that, since $\{x_j\}_{j=1}^\infty$ is dense in $\mathfrak{M}$, if $\{x_j\}_{j=1}^\infty$ were a subset of $\overline{\Gamma(\omega)}$ then it would follow that $\Gamma(\omega)$ is dense in $\mathfrak{M}$. This would imply that $\operatorname{diam}_{\mathfrak{M}}(\Gamma(\omega)) = \operatorname{diam}(\mathfrak{M}) \geqslant \Delta$, in contradiction to assumption (1) of Lemma 116. We have thus checked that the set $\mathcal{G}_\Gamma = \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\}$ is analytic, and hence by Lusin's theorem [189, 192] it is universally measurable. Now,

$$\begin{aligned} \Psi^-(E) &\overset{(3.9)}{=} (\Omega \smallsetminus \mathcal{G}_\Gamma) \\ &\quad \cup \{\omega \in \mathcal{G}_\Gamma : \exists s \in E\ \forall x \in \mathfrak{M},\ d_{\mathfrak{M}}(x,s) \leqslant 2d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x,S) + \Delta\}. \end{aligned}$$

Hence, it remains to prove that the following set is universally measurable:

$$\begin{aligned} &\big\{\omega \in \mathcal{G}_\Gamma : \exists s \in E\ \forall x \in \mathfrak{M},\ d_{\mathfrak{M}}(x,s) \leqslant 2d_{\mathfrak{M}}(x, \Gamma(\omega)) + d_{\mathfrak{M}}(x,S) + \Delta\big\} \\ &\quad = \big\{\omega \in \mathcal{G}_\Gamma : \exists s \in E\ \forall j \in \mathbb{N},\ d_{\mathfrak{M}}(x_j,s) \leqslant 2d_{\mathfrak{M}}(x_j, \Gamma(\omega)) + d_{\mathfrak{M}}(x_j,S) + \Delta\big\}, \end{aligned} \tag{3.12}$$

where we used the fact that $\{x_j\}_{j=1}^\infty$ is dense in $\mathfrak{M}$.

Consider the following subset $\mathcal{C}$ of $\Omega \times E$:

$$\mathcal{C} \stackrel{\mathrm{def}}{=} \big\{(\omega, s) \in \mathcal{G}_\Gamma \times E : \forall j \in \mathbb{N},\ d_{\mathfrak{M}}(x_j, s) \leqslant 2d_{\mathfrak{M}}(x_j, \Gamma(\omega)) + d_{\mathfrak{M}}(x_j, S) + \Delta\big\}.$$

The set in (3.12) is $\pi_1(\mathcal{C})$, where $\pi_1 : \Omega \times E \to \Omega$ is the projection to the first coordinate, i.e., $\pi_1(\omega, s) = \omega$ for every $(\omega, s) \in \Omega \times E$. Since continuous images and preimages of analytic sets are analytic (see, e.g., [154, Proposition 14.4]), by another application of Lusin's theorem it suffices to show that $\mathcal{C}$ is analytic. We already proved that $\mathcal{G}_\Gamma \subseteq \Omega$ is analytic, so there is a Borel subset L of a Polish space $\mathcal{Y}$ and a continuous mapping $\phi : L \to \Omega$ such that $\phi(L) = \mathcal{G}_\Gamma$. Denoting the identity mapping on E by $\mathsf{Id}_E : E \to E$, since ϕ maps L onto $\mathcal{G}_\Gamma$, the set $\mathcal{C}$ is the image under the continuous mapping $\phi \times \mathsf{Id}_E$ of the following subset of $\mathcal{Y} \times E$:

$$\begin{aligned}&\big\{(y, s) \in L \times E : \forall j \in \mathbb{N},\ d_{\mathfrak{M}}(x_j, s) \leqslant 2d_{\mathfrak{M}}\big(x_j, \Gamma(\phi(y))\big) + d_{\mathfrak{M}}(x_j, S) + \Delta\big\}\\ &= \bigcap_{j=1}^{\infty} \big\{(y, s) \in L \times E : d_{\mathfrak{M}}(x_j, s) \leqslant 2d_{\mathfrak{M}}\big(x_j, \Gamma(\phi(y))\big) + d_{\mathfrak{M}}(x_j, S) + \Delta\big\}.\end{aligned}$$

Hence, since continuous images and countable intersections of analytic sets are analytic, by yet another application of Lusin's theorem we see that it suffices to show that for every fixed $x \in \mathfrak{M}$ the following set is analytic, where for every $q \in \mathbb{Q}$ we denote $A_q = \{(y, s) \in L \times E : q < d_{\mathfrak{M}}(x, s)\} = L \times \{s \in E : q < d_{\mathfrak{M}}(x, s)\}$:

$$\begin{aligned}&\big\{(y, s) \in L \times E : d_{\mathfrak{M}}(x, s) \leqslant 2d_{\mathfrak{M}}\big(x, \Gamma(\phi(y))\big) + d_{\mathfrak{M}}(x, S) + \Delta\big\}\\ &= \bigcap_{q \in \mathbb{Q}} \Big(\big((L \times E) \smallsetminus A_q\big)\\ &\qquad \cup \Big(A_q \cap \big\{(y, s) \in L \times E : 2d_{\mathfrak{M}}\big(x, \Gamma(\phi(y))\big) > q - d_{\mathfrak{M}}(x, S) - \Delta\big\}\Big)\Big),\end{aligned}$$

Since A_q is Borel for all $q \in \mathbb{Q}$, it suffices to show that the following set is analytic for every $t \in \mathbb{R}$:

$$\big\{(y, s) \in L \times E : d_{\mathfrak{M}}\big(x, \Gamma(\phi(y))\big) > t\big\} = \phi^{-1}\big(\big\{\omega \in \mathcal{G}_\Gamma : d_{\mathfrak{M}}(x, \Gamma(\omega)) > t\big\}\big) \times E.$$

Since a preimage under a continuous mapping of an analytic set is analytic, the above set is indeed analytic due to assumption (2) of Lemma 116 and the fact that E is closed. ∎

Remark 117. The proof of Lemma 116 used the assumption $\mathrm{diam}(\mathfrak{M}) \geqslant \Delta$ only to deduce that the set

$$\mathcal{G}_\Gamma = \{\omega \in \Omega : \Gamma(\omega) \neq \emptyset\}$$

is analytic from (the case $t = 0$ of) assumption (2) of Lemma 116. Hence, if we add the assumption that $\mathcal{G}_\Gamma$ is analytic to Lemma 116, then we can drop the restriction

$\operatorname{diam}(\mathfrak{M}) \geqslant \Delta$ altogether. Alternatively, recalling equation (3.11) and the paragraph immediately after it, for the above proof of Lemma 116 to go through it suffices to assume that $\Gamma(\omega)$ is not dense in $\mathfrak{M}$ for any $\omega \in \Omega$.

Recalling Definition 113, Lemma 116 and Remark 117 imply the following corollary. Indeed, by Remark 117 we know that we can drop the assumption $\operatorname{diam}(\mathfrak{M}) \geqslant \Delta$ of Lemma 116, and when Γ is a standard set-valued mapping the sets that appears in assumption (2) of Lemma 116 are Borel.

Corollary 118. *Fix $\Delta > 0$. Let $(\mathfrak{M}, d_{\mathfrak{M}})$ and $(\mathcal{Z}, d_{\mathcal{Z}})$ be Polish metric spaces and fix a Borel subset $\Omega \subseteq \mathcal{Z}$. Suppose that $\Gamma : \Omega \to 2^{\mathfrak{M}}$ is a standard set-valued mapping such that $\operatorname{diam}_{\mathfrak{M}}(\Gamma(\omega)) < \Delta$ for every $\omega \in \mathcal{G}_\Gamma$. Then for every closed $\varnothing \neq S \subseteq \mathfrak{M}$ there exists a universally measurable mapping $\gamma : \Omega \to S$ that satisfies*

$$\forall (\omega, x) \in \Omega \times \mathfrak{M}, \quad x \in \Gamma(\omega) \implies d_{\mathfrak{M}}(x, \gamma(\omega)) \leqslant d_{\mathfrak{M}}(x, S) + \Delta.$$

3.3 Measurability of iterative ball partitioning

The following set-valued mapping is a building block of much of the literature on random partitions, including the present investigation. Fix a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$ and $k \in \mathbb{N}$. Define a set-valued mapping $\Gamma : \mathfrak{M}^k \times [0,\infty)^k \to 2^{\mathfrak{M}}$ by setting

$$\Gamma(\vec{x}, \vec{r}) \stackrel{\text{def}}{=} B_{\mathfrak{M}}(x_k, r_k) \smallsetminus \bigcup_{j=1}^{k-1} B_{\mathfrak{M}}(x_j, r_j) \tag{3.13}$$

for $(\vec{x}, \vec{r}) = (x_1, \dots, x_k, r_1, \dots, r_k) \in \mathfrak{M}^k \times [0,\infty)^k$. We can think of Γ as a random subset of $\mathfrak{M}$ if we are given a probability measure **Prob** on $\mathfrak{M}^k \times [0,\infty)^k$. The measure **Prob** can encode the geometry of $(\mathfrak{M}, d_{\mathfrak{M}})$; for example, if $(\mathfrak{M}, d_{\mathfrak{M}})$ is a complete doubling metric space, then in [173] this measure arises from a doubling measure on $\mathfrak{M}$ (see [191, 308]). The measure **Prob** can also have a "smoothing effect" through the randomness of the radii (see, e.g., [1, 30, 71, 96, 173, 208, 238, 239]; choosing a suitable distribution over the random radii is sometimes an important and quite delicate matter, but this intricacy will not arise in the present work. For finite dimensional normed spaces, a random subset as in (3.13) was used in [76, 152]. Note that given $\Delta > 0$, if the measure **Prob** is supported on the set of those $(\vec{x}, \vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k$ for which $r_k \leqslant \Delta/2$, then the mapping Γ takes values in subsets of $\mathfrak{M}$ of diameter at most Δ.

While the definition (3.13) is very simple and natural, in order to use it in the ensuing reasoning we need to know that it satisfies certain measurability requirements. Note first that the set-valued mapping Γ in (3.13) has the following basic measurability property: for every fixed $y \in \mathfrak{M}$ the set $\{(\vec{x}, \vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k : y \in \Gamma(\vec{x}, \vec{r})\}$ is

Borel. Indeed, by definition we have

$$\begin{aligned}\big\{(\vec{x},\vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k : y \in \Gamma(\vec{x},\vec{r})\big\} \\ = \bigcap_{j=1}^{k-1} \big\{(\vec{x},\vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k : d_{\mathfrak{M}}(y,x_j) > r_j\big\} \\ \cap \big\{(\vec{x},\vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k : d_{\mathfrak{M}}(y,x_k) \leqslant r_k\big\}.\end{aligned}$$

In other words, the indicator mapping $(\vec{x},\vec{r}) \mapsto \mathbf{1}_{\Gamma(\vec{x},\vec{r})}(y)$ is Borel measurable for every fixed $y \in \mathfrak{M}$.

Lemma 119. *Fix $k \in \mathbb{N}$. Let $(\mathfrak{M}, d_{\mathfrak{M}})$ be a Polish metric space and suppose that $\Gamma : \mathfrak{M}^k \times [0,\infty)^k \to 2^{\mathfrak{M}}$ be given in (3.13). Then*

$$\Gamma^{-}(S) = \big\{(\vec{x},\vec{r}) \in \mathfrak{M}^k \times [0,\infty)^k : S \cap \Gamma(\vec{x},\vec{r}) \neq \varnothing\big\}$$

is analytic for every analytic subset $S \subseteq \mathfrak{M}$. Consequently, for every complete σ-finite Borel measure μ on $\mathfrak{M}^k \times [0,\infty)^k$, if $\mathcal{F}_\mu$ denotes the σ-algebra of μ-measurable subsets of $\mathfrak{M}^k \times [0,\infty)^k$, then Γ is a strongly measurable set-valued mapping from the measurable space $(\mathfrak{M}^k \times [0,\infty)^k, \mathcal{F}_\mu)$ to $2^{\mathfrak{M}}$.

Proof. Since S is analytic, there exists a Borel subset T of a Polish metric space $\mathfrak{Z}$ and a continuous mapping $\psi : T \to \mathfrak{M}$ such that $\psi(T) = S$. Consider the following Borel subset $\mathfrak{B}$ of the Polish space $\mathfrak{M}^k \times [0,\infty)^k \times \mathfrak{Z}$ ($\mathfrak{B}$ is Borel because it is defined using finitely many continuous inequalities)

$$\begin{aligned}\mathfrak{B} \stackrel{\mathrm{def}}{=} \big\{(\vec{x},\vec{r},t) \in \mathfrak{M}^k \times [0,\infty)^k \times T : d_{\mathfrak{M}}(\psi(t),x_k) \leqslant r_k \\ \wedge \forall j \in \{1,\ldots,k-1\},\ d_{\mathfrak{M}}(\psi(t),x_j) > r_j\big\}.\end{aligned}$$

Then $\Gamma^{-}(S) = \pi(\mathfrak{B})$, where

$$\pi : \mathfrak{M}^k \times [0,\infty)^k \times \mathfrak{Z} \to \mathfrak{M}^k \times [0,\infty)^k$$

is the projection onto the first two coordinates, i.e., $\pi(\vec{x},\vec{r},z) = (\vec{x},\vec{r})$ for $(\vec{x},\vec{r},z) \in \mathfrak{M}^k \times [0,\infty)^k \times \mathfrak{Z}$. Since π is continuous, it follows that $\Gamma^{-}(S)$ is analytic. By Lusin's theorem [189, 192], it follows that $\Gamma^{-}(S)$ is universally measurable. In particular, if μ is a complete σ-finite Borel measure on $\mathfrak{M}^k \times [0,\infty)^k$ and $\mathcal{F}_\mu$ is the σ-algebra of μ-measurable subsets of $\mathfrak{M}^k \times [0,\infty)^k$, then $\Gamma^{-}(E) \in \mathcal{F}_\mu$ for every closed subset $E \subseteq \mathfrak{M}$. Recalling (1.91), this means that Γ is a strongly measurable set-valued mapping from the measurable space $(\mathfrak{M}^k \times [0,\infty)^k, \mathcal{F}_\mu)$ to $2^{\mathfrak{M}}$. ∎

Lemma 120 below contains additional Borel measurability assertions that will be used later. Its assumptions are satisfied, for example, when $\mathfrak{M}$ is a separable normed space, which is the case of interest here. We did not investigate the maximal generality under which the conclusion of Lemma 120 holds.

In what follows, given a metric space $(\mathfrak{M}, d_{\mathfrak{M}})$, for every $x \in \mathfrak{M}$ and $r > 0$ the open ball of radius r centered at x is denoted $B^{\circ}_{\mathfrak{M}}(x,r) = \{y \in \mathfrak{M} : d_{\mathfrak{M}}(x,y) < r\}$.

Lemma 120. *Suppose that $(\mathfrak{M}, d_{\mathfrak{M}})$ is a separable metric space such that*

$$\forall (x,r) \in \mathfrak{M} \times (0,\infty), \quad B_{\mathfrak{M}}(x,r) = \overline{B^{\circ}_{\mathfrak{M}}(x,r)}. \tag{3.14}$$

Fix $k \in \mathbb{N}$ and let $\Gamma : \mathfrak{M}^k \times (0,\infty)^k \to 2^{\mathfrak{M}}$ be given in (3.13). Then the following set is Borel measurable:

$$\mathcal{G}_{\Gamma} = \left\{(\vec{x},\vec{r}) \in \mathfrak{M}^k \times (0,\infty)^k : \Gamma(\vec{x},\vec{r}) \neq \varnothing\right\}.$$

Also, for each $y \in \mathfrak{M}$ the mapping from $\mathcal{G}_{\Gamma}$ to $\mathbb{R}$ that is given by

$$(\vec{x},\vec{r}) \mapsto d_{\mathfrak{M}}(y, \Gamma(x,r))$$

is Borel measurable.

Proof. Let $\mathfrak{D} \subseteq \mathfrak{M}$ be a countable dense subset of $\mathfrak{M}$. The assumption (3.14) implies that $\mathfrak{D} \cap \Gamma(\vec{x},\vec{r})$ is dense in $\Gamma(\vec{x},\vec{r})$ for every $(\vec{x},\vec{r}) \in \mathfrak{M}^k \times (0,\infty)^k$. This is straightforward to check as follows. Fix $y \in \Gamma(\vec{x},\vec{r})$ and $\delta > 0$. We need to find $q \in \mathfrak{D} \cap \Gamma(\vec{x},\vec{r})$ with $d_{\mathfrak{M}}(q,y) < \delta$. Recalling (3.13), since $y \in \Gamma(\vec{x},\vec{r})$ we know that $d_{\mathfrak{M}}(y,x_k) \leqslant r_k$, and also $d_{\mathfrak{M}}(y,x_j) > r_j$ for every $j \in \{1,\ldots,k-1\}$, i.e., $\eta > 0$ where

$$\eta \stackrel{\mathrm{def}}{=} \min\left\{\delta, d_{\mathfrak{M}}(y,x_1) - r_1, \ldots, d_{\mathfrak{M}}(y,x_{k-1}) - r_{k-1}\right\}.$$

By (3.14) there is $z \in B^{\circ}_{\mathfrak{M}}(x_k,r_k)$ with $d_{\mathfrak{M}}(z,y) < \eta/2$. Denote

$$\rho \stackrel{\mathrm{def}}{=} \min\left\{r_k - d_{\mathfrak{M}}(z,x_k), \frac{1}{2}\eta\right\}.$$

Then $\rho > 0$, so the density of $\mathfrak{D}$ in $\mathfrak{M}$ implies that there is $q \in \mathfrak{D}$ with $d_{\mathfrak{M}}(q,z) < \rho$. Consequently,

$$d_{\mathfrak{M}}(q,y) \leqslant d_{\mathfrak{M}}(q,z) + d_{\mathfrak{M}}(z,y) < \rho + \frac{\eta}{2} \leqslant \delta.$$

It remains to observe that $q \in \Gamma(\vec{x},\vec{r})$, because

$$d_{\mathfrak{M}}(q,x_k) \leqslant d_{\mathfrak{M}}(q,z) + d_{\mathfrak{M}}(z,x_k) < \rho + d_{\mathfrak{M}}(z,x_k) \leqslant r_k,$$

and also for every $j \in \{1,\ldots,k-1\}$ we have

$$\begin{aligned} d_{\mathfrak{M}}(q,x_j) &\geqslant d_{\mathfrak{M}}(y,x_j) - d_{\mathfrak{M}}(y,z) - d_{\mathfrak{M}}(z,q) \\ &> d_{\mathfrak{M}}(y,x_j) - \frac{\eta}{2} - \rho \geqslant d_{\mathfrak{M}}(y,x_j) - \eta \geqslant r_j. \end{aligned}$$

For every $(\vec{x}, \vec{r}) \in \mathfrak{M}^k \times (0, \infty)^k$, we have $\Gamma(\vec{x}, \vec{r}) \neq \varnothing$ if and only if $\mathfrak{D} \cap \Gamma(\vec{x}, \vec{r}) \neq \varnothing$. Consequently,

$$\begin{aligned}\mathcal{G}_\Gamma &= \left\{\left(\vec{x}, \vec{r}\right) \in \mathfrak{M}^k \times (0, \infty)^k : \Gamma\left(\vec{x}, \vec{r}\right) \neq \varnothing\right\} \\ &= \bigcup_{q \in \mathfrak{D}} \left\{\left(\vec{x}, \vec{r}\right) \in \mathfrak{M}^k \times (0, \infty)^k : q \in \Gamma\left(\vec{x}, \vec{r}\right)\right\}.\end{aligned}$$

Since $\mathfrak{D}$ is countable and we already checked in the paragraph immediately preceding Lemma 119 that $\{(\vec{x}, \vec{r}) \in \mathfrak{M}^k \times (0, \infty)^k : y \in \Gamma(\vec{x}, \vec{r})\}$ is Borel measurable for every $y \in \mathfrak{M}$, we get that $\mathcal{G}_\Gamma$ is Borel measurable.

Next, $d_{\mathfrak{M}}(y, \Gamma(\vec{x}, \vec{r})) = d_{\mathfrak{M}}(y, \mathfrak{D} \cap \Gamma(\vec{x}, \vec{r}))$ for every $(\vec{x}, \vec{r}) \in \mathcal{G}_\Gamma$ and $y \in \mathfrak{M}$. So, for every $t > 0$ we have

$$\begin{aligned}&\left\{\left(\vec{x}, \vec{r}\right) \in \mathcal{G}_\Gamma : d_{\mathfrak{M}}\left(y, \Gamma\left(\vec{x}, \vec{r}\right)\right) < t\right\} \\ &\quad = \bigcup_{q \in \mathfrak{D} \cap B^{\circ}_{\mathfrak{M}}(y,t)} \left\{\left(\vec{x}, \vec{r}\right) \in \mathfrak{M}^k \times (0, \infty)^k : q \in \Gamma\left(\vec{x}, \vec{r}\right)\right\}.\end{aligned}$$

It follows that $\{(\vec{x}, \vec{r}) \in \mathcal{G}_\Gamma : d_{\mathfrak{M}}(y, \Gamma(\vec{x}, \vec{r})) < t\}$ is Borel measurable for every $t \in \mathbb{R}$. ∎

Corollary 121 below follows directly from the definition of a standard set-valued mapping due to Lemma 120 and the discussion in the paragraph immediately preceding Lemma 119.

Corollary 121. *Let* $(\mathfrak{M}, d_{\mathfrak{M}})$ *be a Polish metric space satisfying* (3.14)*. Then, for every* $k \in \mathbb{N}$ *the set-valued mapping* $\Gamma : \mathfrak{M}^k \times (0, \infty)^k \to 2^{\mathfrak{M}}$ *in* (3.13) *is standard.*