Chapter 6

Volume computations

In this section we will prove volume estimates that occur in our bounds on the separation modulus.

6.1 Direct sums

Fix $n \in \mathbb{N}$ and a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. Throughout what follows, the (normalized) *cone measure* [120] on $\partial B_{\mathbf{X}}$ will be denoted $\kappa_{\mathbf{X}}$. Thus, for every measurable $A \subseteq \partial B_{\mathbf{X}}$,

$$\kappa_{\mathbf{X}}(A) \stackrel{\mathrm{def}}{=} \frac{\mathrm{vol}_n([0,1]A)}{\mathrm{vol}_n(B_{\mathbf{X}})} = \frac{\mathrm{vol}_n(\{sv : (s,v) \in [0,1] \times A\})}{\mathrm{vol}_n(B_{\mathbf{X}})}. \tag{6.1}$$

The probability measure $\kappa_{\mathbf{X}}$ is characterized by the following "generalized polar coordinates" identity, which holds for every $f \in L_1(\mathbb{R}^n)$; see, e.g., [242, Proposition 1]:

$$\int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = n\,\mathrm{vol}_n(B_{\mathbf{X}}) \int_0^\infty r^{n-1} \bigg(\int_{\partial B_{\mathbf{X}}} f(r\theta)\,\mathrm{d}\kappa_{\mathbf{X}}(\theta) \bigg)\,\mathrm{d}r. \tag{6.2}$$

As a quick application of (6.2), we will next record for ease of later reference the following computation of the volume of the unit ball of an ℓ_p direct sum of normed spaces.

Lemma 149. *Fix $n, m_1, \ldots, m_n \in \mathbb{N}$ and normed spaces $\{\mathbf{X}_j = (\mathbb{R}^{m_1}, \|\cdot\|_{\mathbf{X}_{m_j}})\}_{j=1}^n$. Then*

$$\forall\, p \in [1,\infty], \quad \mathrm{vol}_{m_1+\cdots+m_n}(B_{\mathbf{X}_1 \oplus_p \cdots \oplus_p \mathbf{X}_n}) = \frac{\prod_{j=1}^n \Gamma\big(1 + \frac{m_j}{p}\big)\,\mathrm{vol}_{m_j}\big(B_{\mathbf{X}_j}\big)}{\Gamma\big(1 + \frac{m_1+\cdots+m_n}{p}\big)}. \tag{6.3}$$

Proof. This follows by induction on n from the following identity (direct application of Fubini), which holds for every $a, b \in \mathbb{N}$ and any two normed spaces $\mathbf{X} = (\mathbb{R}^a, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^b, \|\cdot\|_{\mathbf{Y}})$:

$$\begin{aligned} \mathrm{vol}_{a+b}(B_{\mathbf{X} \oplus_p \mathbf{Y}}) &= \int_{B_{\mathbf{X}}} \mathrm{vol}_b\Big(\big(1 - \|x\|_{\mathbf{X}}^p\big)^{\frac{1}{p}} B_{\mathbf{Y}}\Big)\,\mathrm{d}x \\ &= \mathrm{vol}_b(B_{\mathbf{Y}}) \int_{B_{\mathbf{X}}} \big(1 - \|x\|_{\mathbf{X}}^p\big)^{\frac{b}{p}}\,\mathrm{d}x \end{aligned}$$

$$\overset{(6.2)}{=} \mathrm{vol}_a(B_{\mathbf{X}})\,\mathrm{vol}_b(B_{\mathbf{Y}}) \int_0^1 ar^{a-1}\big(1-r^p\big)^{\frac{b}{p}}\,\mathrm{d}r$$

$$= \mathrm{vol}_a(B_{\mathbf{X}})\,\mathrm{vol}_b(B_{\mathbf{Y}}) \frac{\Gamma\big(1+\frac{b}{p}\big)\Gamma\big(1+\frac{a}{p}\big)}{\Gamma\big(1+\frac{a+b}{p}\big)}. \qquad \blacksquare$$

By Lemma 149, for every $m \in \mathbb{N}$, every normed space $\mathbf{X} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{X}})$ satisfies

$$\mathrm{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big) = \frac{\Gamma\big(1+\frac{m}{p}\big)^n}{\Gamma\big(1+\frac{nm}{p}\big)} \mathrm{vol}_m(B_{\mathbf{X}})^n, \tag{6.4}$$

and hence,

$$\mathrm{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)^{\frac{1}{nm}} \asymp \frac{\mathrm{vol}_m(B_{\mathbf{X}})^{\frac{1}{m}}}{n^{\frac{1}{p}}}. \tag{6.5}$$

In particular, for every $m, n \in \mathbb{N}$ and $1 \leqslant p, q \leqslant \infty$ we have

$$\mathrm{vol}_{nm}\big(B_{\ell_p^n(\ell_q^m)}\big) = \frac{2^{nm}\Gamma\big(1+\frac{1}{q}\big)^{nm}\Gamma\big(1+\frac{m}{p}\big)^n}{\Gamma\big(1+\frac{m}{q}\big)^n\Gamma\big(1+\frac{nm}{p}\big)}, \tag{6.6}$$

and hence,

$$\mathrm{vol}_{nm}\big(B_{\ell_p^n(\ell_q^m)}\big)^{\frac{1}{nm}} \asymp \frac{1}{n^{\frac{1}{p}}m^{\frac{1}{q}}}. \tag{6.7}$$

The following simple lemma records an extension of (6.5) to m-fold iterations of the operation $\mathbf{X} \mapsto \ell_p^n(\mathbf{X})$, i.e., to spaces of the form

$$\ell_{p_m}^{n_m}\Big(\ell_{p_{m-1}}^{n_{m-1}}\big(\cdots \ell_{p_1}^{n_1}(\mathbf{X})\cdots\big)\Big);$$

the main point for us here is that the implicit constants remain bounded as $m \to \infty$.

Lemma 150. *Fix* $\{n_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ *and* $\{p_k\}_{k=1}^{\infty} \subseteq [1,\infty]$. *Let* $\mathbf{X} = (\mathbb{R}^{n_0}, \|\cdot\|_{\mathbf{X}})$ *be a normed space and define*

$$\forall k \in \mathbb{N} \cup \{0\}, \quad \mathbf{X}_{k+1} = \ell_{p_k}^{n_k}(\mathbf{X}_k), \quad \text{where } \mathbf{X}_0 = \mathbf{X}.$$

Then, for every $m \in \mathbb{N}$ *we have*

$$\mathrm{vol}_{n_0\cdots n_m}\big(B_{\mathbf{X}_m}\big)^{\frac{1}{n_0\cdots n_k}} \asymp \frac{\mathrm{vol}_{n_0}\big(B_{\mathbf{X}}\big)^{\frac{1}{n_0}}}{\prod_{k=1}^m n_k^{\frac{1}{p_k}}}.$$

Proof. With the convention that an empty product equals 1, by applying (6.4) inductively we see that

$$\mathrm{vol}_{n_0\cdots n_m}\big(B_{\mathbf{X}_m}\big) = \mathrm{vol}_{n_0}\big(B_{\mathbf{X}}\big)^{n_1\cdots n_m} \prod_{k=1}^m \frac{\Gamma\big(1+\frac{n_0\cdots n_{k-1}}{p_k}\big)^{n_k\cdots n_m}}{\Gamma\big(1+\frac{n_0\cdots n_k}{p_k}\big)^{n_{k+1}\cdots n_m}}.$$

Hence,

$$\frac{\mathrm{vol}_{n_0\cdots n_m}(B_{\mathbf{X}_m})^{\frac{1}{n_0\cdots n_k}}\prod_{k=1}^m n_k^{\frac{1}{p_k}}}{\mathrm{vol}_{n_0}(B_{\mathbf{X}})^{\frac{1}{n_0}}} = \prod_{k=1}^m \frac{\Gamma\big(1+\frac{n_0\cdots n_{k-1}}{p_k}\big)^{\frac{1}{n_0\cdots n_{k-1}}}}{\Gamma\big(1+\frac{n_0\cdots n_k}{p_k}\big)^{\frac{1}{n_0\cdots n_k}}} n_k^{\frac{1}{p_k}} = \prod_{k=1}^m f_{n_0\cdots n_{k-1},n_k}\left(\frac{1}{p_k}\right), \tag{6.8}$$

where for $u, v, t > 0$ we denote

$$f_{u,v}(t) \stackrel{\mathrm{def}}{=} \frac{\Gamma(1+ut)^{\frac{1}{u}}}{\Gamma(1+uvt)^{\frac{1}{uv}}} v^t.$$

Since $(\log\Gamma(z))' = \int_0^\infty \frac{se^{-zs}}{1-e^{-s}}\,\mathrm{d}s$ for $z > 0$ (see, e.g., [313, Chapter XII]), if $u, t > 0$ and $v \geqslant 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\log f_{u,v}(t) = \log v + \int_0^\infty \left(e^{-uts} - e^{-uvts}\right)\frac{se^{-s}}{1-e^{-s}}\,\mathrm{d}s \geqslant 0.$$

Thus, $f_{u,v}$ is increasing on $[0,\infty)$, and therefore we get from (6.8) that

$$1 = \prod_{k=1}^m f_{n_0\cdots n_{k-1},n_k}(0) \leqslant \frac{\mathrm{vol}_{n_0\cdots n_m}(B_{\mathbf{X}_m})^{\frac{1}{n_0\cdots n_k}}\prod_{k=1}^m n_k^{\frac{1}{p_k}}}{\mathrm{vol}_{n_0}(B_{\mathbf{X}})^{\frac{1}{n_0}}} \leqslant \prod_{k=1}^m f_{n_0\cdots n_{k-1},n_k}(1) = \frac{(n_0!)^{\frac{1}{n_0}} n_1\cdots n_m}{\big((n_0\cdots n_m)!\big)^{\frac{1}{n_0\cdots n_m}}} \leqslant e. \qquad \blacksquare$$

The first part of Lemma 151 below is a restatement of Lemma 37 from the Introduction. Qualitatively, it shows that the class of spaces for which Conjecture 10 holds is closed under unconditional composition, namely, norms of the form (6.9) below. The second part of Lemma 151 is further information that pertains to Conjecture 49, i.e., to the symmetric version of the weak reverse isoperimetric conjecture, for which we want the operator S to be the identity mapping (i.e., weak reverse isoperimetry holds without the need to first change the "position" of the given normed space).

Lemma 151. *Fix $n, m_1, \ldots, m_n \in \mathbb{N}$. Let*

$$\mathbf{X}_1 = (\mathbb{R}^{m_1}, \|\cdot\|_{\mathbf{X}_1}), \ldots, \mathbf{X}_n = (\mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}_n})$$

be normed spaces. Also, let $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ be an unconditional normed space. Define a normed space $\mathbf{X} = (\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}})$ by

$$\forall x = (x_1,\ldots,x_n) \in \mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n}, \quad \|x\|_{\mathbf{X}} \stackrel{\mathrm{def}}{=} \big\|\big(\|x_1\|_{\mathbf{X}_1},\ldots,\|x_n\|_{\mathbf{X}_n}\big)\big\|_{\mathbf{E}}. \tag{6.9}$$

Then, Conjecture 10 (equivalently, Conjecture 35) holds for the space $\mathbf{X}$ *if it holds for all of the spaces* $\mathbf{X}_1, \ldots, \mathbf{X}_n$.

More precisely, suppose that there exist $S_1 \in \mathsf{SL}_{m_1}(\mathbb{R}), \ldots, S_n \in \mathsf{SL}_{m_n}(\mathbb{R})$, *normed spaces* $\mathbf{Y}_1 = (\mathbb{R}^{m_1}, \|\cdot\|_{\mathbf{Y}_1}), \ldots, \mathbf{Y}_n = (\mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{Y}_n})$, *and* $\alpha > 0$ *such that for every* $k \in \{1, \ldots, n\}$ *we have*

$$B_{\mathbf{Y}_k} \subseteq S_k B_{\mathbf{X}_k} \quad \text{and} \quad \frac{\mathrm{iq}(B_{\mathbf{Y}_k})}{\sqrt{m_k}} \left(\frac{\mathrm{vol}_{m_k}(B_{\mathbf{X}_k})}{\mathrm{vol}_{m_k}(B_{\mathbf{Y}_k})} \right)^{\frac{1}{m_k}} \leqslant \alpha. \tag{6.10}$$

Then, there exist a normed space $\mathbf{Y} = (\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}, \|\cdot\|_{\mathbf{X}})$ *and a linear transformation* $S \in \mathsf{SL}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})$ *such that*

$$B_{\mathbf{Y}} \subseteq S B_{\mathbf{X}} \quad \text{and} \quad \frac{\mathrm{iq}(B_{\mathbf{Y}})}{\sqrt{m_1 + \cdots + m_n}} \left(\frac{\mathrm{vol}_{m_1+\cdots+m_n}(B_{\mathbf{X}})}{\mathrm{vol}_{m_1+\cdots+m_n}(B_{\mathbf{Y}})} \right)^{\frac{1}{m_1+\cdots+m_n}} \lesssim \alpha. \tag{6.11}$$

If furthermore $S_1, \ldots, S_n$ *are all identity mappings (of the respective dimensions), then* S *can be taken to be the identity mapping provided the following two conditions hold:*

$$\left\| \sum_{i=1}^n e_i \right\|_{\mathbf{E}} \left\| \sum_{i=1}^n e_i \right\|_{\mathbf{E}^*} \lesssim n \tag{6.12}$$

and

$$\left(\prod_{k=1}^n m_k^{m_k} \mathrm{vol}_{m_k}(B_{\mathbf{X}_k}) \right)^{\frac{1}{m_1+\cdots+m_n}} \lesssim \frac{m_1 + \cdots + m_n}{n} \min_{k \in \{1,\ldots,n\}} \mathrm{vol}_{m_k}(B_{\mathbf{X}_k})^{\frac{1}{m_k}}. \tag{6.13}$$

Note that (6.13) is satisfied in particular if $m_i \asymp m_j$ *and*

$$\mathrm{vol}_{m_i}(B_{\mathbf{X}_i})^{\frac{1}{m_i}} \asymp \mathrm{vol}_{m_j}(B_{\mathbf{X}_j})^{\frac{1}{m_j}}$$

for every $i, j \in \{1, \ldots, n\}$.

Prior to proving Lemma 151 we will make some basic observations. Firstly, (6.9) indeed defines a norm because it is well known that the requirement that $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ is an unconditional normed space is equivalent to (see, e.g., [181, Proposition 1.c.7]) the following "contraction property":

$$\forall a, x \in \mathbb{R}^n, \quad \|(a_1 x_1, \ldots, a_n x_n)\|_{\mathbf{E}} \leqslant \|a\|_{\ell_\infty^n} \|x\|_{\mathbf{E}}. \tag{6.14}$$

Thus, $\|x\|_{\mathbf{E}} \leqslant \|y\|_{\mathbf{E}}$ if $x, y \in \mathbb{R}^n$ satisfy $|x_i| \leqslant |y_i|$ for every $i \in \{1, \ldots, n\}$, so the triangle inequality for (6.9) follows from applying the triangle inequalities entry-wise for each of the norms $\{\|\cdot\|_{\mathbf{X}_i}\}_{i=1}^n$, using this monotonicity property, and then applying the triangle inequality for $\|\cdot\|_{\mathbf{E}}$.

It is well known that condition (6.12) holds (as an equality) when $\mathbf{E}$ is a symmetric normed space (see, e.g., [182, Proposition 3.a.6]). More generally, condition (6.12) holds (also as an equality) in the setting of the following simple averaging lemma, which shows in particular that Lemma 151 implies Lemma 53.

Lemma 152. *Let* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ *be a normed space such that for every two indices* $j,k \in \{1,\ldots,n\}$ *there exists a permutation* $\pi = \pi_{jk} \in S_n$ *with* $\pi(j) = k$ *such that* $\|\sum_{i=1}^n a_{\pi(i)} e_i\|_{\mathbf{X}} = \|\sum_{i=1}^n a_i e_i\|_{\mathbf{X}}$ *for every* $a_1,\ldots,a_n \in \mathbb{R}$. *Then,*

$$\Big\|\sum_{i=1}^n e_i\Big\|_{\mathbf{X}} \Big\|\sum_{i=1}^n e_i\Big\|_{\mathbf{X}^*} = n.$$

Proof. Denote $\mathfrak{S}(\mathbf{X}) = \{\pi \in S_n : T_\pi \in \mathsf{Isom}(\mathbf{X})\}$, where $T_\pi \in \mathsf{GL}_n(\mathbb{R})$ was defined in Example 40 for each $\pi \in S_n$. Then, $\mathfrak{S}(\mathbf{X})$ is a subgroup of S_n that we are assuming acts transitively on $\{1,\ldots,n\}$. Consequently,

$$\forall i,j \in \{1,\ldots,n\}, \quad |\{\pi \in \mathfrak{S}(\mathbf{X}) : \pi(i) = j\}| = \frac{|\mathfrak{S}(\mathbf{X})|}{n}. \tag{6.15}$$

For every $a_1,\ldots,a_n \in \mathbb{R}$ we have

$$\begin{aligned}\frac{1}{|\mathfrak{S}(\mathbf{X})|}\sum_{\pi\in\mathfrak{S}(\mathbf{X})}\sum_{i=1}^n a_{\pi(i)}e_i &= \sum_{i=1}^n\left(\sum_{j=1}^n \frac{|\{\pi \in \mathfrak{S}(\mathbf{X}) : \pi(i) = j\}|}{|\mathfrak{S}(\mathbf{X})|}a_j\right)e_i\\ &\overset{(6.15)}{=} \frac{\sum_{j=1}^n a_j}{n}\sum_{i=1}^n e_i.\end{aligned}$$

Hence,

$$\begin{aligned}\left|\left\langle \sum_{j=1}^n e_j, \sum_{j=1}^n a_j e_j\right\rangle\right| &= \left|\sum_{j=1}^n a_j\right|\\ &= \frac{n\left\|\frac{1}{|\mathfrak{S}(\mathbf{X})|}\sum_{\pi\in\mathfrak{S}(\mathbf{X})}\sum_{i=1}^n a_{\pi(i)}e_i\right\|_{\mathbf{X}}}{\left\|\sum_{i=1}^n e_i\right\|_{\mathbf{X}}}\\ &\leqslant \frac{\frac{n}{|\mathfrak{S}(\mathbf{X})|}\sum_{\pi\in\mathfrak{S}(\mathbf{X})}\left\|\sum_{i=1}^n a_{\pi(i)}e_i\right\|_{\mathbf{X}}}{\left\|\sum_{i=1}^n e_i\right\|_{\mathbf{X}}}\\ &= \frac{n\left\|\sum_{i=1}^n a_i e_i\right\|_{\mathbf{X}}}{\left\|\sum_{i=1}^n e_i\right\|_{\mathbf{X}}},\end{aligned}$$

where the penultimate step uses convexity and the final step uses the assumption that T_π is an isometry of $\mathbf{X}$ for every $\pi \in \mathfrak{S}(\mathbf{X})$. Since this holds for every $a_1,\ldots,a_n \in \mathbb{R}$, we have $\|\sum_{i=1}^n e_i\|_{\mathbf{X}^*} \leqslant n/\|\sum_{i=1}^n e_i\|_{\mathbf{X}}$. The reverse inequality holds for any normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ because $\langle\sum_{i=1}^n e_i, \sum_{i=1}^n e_i\rangle = n$. ∎

By combining Lemmas 151 and 152 we obtain the following corollary that establishes Conjecture 49 for the iteratively nested ℓ_p spaces of Lemma 150, provided it holds for the initial space $\mathbf{X}$.

Corollary 153. *Fix* $\{n_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ *and* $\{p_k\}_{k=1}^{\infty} \subseteq [1,\infty]$. *Let* $\mathbf{X} = (\mathbb{R}^{n_0}, \|\cdot\|_{\mathbf{X}})$ *be a normed space and define*

$$\forall k \in \mathbb{N}, \quad \mathbf{X}_{k+1} = \ell_{p_k}^{n_k}(\mathbf{X}_k), \quad \text{where } \mathbf{X}_0 = \mathbf{X}.$$

Suppose that $\alpha > 0$ *and there exists a normed space* $\mathbf{Y} = (\mathbb{R}^{n_0}, \|\cdot\|_{\mathbf{Y}})$ *with* $B_{\mathbf{Y}} \subseteq B_{\mathbf{X}}$ *and that satisfies*

$$\frac{\mathrm{iq}(B_{\mathbf{Y}})}{\sqrt{n_0}} \left(\frac{\mathrm{vol}_{n_0}(B_{\mathbf{X}})}{\mathrm{vol}_{n_0}(B_{\mathbf{Y}})}\right)^{\frac{1}{n_0}} \leqslant \alpha. \tag{6.16}$$

Then, for every $m \in \mathbb{N}$ *there is a normed space* $\mathbf{Y}_m = (\mathbb{R}^{n_0 \cdots n_m}, \|\cdot\|_{\mathbf{Y}_m})$ *that satisfies* $B_{\mathbf{Y}_m} \subseteq B_{\mathbf{X}_m}$ *and*

$$\frac{\mathrm{iq}\big(B_{\mathbf{Y}_m}\big)}{\sqrt{n_0 \cdots n_m}} \left(\frac{\mathrm{vol}_{n_0\cdots n_m}\big(B_{\mathbf{X}_m}\big)}{\mathrm{vol}_{n_0\cdots n_m}\big(B_{\mathbf{Y}_m}\big)}\right)^{\frac{1}{n_0\cdots n_m}} \lesssim \alpha,$$

To see why Corollary 153 indeed follows from Lemmas 151 and 152, observe that if we start with $\mathbf{E}_0 = \mathbb{R}$ and define inductively $\mathbf{E}_{k+1} = \ell_{p_k}^{n_k}(\mathbf{E}_k)$, then for each $m \in \mathbb{N}$ the space $\mathbf{E}_m$ is unconditional and satisfies the assumptions of Lemma 152. The space $\mathbf{Y}_m$ of Corollary 153 is the same space that is defined in Lemma 151 if we take $\mathbf{E} = \mathbf{E}_m$, and also $\mathbf{X}_1 = \cdots = \mathbf{X}_m = \mathbf{X}$, which ensures that (6.13) holds.

Proof of Lemma 151. Denote

$$M \stackrel{\mathrm{def}}{=} \sum_{k=1}^{n} m_k = \dim(\mathbf{X}) \quad \text{and} \quad \forall k \in \{1,\ldots,n\}, \quad \rho_k \stackrel{\mathrm{def}}{=} \mathrm{vol}_{m_k}(B_{\mathbf{X}_k})^{\frac{1}{m_k}}. \tag{6.17}$$

Fix numbers $c, C_1, \ldots, C_n, \gamma_1, \ldots, \gamma_n, w_1, \ldots, w_n, w_1^*, \ldots, w_n^*, \beta_1, \ldots, \beta_n > 0$ that satisfy the following conditions (their values will be specified later). Firstly, we require that

$$\Big\|\sum_{i=1}^{n} w_i e_i\Big\|_{\mathbf{E}} = \Big\|\sum_{i=1}^{n} w_i^* e_i\Big\|_{\mathbf{E}^*} = 1. \tag{6.18}$$

Secondly, we require that

$$\forall k \in \{1,\ldots,n\}, \quad w_k w_k^* \geqslant \frac{m_k}{\gamma_k M}. \tag{6.19}$$

Finally, we require that

$$\forall k \in \{1,\ldots,n\}, \quad \frac{1}{c w_k \rho_k} \leqslant \beta_k \leqslant \frac{C_k}{w_k \rho_k}, \tag{6.20}$$

Denote

$$D \stackrel{\text{def}}{=} \left(\prod_{k=1}^{n} \beta_k^{m_k}\right)^{\frac{1}{M}}.$$

Consider the block diagonal linear operator $S : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n} \to \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$ that is given by

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}, \quad Sx \stackrel{\text{def}}{=} \frac{1}{D}\big(\beta_1 S_1 x_1, \ldots, \beta_n S_n x_n\big). \tag{6.21}$$

The normalization by D in (6.21) ensures that $S \in \mathsf{SL}(\mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n})$.

Since $\sum_{k=1}^{n} w_k^* e_k$ is a unit functional in $\mathbf{E}^*$, we have

$$\begin{aligned}
\left\|S^{-1}x\right\|_{\mathbf{X}} &\stackrel{(6.9)\wedge(6.21)}{=} D\left\|\sum_{k=1}^{n} \frac{\|S_k^{-1}x_k\|_{\mathbf{X}_k}}{\beta_k} e_k\right\|_{\mathbf{E}} \\
&\stackrel{(6.18)}{\geqslant} D\left\langle \sum_{k=1}^{n} w_k^* e_k, \sum_{k=1}^{n} \frac{\|S_k^{-1}x_k\|_{\mathbf{X}_k}}{\beta_k} e_k \right\rangle \\
&\stackrel{(6.19)}{\geqslant} \frac{D}{M} \sum_{k=1}^{n} \frac{m_k \|S_k^{-1}x_k\|_{\mathbf{X}_k}}{\gamma_k w_k \beta_k},
\end{aligned}$$

for every $x = (x_1, \ldots, x_n) \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n}$. This shows that

$$\begin{aligned}
SB_{\mathbf{X}} &\subseteq \left\{x \in \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_n} : \sum_{k=1}^{n} \frac{m_k \|S_k^{-1}x_k\|_{\mathbf{X}_k}}{\gamma_k w_k \beta_k} \leqslant \frac{M}{D}\right\} \\
&= \frac{M}{D} B_{\left(\frac{\gamma_1 w_1 \beta_1}{m_1} S_1 \mathbf{X}_1\right) \oplus_1 \cdots \oplus_1 \left(\frac{\gamma_n w_n \beta_n}{m_n} S_n \mathbf{X}_n\right)}.
\end{aligned}$$

Using Lemma 149, we therefore have

$$\begin{aligned}
\mathrm{vol}_M(B_{\mathbf{X}})^{\frac{1}{M}} &\leqslant \frac{M}{D} \mathrm{vol}_M\left(B_{\left(\frac{\gamma_1 w_1 \beta_1}{m_1} S_1 \mathbf{X}_1\right) \oplus_1 \cdots \oplus_1 \left(\frac{\gamma_n w_n \beta_n}{m_n} S_n \mathbf{X}_n\right)}\right)^{\frac{1}{M}} \\
&\stackrel{(6.3)}{=} \frac{1}{D}\left(\frac{M^M}{M!} \prod_{k=1}^{n} m_k! \left(\frac{\gamma_k w_k \beta_k \rho_k}{m_k}\right)^{m_k}\right)^{\frac{1}{M}} \\
&\stackrel{(6.20)}{\leqslant} \frac{1}{D}\left(\frac{M^M}{M!} \prod_{k=1}^{n} \frac{m_k!}{m_k^{m_k}} (\gamma_k C_k)^{m_k}\right)^{\frac{1}{M}} \\
&\leqslant \frac{e}{D}\left(\prod_{k=1}^{n} (\gamma_k C_k)^{m_k}\right)^{\frac{1}{M}}.
\end{aligned} \tag{6.22}$$

Next, for every $x=(x_1,\ldots,x_n)\in\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n}$ we have

$$\begin{aligned}\|S^{-1}x\|_{\mathbf{X}} &\overset{(6.9)\wedge(6.21)}{=} D\left\|\sum_{k=1}^{n}\frac{\|S_k^{-1}x_k\|_{\mathbf{X}_k}}{\beta_k}e_k\right\|_{\mathbf{E}}\\ &\overset{(6.14)}{\leqslant} D\left(\max_{k\in\{1,\ldots,n\}}\frac{\|S_k^{-1}x_k\|_{\mathbf{X}_k}}{w_k\beta_k}\right)\left\|\sum_{k=1}^{n}w_ke_k\right\|_{\mathbf{E}}\\ &\overset{(6.18)}{=} D\max_{k\in\{1,\ldots,n\}}\frac{\|S_k^{-1}x_k\|_{\mathbf{X}_k}}{w_k\beta_k}.\end{aligned}$$

This establishes the following inclusion:

$$SB_{\mathbf{X}}\supseteq\frac{1}{D}\prod_{k=1}^{n}w_k\beta_kS_kB_{\mathbf{X}_k}\overset{\mathrm{def}}{=}\Omega. \tag{6.23}$$

Thanks to (1.62), the assumption (6.10) of Lemma 151 implies that

$$\forall k\in\{1,\ldots,n\},\quad \lambda\big(S_kB_{\mathbf{X}_k}\big)\rho_k^2\overset{(6.17)}{=}\lambda\big(S_kB_{\mathbf{X}_k}\big)\,\mathrm{vol}_{m_k}\big(B_{\mathbf{X}_k}\big)^{\frac{2}{m_k}}\lesssim\alpha^2m_k. \tag{6.24}$$

For each $k\in\{1,\ldots,n\}$ take $f_k:S_kB_{\mathbf{X}_k}\to\mathbb{R}$ that is smooth on the interior of $S_kB_{\mathbf{X}_k}$, vanishes on $\partial S_kB_{\mathbf{X}_k}$, and satisfies $\Delta f_k=-\lambda(S_kB_{\mathbf{X}_k})f_k$ on the interior of $S_kB_{\mathbf{X}_k}$. Define $f:\Omega\to\mathbb{R}$ by

$$\forall x=(x_1,\ldots,x_n)\in\Omega=\frac{1}{D}\prod_{k=1}^{n}w_k\beta_kS_kB_{\mathbf{X}_k},\qquad f(x)\overset{\mathrm{def}}{=}\prod_{k=1}^{n}f_k\left(\frac{D}{w_k\beta_k}x_k\right),$$

Thus $f\equiv0$ on the boundary of Ω and on the interior of Ω it is smooth and satisfies

$$\Delta f=-D^2\left(\sum_{k=1}^{n}\frac{\lambda\big(S_kB_{\mathbf{X}_k}\big)}{(w_k\beta_k)^2}\right)f \tag{6.25}$$

Hence,

$$\begin{aligned}\lambda(S\mathbf{X})=\lambda(SB_{\mathbf{X}})&\overset{(6.23)}{\leqslant}\lambda(\Omega)\overset{(6.25)}{\leqslant}D^2\left(\sum_{k=1}^{n}\frac{\lambda\big(S_kB_{\mathbf{X}_k}\big)}{(w_k\beta_k)^2}\right)\\ &\overset{(6.20)}{\leqslant}(cD)^2\left(\sum_{k=1}^{n}\lambda\big(S_kB_{\mathbf{X}_k}\big)\rho_k^2\right)\overset{(6.24)}{\lesssim}(c\alpha D)^2M.\end{aligned} \tag{6.26}$$

By combining (6.22) and (6.26) we see that

$$\lambda(S\mathbf{X})\,\mathrm{vol}_M(B_{\mathbf{X}})^{\frac{2}{M}}\lesssim c^2\left(\prod_{k=1}^{n}(\gamma_kC_k)^{m_k}\right)^{\frac{2}{M}}\alpha^2M.$$

Another application of (1.62) now shows that the desired conclusion (6.11) holds with $\mathbf{Y} = \mathrm{Ch}\, S\mathbf{X}$ (recall the definition of Cheeger space in Section 1.6.1) provided

$$c\left(\prod_{k=1}^{n}(\gamma_k C_k)^{m_k}\right)^{\frac{1}{M}} \lesssim 1. \tag{6.27}$$

To get (6.11), by the Lozanovskiĭ factorization theorem [186] there exist weights $w_1, \ldots, w_n, w_1^*, \ldots, w_n^* > 0$ such that (6.18) holds and also $w_k w_k^* = m_k/M$ for every $k \in \{1, \ldots, n\}$. Thus (6.19) holds (as equality) if we choose $\gamma_1 = \cdots = \gamma_n = 1$. If we take $c = C_1 = \cdots = C_n = 1$ and $\beta_k = 1/(w_k\rho_k)$ for each $k \in \{1, \ldots, n\}$, then both (6.20) and (6.27) also hold (as equalities). With these choices, (6.11) holds.

Suppose that the additional assumptions (6.12) and (6.13) hold. Denote

$$\eta = \frac{1}{n}\left\|\sum_{i=1}^{n} e_i\right\|_{\mathbf{E}}\left\|\sum_{i=1}^{n} e_i\right\|_{\mathbf{E}^*}.$$

Thus, $\eta = O(1)$ by (6.12). Consider the weights $w_1 = \cdots = w_n = 1/\|\sum_{i=1}^n e_i\|_{\mathbf{E}}$ and $w_1^* = \cdots = w_n^* = 1/\|\sum_{i=1}^n e_i\|_{\mathbf{E}^*}$, so that (6.18) holds by design. This choice also ensures that if we take $\gamma_k = m_k/(\eta M)$ for each $k \in \{1, \ldots, n\}$, then (6.19) holds (as an equality). Next, choose $C_k = \rho_k$ for each $k \in \{1, \ldots, n\}$, as well as $\beta_1 = \cdots = \beta_n = \|\sum_{i=1}^n e_i\|_{\mathbf{E}}$ and $c = 1/\min_{k\in\{1,\ldots,n\}} \rho_k$. This ensures that (6.20) holds, and also that (6.27) coincides with the assumption (6.13), since $\eta = O(1)$. The desired conclusion (6.11) therefore holds with $Sx = (S_1x_1, \ldots, S_nx_n)$ in (6.21). In particular, if $S_k = \mathsf{Id}_{m_k}$ for every $k \in \{1, \ldots, n\}$, then we can take $S = \mathsf{Id}_{\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_n}}$ in (6.11). ■

The following lemma provides a formula for the cone measure of Orlicz spaces. Fix a convex increasing function $\psi : [0,\infty) \to [0,\infty]$ that satisfies $\psi(0) = 0$ and $\lim_{x\to\infty}\psi(x) = \infty$ (so, if $\lim_{x\to a^-}\psi(x) = \infty$ for some $a \in (0,\infty)$, then we require that $\psi(x) = \infty$ for every $x \geqslant a$). Henceforth, the associated Orlicz space (see, e.g., [268]) $\ell_\psi^n = (\mathbb{R}^n, \|\cdot\|_{\ell_\psi^n})$ will always be endowed with the Luxemburg norm that is given by

$$\forall x \in \mathbb{R}^n, \quad \|x\|_{\ell_\psi^n} = \inf\left\{s > 0 : \sum_{i=1}^{n}\psi\left(\frac{|x_i|}{s}\right) \leqslant 1\right\}. \tag{6.28}$$

Lemma 154. *Fix $n \in \mathbb{N}$. Suppose that $\psi : [0,\infty) \to [0,\infty]$ is convex, increasing, continuously differentiable on the set $\{x \in (0,\infty) : \psi(x) < \infty\}$, and satisfies $\lim_{x\to\infty}\psi(x) = \infty$ and $\psi(0) = 0$. Define a function $\varphi_\psi^n : \mathbb{R}^n \to [0,\infty)$ by setting*

$$\forall \tau = (\tau_1, \ldots, \tau_n) \in \mathbb{R}^n, \quad \varphi_\psi^n(\tau) \stackrel{\mathrm{def}}{=} \frac{\sum_{i=1}^n \psi^{-1}(|\tau_i|)\psi'\big(\psi^{-1}(|\tau_i|)\big)}{\prod_{i=1}^n \psi'\big(\psi^{-1}(|\tau_i|)\big)}. \tag{6.29}$$

Then, for every $g \in L_1(\kappa_{\ell_\psi^n})$ *we have*

$$\frac{n!}{2^n}\operatorname{vol}_n\big(B_{\ell_\psi^n}\big)\int_{\partial B_{\ell_\psi^n}} g(\theta)\,\mathrm{d}\kappa_{\ell_\psi^n}(\theta)$$
$$= \int_{\partial B_{\ell_1^n}} g\big(\psi^{-1}(|\tau_1|)\operatorname{sign}(\tau_1),\ldots,\psi^{-1}(|\tau_n|)\operatorname{sign}(\tau_n)\big)\varphi_\psi^n(\tau)\,\mathrm{d}\kappa_{\ell_1^n}(\tau). \quad (6.30)$$

For example, when $\psi(t) = t^p$ for some $p \geqslant 1$ and every $t \geqslant 0$, in which case $\ell_\psi^n = \ell_p^n$, Lemma 154 gives

$$\int_{\partial B_{\ell_p^n}} g\,\mathrm{d}\kappa_{\ell_\psi^n} = \frac{\Gamma\big(1+\frac{n}{p}\big)}{n!\Gamma\big(1+\frac{1}{p}\big)^n}\int_{\partial B_{\ell_1^n}} \frac{g\circ M_{1\to p}^n(\tau)}{|\tau_1\cdots\tau_n|^{1-\frac{1}{p}}}\,\mathrm{d}\kappa_{\ell_1^n}(\tau),$$

where $M_{1\to p}:\mathbb{R}^n\to\mathbb{R}^n$ is the Mazur map [205] from ℓ_1^n to ℓ_p^n, i.e.,

$$\forall x\in\mathbb{R}^n,\quad M_{1\to p}^n(x_1,\ldots,x_n) = \big(|x_1|^{\frac{1}{p}}\operatorname{sign}(x_1),\ldots,|x_n|^{\frac{1}{p}}\operatorname{sign}(x_n)\big).$$

As another special case of Lemma 154, consider the following family of Orlicz spaces $\Omega_\beta^n = (\mathbb{R}^n, \|\cdot\|_{\Omega_\beta^n})$:

$$\forall \beta>0,\quad \Omega_\beta^n \stackrel{\mathrm{def}}{=} \ell_{\psi_\beta}^n, \quad (6.31)$$

where

$$t\geqslant 0,\quad \psi_\beta(t) \stackrel{\mathrm{def}}{=} \begin{cases} \frac{1}{\beta}\log\big(\frac{1}{1-t}\big) & \text{if } 0\leqslant t<1,\\ \infty & \text{if } t\geqslant 1.\end{cases} \quad (6.32)$$

Observe that by considering the case $g\equiv 1$ of (6.30) we obtain the following identity:

$$\int_{\partial B_{\ell_\psi^n}} g\,\mathrm{d}\kappa_{\ell_\psi^n}$$
$$= \frac{\int_{\partial B_{\ell_1^n}} g\big(\psi^{-1}(|\tau_i|)\operatorname{sign}(\tau_1),\ldots,\psi^{-1}(|\tau_n|)\operatorname{sign}(\tau_n)\big)\varphi_\psi^n(\tau)\,\mathrm{d}\kappa_{\ell_1^n}(\tau)}{\int_{\partial B_{\ell_1^n}} \varphi_\psi^n(\tau)\,\mathrm{d}\kappa_{\ell_1^n}(\tau)}, \quad (6.33)$$

where we recall that φ_ψ^n is defined in (6.29). When $\psi=\psi_\beta$ as in (6.32) for some $\beta>0$ (we will eventually need to work with $\beta\asymp n$), for every $\tau\in\partial B_{\ell_1^n}$ we have

$$\varphi_{\psi_\beta}^n(\tau) = \frac{\sum_{i=1}^n \psi_\beta^{-1}(|\tau_i|)\psi_\beta'\big(\psi_\beta^{-1}(|\tau_i|)\big)}{\prod_{i=1}^n \psi_\beta'\big(\psi_\beta^{-1}(|\tau_i|)\big)} = \frac{\sum_{i=1}^n\big(1-e^{-\beta|\tau_i|}\big)\frac{e^{\beta|\tau_i|}}{\beta}}{\prod_{i=1}^n\frac{e^{\beta|\tau_i|}}{\beta}}$$
$$= \frac{\beta^{n-1}\sum_{i=1}^n\big(e^{\beta|\tau_i|}-1\big)}{e^{\beta\|\tau\|_{\ell_1^n}}} = \frac{\beta^{n-1}}{e^\beta}\sum_{i=1}^n\big(e^{\beta|\tau_i|}-1\big). \quad (6.34)$$

Consequently, (6.33) gives the following identity, which we will need later:

$$\int_{\partial B_{\Omega_\beta^n}} g \,\mathrm{d}\kappa_{\Omega_\beta^n} = \frac{\int_{\partial B_{\ell_1^n}} g\big((e^{\beta|\tau_1|}-1)\operatorname{sign}(\tau_1),\ldots,(e^{\beta|\tau_n|}-1)\operatorname{sign}(\tau_n)\big)\sum_{i=1}^n\big(e^{\beta|\tau_i|}-1\big)\,\mathrm{d}\kappa_{\ell_1^n}(\tau)}{\int_{\partial B_{\ell_1^n}}\sum_{i=1}^n\big(e^{\beta|\tau_i|}-1\big)\,\mathrm{d}\kappa_{\ell_1^n}(\tau)}.$$

Proof of Lemma 154. For each $i\in\{1,\ldots,n\}$ define $f_i:\mathbb{R}^n\to\mathbb{R}$ by setting $f_i(0)=0$ and

$$\forall y\in\mathbb{R}^n\smallsetminus\{0\},\qquad f_i(y)=\|y\|_{\ell_1^n}\psi^{-1}\bigg(\frac{|y_i|}{\|y\|_{\ell_1^n}}\bigg)\operatorname{sign}(y_i).$$

Consider $f=(f_1,\ldots,f_n):\mathbb{R}^n\to\mathbb{R}^n$. Then, $\|f(y)\|_{\ell_\psi^n}=\|y\|_{\ell_1^n}$ for every $y\in\mathbb{R}^n$. Hence, $f(B_{\ell_1^n})=B_{\ell_\psi^n}$. Now,

$$\begin{aligned}\int_{\partial B_{\ell_\psi^n}} g(\theta)\,\mathrm{d}\kappa_{\ell_\psi^n}(\theta) &\overset{(6.2)}{=} \frac{1}{\operatorname{vol}_n\big(B_{\ell_\psi^n}\big)}\int_{f(B_{\ell_1^n})} g\bigg(\frac{1}{\|x\|_{\ell_\psi^n}}x\bigg)\,\mathrm{d}x\\ &= \frac{1}{\operatorname{vol}_n\big(B_{\ell_\psi^n}\big)}\int_{B_{\ell_1^n}} g\bigg(\frac{1}{\|f(y)\|_{\ell_\psi^n}}f(y)\bigg)|\det f'(y)|\,\mathrm{d}y\\ &\overset{(6.2)}{=} \frac{\operatorname{vol}_n\big(B_{\ell_1^n}\big)}{\operatorname{vol}_n\big(B_{\ell_\psi^n}\big)}\int_{\partial B_{\ell_1^n}} g\big(f(\tau)\big)|\det f'(\tau)|\,\mathrm{d}\kappa_{\ell_1^n}(\tau),\end{aligned}$$

where in the final step we used the fact f is positively homogeneous of order 1, and hence its derivative is homogeneous of order 0 almost everywhere (f is continuously differentiable on $\{y\in\mathbb{R}^n;\ y_1,\ldots,y_n\neq 0\}$). Since the volume of the unit ball of ℓ_1^n equals $2^n/n!$, it remains to check that the Jacobian of f satisfies

$$\det f'(\tau)=\frac{\sum_{i=1}^n\psi^{-1}(|\tau_i|)\psi'\big(\psi^{-1}(|\tau_i|)\big)}{\prod_{i=1}^n\psi'\big(\psi^{-1}(|\tau_i|)\big)}=\varphi_\psi^n(\tau),$$

for every $\tau\in\partial B_{\ell_1^n}$ with $\tau_1,\ldots,\tau_n\neq 0$. This indeed holds because for every such τ and $i,j\in\{1,\ldots,n\}$ we have

$$\partial_j f_i(\tau)=\frac{\delta_{ij}-\tau_i\operatorname{sign}(\tau_j)}{\psi'\big(\psi^{-1}(|\tau_i|)\big)}+\psi^{-1}(|\tau_i|)\operatorname{sign}(\tau_i)\operatorname{sign}(\tau_j).$$

Hence, $f'(\tau)=A(\tau)+u(\tau)\otimes v(\tau)$, where $A(\tau)\in\mathsf{M}_n(\mathbb{R})$ is the diagonal matrix $\operatorname{Diag}((1/\psi'(\psi^{-1}(|\tau_i|)))_{i=1}^n)$ and the vectors $u(\tau),v(\tau)\in\mathbb{R}^n$ are defined by setting

$$u(\tau)=\bigg(\psi^{-1}(|\tau_i|)\operatorname{sign}(\tau_i)-\frac{\tau_i}{\psi'\big(\psi^{-1}(|\tau_i|)\big)}\bigg)_{i=1}^n,\qquad v(\tau)=\big(\operatorname{sign}(\tau_i)\big)_{i=1}^n\in\mathbb{R}^n.$$

By the textbook formula for the determinant of a rank-1 perturbation of an invertible matrix (e.g., [214, Section 6.2]), it follows that

$$\det f'(\tau) = \big(1 + \langle A(\tau)^{-1}u(\tau), v(\tau)\rangle\big) \det A(\tau)$$
$$= \frac{1+\sum_{i=1}^n \psi'\big(\psi^{-1}(|\tau_i|)\big)\big(\psi^{-1}(|\tau_i|)\operatorname{sign}(\tau_i)-\frac{\tau_i}{\psi'(\psi^{-1}(|\tau_i|))}\big)\operatorname{sign}(\tau_i)}{\prod_{i=1}^n \psi'\big(\psi^{-1}(|\tau_i|)\big)}$$
$$= \frac{\sum_{i=1}^n \psi^{-1}(|\tau_i|)\psi'\big(\psi^{-1}(|\tau_i|)\big)}{\prod_{i=1}^n \psi'\big(\psi^{-1}(|\tau_i|)\big)}. \qquad \blacksquare$$

Another description of $\kappa_{\mathbf{X}}$ is the fact (see, e.g., [242, Lemma 1]) that the Radon–Nikodým derivative of the $(n-1)$-dimensional Hausdorff (non-normalized surface area) measure on $\partial B_{\mathbf{X}}$ with respect to the (non-normalized cone) measure $\mathrm{vol}_n(B_{\mathbf{X}})\kappa_{\mathbf{X}}$ is equal at almost every $x \in \partial B_{\mathbf{X}}$ to n times the Euclidean length of the gradient at x of the function $u \mapsto \|u\|_{\mathbf{X}}$. In other words, for any $g \in L_1(\partial B_{\mathbf{X}})$,

$$\int_{\partial B_{\mathbf{X}}} g(x)\,\mathrm{d}x = n\,\mathrm{vol}_n(B_{\mathbf{X}}) \int_{\partial B_{\mathbf{X}}} g(x)\big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}\,\mathrm{d}\kappa_{\mathbf{X}}(x). \tag{6.35}$$

The special case $g \equiv 1$ of (6.35) gives the following identity:

$$\frac{\mathrm{vol}_{n-1}(\partial B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})} = n\int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}\,\mathrm{d}\kappa_{\mathbf{X}}(x)$$
$$= \fint_{B_{\mathbf{X}}} \frac{\|\nabla\|\cdot\|_{\mathbf{X}}(x)\|_{\ell_2^n}}{\|x\|_{\mathbf{X}}^{n-1}}\,\mathrm{d}x, \tag{6.36}$$

where the second equality in (6.36) is an application of (6.2) because it is straightforward to check that $\|\nabla\|\cdot\|_{\mathbf{X}}(rx)\|_{\ell_2^n} = \|\nabla\|\cdot\|_{\mathbf{X}}(x)\|_{\ell_2^n}$ for any $r > 0$ and $x \in \mathbb{R}^n$ at which the norm $\|\cdot\|_{\mathbf{X}}$ is smooth.

Remark 155. By applying Cauchy–Schwarz to the first equality in (6.36), we see that

$$\frac{\mathrm{vol}_{n-1}(\partial B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})} \leqslant n\bigg(\int_{\partial B_{\mathbf{X}}} \|\nabla\|\cdot\|_{\mathbf{X}}(x)\|_{\ell_2^n}^2\,\mathrm{d}\kappa_{\mathbf{X}}(x)\bigg)^{\frac12}$$
$$= \bigg(\frac{n}{\mathrm{vol}_n(B_{\mathbf{X}})}\int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}\,\mathrm{d}x\bigg)^{\frac12}, \tag{6.37}$$

where the final step of (6.37) is an application of (6.35) with $g(x) = \|\nabla\|\cdot\|_{\mathbf{X}}(x)\|_{\ell_2^n}$. If $\|\cdot\|_{\mathbf{X}}$ is twice continuously differentiable on $\mathbb{R}^n \smallsetminus \{0\}$ and $\varphi : \mathbb{R} \to [0,\infty)$ is twice continuously differentiable with $\varphi'(1) > 0$ and $\varphi''(0) = 0$, then because for every $x \in \partial B_{\mathbf{X}}$ the vector $\nabla\|\cdot\|_{\mathbf{X}}(x)/\|\nabla\|\cdot\|_{\mathbf{X}}(x)\|_{\ell_2^n}$ is the unit outer normal to $\partial B_{\mathbf{X}}$ at x, by the divergence theorem we have

$$\begin{aligned}\int_{\partial B_{\mathbf{X}}} \Delta\big(\varphi \circ \|\cdot\|_{\mathbf{X}}\big)(x)\,\mathrm{d}x &= \int_{\partial B_{\mathbf{X}}} \operatorname{div}\nabla\big(\varphi \circ \|\cdot\|_{\mathbf{X}}\big)(x)\,\mathrm{d}x \\ &= \int_{\partial B_{\mathbf{X}}} \frac{\langle \nabla(\varphi \circ \|\cdot\|_{\mathbf{X}})(x), \nabla\|\cdot\|_{\mathbf{X}}(x)\rangle}{\big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}}\,\mathrm{d}x \\ &= \int_{\partial B_{\mathbf{X}}} \varphi'\big(\|x\|_{\mathbf{X}}\big)\big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}\,\mathrm{d}x \\ &= \varphi'(1)\int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}(x)\big\|_{\ell_2^n}\,\mathrm{d}x.\end{aligned}$$

A substitution of this identity into (6.37) give the following bound:

$$\frac{\operatorname{vol}_{n-1}(\partial B_{\mathbf{X}})}{\operatorname{vol}_n(B_{\mathbf{X}})} \leqslant \frac{\sqrt{n}}{\sqrt{\varphi'(1)}}\left(\fint_{\partial B_{\mathbf{X}}} \Delta\big(\varphi \circ \|\cdot\|_{\mathbf{X}}\big)(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{6.38}$$

In particular, for every $p > 2$ we have

$$\frac{\operatorname{vol}_{n-1}(\partial B_{\mathbf{X}})}{\operatorname{vol}_n(B_{\mathbf{X}})} \leqslant \sqrt{\frac{n}{p}}\left(\fint_{B_{\mathbf{X}}} \Delta\big(\|\cdot\|_{\mathbf{X}}^p\big)(x)\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{6.39}$$

It is worthwhile to record (6.38) separately because this estimate is sometimes convenient for getting good bounds on $\operatorname{vol}_{n-1}(\partial B_{\mathbf{X}})$. In particular, by using (6.39) when $\mathbf{X}$ is an ℓ_p direct sum one can obtain an alternative derivation of some of the ensuing estimates. Another noteworthy consequence of (6.37) is when there is a transitive subgroup of permutations $G \leqslant S_n$ such that $\|(x_{\pi(1)}, \ldots, x_{\pi(n)})\|_{\mathbf{X}} = \|x\|_{\mathbf{X}}$ for all $x \in \mathbb{R}^n$ and $\pi \in G$. Under this further symmetry assumption, the first inequality of (6.37) becomes

$$\frac{\operatorname{vol}_{n-1}(\partial B_{\mathbf{X}})}{\operatorname{vol}_n(B_{\mathbf{X}})} \leqslant n^{\frac{3}{2}}\left(\int_{\partial B_{\mathbf{X}}} \left(\frac{\partial\|\cdot\|_{\mathbf{X}}}{\partial x_1}(x)\right)^2 \mathrm{d}\kappa_{\mathbf{X}}(x)\right)^{\frac{1}{2}}.$$

The following lemma provides a probabilistic interpretation of the cone measure which generalizes the treatment of the special case $\mathbf{X} = \ell_p^n$ by Schechtman–Zinn [279] and Rachev–Rüschendorf [266].

Lemma 156 (Probabilistic representation of cone measure). *Fix $n \in \mathbb{N}$ and let $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ be a normed space. Suppose that $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(0) = 0$, $\varphi(t) > 0$ when $t > 0$ and $\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r < \infty$. Let V be a random vector in $\mathbb{R}^n$ whose density at each $x \in \mathbb{R}^n$ is equal to*

$$\frac{1}{n\operatorname{vol}_n(B_{\mathbf{X}})\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r}\varphi\big(\|x\|_{\mathbf{X}}\big), \tag{6.40}$$

where we note that (6.40) *in indeed a probability density by* (6.2)*. Then, the density of* $\|\mathsf{V}\|_{\mathbf{X}}$ *at* $s \in [0, \infty)$ *is equal to* $s^{n-1}\varphi(s)/\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r$*. Moreover, the following two assertions hold:*

- $\mathsf{V}/\|\mathsf{V}\|_{\mathbf{X}}$ *is distributed according to the cone measure* $\kappa_{\mathbf{X}}$*,*
- $\|\mathsf{V}\|_{\mathbf{X}}$ *and* $\mathsf{V}/\|\mathsf{V}\|_{\mathbf{X}}$ *are (stochastically) independent.*

Proof. The density of $\|\mathsf{V}\|_{\mathbf{X}}$ at $s \in [0, \infty)$ is equal to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{Prob}\big[\|\mathsf{V}\|_{\mathbf{X}} \leqslant s\big] &\overset{(6.40)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{n\,\mathrm{vol}_n(B_{\mathbf{X}})\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r}\int_{sB_{\mathbf{X}}}\varphi\big(\|x\|_{\mathbf{X}}\big)\,\mathrm{d}x\right)\\ &\overset{(6.2)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\int_0^s t\,r^{n-1}\varphi(r)\,\mathrm{d}r}{\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r}\right)\\ &= \frac{s^{n-1}\varphi(s)}{\int_0^\infty r^{n-1}\varphi(r)\,\mathrm{d}r}.\end{aligned}$$

The rest of Lemma 156 is equivalent to showing that for every measurable $A \subseteq \partial B_{\mathbf{X}}$ and $\rho > 0$,

$$\mathbf{Prob}\left[\frac{\mathsf{V}}{\|\mathsf{V}\|_{\mathbf{X}}} \in A \mid \|\mathsf{V}\|_{\mathbf{X}} = \rho\right] = \kappa_{\mathbf{X}}(A).$$

To prove this identity, observe first that for every $a, b \in \mathbb{R}$ with $a < b$ we have

$$\begin{aligned}\mathrm{vol}_n([a,b]A) &= \mathrm{vol}_n\Big(b\Big(\big([0,1]A\big) \smallsetminus \Big(\frac{a}{b}[0,1]A\Big)\Big)\Big)\\ &= (b^n - a^n)\,\mathrm{vol}_n([0,1]A).\end{aligned}$$

Hence, it follows from the definition (6.1) that

$$\kappa_{\mathbf{X}}(A) = \frac{\mathrm{vol}_n([a,b]A)}{\mathrm{vol}_n([a,b]\partial B_{\mathbf{X}})}. \tag{6.41}$$

Consequently,

$$\begin{aligned}\mathbf{Prob}\left[\frac{\mathsf{V}}{\|\mathsf{V}\|_{\mathbf{X}}} \in A \mid \|\mathsf{V}\|_{\mathbf{X}} = \rho\right] &= \lim_{\varepsilon\to 0}\frac{\mathbf{Prob}[\mathsf{V}\in\|\mathsf{V}\|_{\mathbf{X}}A \text{ and } \rho-\varepsilon \leqslant \|\mathsf{V}\|_{\mathbf{X}} \leqslant \rho+\varepsilon]}{\mathbf{Prob}[\rho-\varepsilon \leqslant \|\mathsf{V}\|_{\mathbf{X}} \leqslant \rho+\varepsilon]}\\ &= \lim_{\varepsilon\to 0}\frac{\int_{([0,\infty)A)\cap([\rho-\varepsilon,\rho+\varepsilon]\partial B_{\mathbf{X}})}\varphi(\|x\|_{\mathbf{X}})\,\mathrm{d}x}{\int_{[\rho-\varepsilon,\rho+\varepsilon]\partial B_{\mathbf{X}}}\varphi(\|x\|_{\mathbf{X}})\,\mathrm{d}x}\\ &= \lim_{\varepsilon\to 0}\frac{\mathrm{vol}_n([\rho-\varepsilon,\rho+\varepsilon]A)}{\mathrm{vol}_n([\rho-\varepsilon,\rho+\varepsilon]\partial B_{\mathbf{X}})}\\ &= \kappa_{\mathbf{X}}(A),\end{aligned}$$

where the penultimate step holds as φ is continuous at ρ and $\varphi(\rho) > 0$, and the final step uses (6.41). ■

Lemma 157. *Fix $m, n \in \mathbb{N}$ and $p \in (1,\infty)$. Suppose that $\mathbf{X} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{X}})$ is a normed space. Let $\mathsf{R}_1,\ldots,\mathsf{R}_n$ be i.i.d. random variables taking values in $[0,\infty)$ whose density at each $t \in (0,\infty)$ is equal to*

$$\frac{p}{2(p-1)\Gamma\big(\frac{m}{p}\big)} t^{\frac{m}{2p-2}-1} e^{-t^{\frac{p}{2p-2}}}. \tag{6.42}$$

Then,

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} = \frac{p\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)} \int_{(\partial B_{\mathbf{X}})^n} \mathbb{E}\left[\left(\sum_{i=1}^n \mathsf{R}_i \|\nabla\|\cdot\|_{\mathbf{X}}(x_i)\|_{\ell_2^m}^2\right)^{\frac12}\right] \mathrm{d}\kappa_{\mathbf{X}}^{\otimes n}(x_1,\ldots,x_n). \tag{6.43}$$

Furthermore,

$$\int_{\partial B_{\ell_p^n(\mathbf{X})}} \|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\|_{\ell_2^n(\ell_2^m)}^2 \,\mathrm{d}\kappa_{\ell_p^n(\mathbf{X})} = \frac{n\Gamma\big(\frac{nm}{p}\big)\Gamma\big(\frac{m+2p-2}{p}\big)}{\Gamma\big(\frac{m}{p}\big)\Gamma\big(\frac{nm+2p-2}{p}\big)} \int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}\big\|_{\ell_2^m}^2 \,\mathrm{d}\kappa_{\mathbf{X}}. \tag{6.44}$$

Proof. For almost every $x = (x_1,\ldots,x_n) \in \ell_p^n(\mathbf{X})$ we have

$$\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}(x) = \frac{1}{\|x\|_{\ell_p^n(\mathbf{X})}^{p-1}}\big(\|x_1\|_{\mathbf{X}}^{p-1}\nabla\|\cdot\|_{\mathbf{X}}(x_1),\ldots,\|x_n\|_{\mathbf{X}}^{p-1}\nabla\|\cdot\|_{\mathbf{X}}(x_n)\big).$$

Consequently,

$$\begin{aligned}
&\|x\|_{\ell_p^n(\mathbf{X})}^{p-1}\Big\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\Big(\frac{x}{\|x\|_{\ell_p^n(\mathbf{X})}}\Big)\Big\|_{\ell_2^n(\ell_2^m)}\\
&= \left(\sum_{i=1}^n \|x_i\|_{\mathbf{X}}^{2p-2}\Big\|\nabla\|\cdot\|_{\mathbf{X}}\Big(\frac{x_i}{\|x\|_{\ell_p^n(\mathbf{X})}}\Big)\Big\|_{\ell_2^m}^2\right)^{\frac12}\\
&= \left(\sum_{i=1}^n \|x_i\|_{\mathbf{X}}^{2p-2}\Big\|\nabla\|\cdot\|_{\mathbf{X}}\Big(\frac{x_i}{\|x_i\|_{\mathbf{X}}}\Big)\Big\|_{\ell_2^m}^2\right)^{\frac12},
\end{aligned} \tag{6.45}$$

where we used the straightforward fact that the gradient of any (finite dimensional) norm is homogeneous of order 0 (on its domain of definition, which is almost everywhere).

Let

$$\mathsf{V} = (\mathsf{V}_1,\ldots,\mathsf{V}_n)$$

be a random vector on $\ell_p^n(\mathbf{X})$ whose density at $x = (x_1, \ldots, x_n) \in \ell_p^n(\mathbf{X})$ is

$$\frac{1}{\Gamma\big(1+\frac{nm}{p}\big)\operatorname{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} e^{-\|x\|_{\ell_p^n(\mathbf{X})}^p} = \frac{1}{\Gamma\big(1+\frac{nm}{p}\big)\operatorname{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} \prod_{i=1}^n e^{-\|x_i\|_{\mathbf{X}}^p}. \tag{6.46}$$

By combining Lemma 156 with the first equality in (6.36), we see that

$$\frac{\operatorname{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\operatorname{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} = nm\mathbb{E}\left[\left\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\left(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\right)\right\|_{\ell_2^n(\ell_2^m)}\right]. \tag{6.47}$$

Also, using the formula from Lemma 156 for the density of $\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}$, for $q > -nm$ we have

$$\mathbb{E}\big[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^q\big] = \frac{\int_0^\infty s^{nm+q-1}e^{-s^p}\,ds}{\int_0^\infty r^{nm-1}e^{-r^p}\,dr} = \frac{\Gamma\big(\frac{nm+q}{p}\big)}{\Gamma\big(\frac{nm}{p}\big)}. \tag{6.48}$$

Consequently,

$$\begin{aligned}
&\mathbb{E}\left[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^{p-1}\left\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\left(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\right)\right\|_{\ell_2^n(\ell_2^m)}\right]\\
&\quad= \mathbb{E}\big[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^{p-1}\big]\mathbb{E}\left[\left\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\left(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\right)\right\|_{\ell_2^n(\ell_2^m)}\right]\\
&\quad= \frac{\Gamma\big(\frac{nm+p-1}{p}\big)}{nm\Gamma\big(\frac{nm}{p}\big)}\cdot\frac{\operatorname{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\operatorname{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)},
\end{aligned} \tag{6.49}$$

where the first step of (6.49) uses the independence of $\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}$ and $\mathsf{V}/\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}$, by Lemma 156, and the final step of (6.49) is a substitution of (6.47) and the case $q = p-1$ of (6.48). Hence,

$$\begin{aligned}
&\frac{\operatorname{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\operatorname{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)}\\
&\quad= \frac{nm\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)}\mathbb{E}\left[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^{p-1}\left\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\left(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\right)\right\|_{\ell_2^n(\ell_2^m)}\right]\\
&\quad= \frac{p\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)}\mathbb{E}\left[\left(\sum_{i=1}^n \|\mathsf{V}_i\|_{\mathbf{X}}^{2p-2}\left\|\nabla\|\cdot\|_{\mathbf{X}}\left(\frac{\mathsf{V}_i}{\|\mathsf{V}_i\|_{\mathbf{X}}}\right)\right\|_{\ell_2^m}^2\right)^{\frac12}\right],
\end{aligned} \tag{6.50}$$

where in the last step we used the identity (6.45).

The product structure of the density of V in (6.46) means that $\mathsf{V}_1, \ldots, \mathsf{V}_n$ are (stochastically) independent. By Lemma 156, for each $i \in \{1, \ldots, n\}$ the random vector $\mathsf{V}_i/\|\mathsf{V}_i\|_{\mathbf{X}}$ is distributed on $\partial B_{\mathbf{X}}$ according to the cone measure $\kappa_{\mathbf{X}}$, and it is independent of the random variable

$$\mathsf{R}_i \stackrel{\text{def}}{=} \|\mathsf{V}_i\|_{\mathbf{X}}^{2p-2}, \tag{6.51}$$

whose density at $t \in (0,\infty)$ is equal (using Lemma 156 once more) to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{Prob}\big[\|\mathsf{V}_i\|_{\mathbf{X}} \leqslant t^{\frac{1}{2p-2}}\big] = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^{t^{\frac{1}{2p-2}}} \frac{s^{m-1}e^{-s^p}}{\int_0^\infty r^{m-1}e^{-r^p}\,\mathrm{d}r}\,\mathrm{d}s$$
$$= \frac{p}{2(p-1)\Gamma\big(\frac{m}{p}\big)} t^{\frac{m}{2p-2}-1}e^{-t^{\frac{p}{2p-2}}}.$$

Hence, the identity (6.50) which we established above coincides with the desired identity (6.43).

To prove the identity (6.44), let R be a random variable whose density at each $t \in (0,\infty)$ is given by (6.42), i.e., $\mathsf{R}_1,\ldots,\mathsf{R}_n$ are independent copies of R. Then, for every $\alpha > -m/(2p-2)$ we have

$$\mathbb{E}\big[\mathsf{R}^\alpha\big] = \frac{p}{2(p-1)\Gamma\big(\frac{m}{p}\big)}\int_0^\infty t^{\frac{m}{2p-2}+\alpha-1}e^{-t^{\frac{p}{2p-2}}}\,\mathrm{d}t = \frac{\Gamma\big(2\alpha+\frac{m-2\alpha}{p}\big)}{\Gamma\big(\frac{m}{p}\big)}. \tag{6.52}$$

Using Lemma 156 (including the independence of $\mathsf{V}_i/\|\mathsf{V}_i\|_{\mathbf{X}}$ and $\|\mathsf{V}_i\|_{\mathbf{X}}$), we have

$$\mathbb{E}\Bigg[\sum_{i=1}^n \|\mathsf{V}_i\|_{\mathbf{X}}^{2p-2}\bigg\|\nabla\|\cdot\|_{\mathbf{X}}\Big(\frac{\mathsf{V}_i}{\|\mathsf{V}_i\|_{\mathbf{X}}}\Big)\bigg\|_{\ell_2^m}^2\Bigg]$$
$$= n\mathbb{E}[\mathsf{R}]\int_{\partial B_{\mathbf{X}}}\big\|\nabla\|\cdot\|_{\mathbf{X}}\big\|_{\ell_2^m}^2\,\mathrm{d}\kappa_{\mathbf{X}}$$
$$= \frac{n\Gamma\big(\frac{m+2p-2}{p}\big)}{\Gamma\big(\frac{m}{p}\big)}\int_{\partial B_{\mathbf{X}}}\big\|\nabla\|\cdot\|_{\mathbf{X}}\big\|_{\ell_2^m}^2\,\mathrm{d}\kappa_{\mathbf{X}}, \tag{6.53}$$

where we recall (6.51) and the last step of (6.53) is the case $\alpha = 1$ of (6.52). At the same time,

$$\mathbb{E}\Bigg[\sum_{i=1}^n \|\mathsf{V}_i\|_{\mathbf{X}}^{2p-2}\bigg\|\nabla\|\cdot\|_{\mathbf{X}}\Big(\frac{\mathsf{V}_i}{\|\mathsf{V}_i\|_{\mathbf{X}}}\Big)\bigg\|_{\ell_2^m}^2\Bigg]$$
$$= \mathbb{E}\Bigg[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^{2p-2}\bigg\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\Big(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\Big)\bigg\|_{\ell_2^n(\ell_2^m)}^2\Bigg]$$
$$= \mathbb{E}\big[\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}^{2p-2}\big]\mathbb{E}\Bigg[\bigg\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\Big(\frac{\mathsf{V}}{\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}}\Big)\bigg\|_{\ell_2^n(\ell_2^m)}^2\Bigg]$$
$$= \frac{\Gamma\big(\frac{nm+2p-2}{p}\big)}{\Gamma\big(\frac{nm}{p}\big)}\int_{\partial B_{\ell_p^n(\mathbf{X})}}\big\|\nabla\|\cdot\|_{\ell_p^n(\mathbf{X})}\big\|_{\ell_2^n(\ell_2^m)}^2\,\mathrm{d}\kappa_{\ell_p^n(\mathbf{X})}, \tag{6.54}$$

where the first step of (6.54) uses the identity (6.45), the second step of (6.54) uses the independence of $\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}$ and $\mathsf{V}/\|\mathsf{V}\|_{\ell_p^n(\mathbf{X})}$ per Lemma 156, and the final step of uses the case $q = 2p - 2$ of (6.48) and Lemma 156. The desired identity (6.44) now follows by substituting (6.54) into (6.53). ∎

The following lemma will have a central role in the proof of Theorems 24 and 48.

Lemma 158. *Suppose that $n, m \in \mathbb{N}$ and $\beta > 0$ satisfy $\beta \leqslant \frac{m-1}{2}$. Then,*

$$\forall 1 \leqslant p \leqslant m, \quad \mathrm{iq}\big(B_{\ell_p^n(\Omega_\beta^m)}\big) \asymp \sqrt{nm} = \sqrt{\dim\big(\ell_p^n(\Omega_\beta^m)\big)}.$$

Recall that the normed space $\Omega_\beta^m = (\mathbb{R}^m, \|\cdot\|_{\Omega_\beta^m})$ was defined in (6.31) and (6.32).

Prior to proving Lemma 158, we will show how it implies Theorem 48, and then deduce Theorem 24.

Proof of Theorem 48 assuming Lemma 158. By the assumption (1.73) of Theorem 48, write $n = km$ for some $k, m \in \mathbb{N}$ with $\max\{2, p\} \leqslant m \leqslant e^p$. Then $(m-1)/2 > 0$ and $m \geqslant p$, so we may apply Lemma 158 with n replaced by k and $\beta = (m-1)/2$. Denoting $\mathbf{Y} = \ell_p^k(\Omega_\beta^m)$, the conclusion of Lemma 158 is that $\mathrm{iq}(B_{\mathbf{Y}}) \asymp \sqrt{n}$.

$\mathbf{Y}$ is canonically positioned (it is a space from Example 40). To prove Theorem 48, it remains to check that $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\ell_p^n}$, where, since $n = km$, we identify $\mathbb{R}^n$ with $\mathsf{M}_{k\times n}(\mathbb{R})$, namely we identify ℓ_p^n with $\ell_p^k(\ell_p^m)$.

In fact, for any $\beta > 0$ (not only our choice $\beta = (m-1)/2$ above) we will check that

$$\forall x \in \mathbb{R}^m, \quad \big(1 - e^{-\frac{\beta}{m}}\big)\|x\|_{\Omega_\beta^m} \leqslant \|x\|_{\ell_\infty^m} \leqslant \|x\|_{\Omega_\beta^m}. \tag{6.55}$$

It follows from (6.55) that $\|\cdot\|_{\Omega_\beta^m} \asymp \|\cdot\|_{\ell_\infty^m}$ when $\beta \asymp m$. But, $\|\cdot\|_{\ell_p^m} \asymp \|\cdot\|_{\ell_\infty^m}$ by the assumption $e^p \geqslant m$. So,

$$\beta \asymp n \implies \|\cdot\|_{\mathbf{Y}} = \|\cdot\|_{\ell_p^k(\Omega_\beta^m)} \asymp \|\cdot\|_{\ell_p^k(\ell_\infty^m)} \asymp \|\cdot\|_{\ell_p^k(\ell_p^m)} = \|\cdot\|_{\ell_p^n}.$$

Fix $x \in \mathbb{R}^m$. To verify the second inequality in (6.55), the definition (6.32) gives $\sum_{i=1}^m \psi_\beta(|x_i|/s) = \infty$ when $0 < s \leqslant \|x\|_{\ell_\infty^m}$, so $\|x\|_{\Omega_\beta^m} \geqslant \|x\|_{\ell_\infty^m}$ by (6.28) and (6.31). For the first inequality in (6.55), by direct differentiation it is elementary to verify that the function $u \mapsto \log(1/(1-u))/u$ is increasing on the interval $[0, 1)$. Thus,

$$0 \leqslant t \leqslant \alpha < 1 \implies \psi_\beta(t) = \frac{1}{\beta}\log\left(\frac{1}{1-t}\right) \leqslant \frac{\log\left(\frac{1}{1-\alpha}\right)}{\alpha\beta} t.$$

Hence, for every fixed $0 < \alpha < 1$,

$$s \geqslant \frac{1}{\alpha}\|x\|_{\ell_\infty^m} \implies \sum_{i=1}^m \psi_\beta\left(\frac{|x_i|}{s}\right) \leqslant \sum_{i=1}^m \frac{\log\left(\frac{1}{1-\alpha}\right)}{\alpha\beta s}|x_i| \leqslant \frac{m\log\left(\frac{1}{1-\alpha}\right)}{\alpha\beta s}\|x\|_{\ell_\infty^m}. \tag{6.56}$$

Provided $\alpha \geqslant 1 - e^{-\beta/m}$, the choice $s = m\log(1/(1-\alpha))\|x\|_{\ell_\infty^m}/(\alpha\beta)$ satisfies the requirement $s \geqslant \|x\|_{\ell_\infty^m}/\alpha$, so we get from (6.28) and (6.56) that

$$\|x\|_{\Omega_\beta^m} \leqslant \frac{m\log\left(\frac{1}{1-\alpha}\right)}{\alpha\beta}\|x\|_{\ell_\infty^m}. \tag{6.57}$$

The optimal choice of α in (6.57) is $\alpha = 1 - e^{-\beta/m}$, giving the first inequality in (6.55). ∎

Having proved Theorem 48 (assuming Lemma 158, which we will soon prove), we have also already established Theorem 24 provided $n \in \mathbb{N}$ and $p \geqslant 1$ satisfy the divisor condition (1.73). Indeed, the space $\mathbf{Y}$ that Theorem 48 provides is canonically positioned and hence by the discussion in Section 1.6.2 it is also in its minimum surface area position, so by [104, Proposition 3.1] we have

$$\frac{\operatorname{MaxProj}(B_{\mathbf{Y}})}{\operatorname{vol}_n(B_{\mathbf{Y}})} \asymp \frac{\operatorname{vol}_{n-1}(\partial B_{\mathbf{Y}})}{\operatorname{vol}_n(B_{\mathbf{Y}})\sqrt{n}} = \left(\frac{\operatorname{iq}(B_{\mathbf{Y}})}{\sqrt{n}}\right)\frac{1}{\operatorname{vol}_n(B_{\mathbf{Y}})^{\frac{1}{n}}} \asymp \frac{1}{\operatorname{vol}_n(B_{\ell_p^n})^{\frac{1}{n}}} \overset{(6.4)}{\asymp} n^{\frac{1}{p}},$$

where the penultimate step uses the fact that $\operatorname{iq}(B_{\mathbf{Y}}) \asymp \sqrt{n}$ by Theorem 48, and also that by Theorem 48 we have $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\ell_p^n}$, which implies that the nth root of the volume of the unit ball of $\mathbf{Y}$ is proportional to the nth root of the volume of the unit ball of ℓ_p^n.

The deduction of Theorem 24 for the remaining values of $p \geqslant 1$ and $n \in \mathbb{N}$ uses the following identity, which we will also use in the proof of Proposition 164 below.

Lemma 159. *Fix $n, m \in \mathbb{N}$. Suppose that $K \subseteq \mathbb{R}^n$ and $L \subseteq \mathbb{R}^m$ are convex bodies. Then,*

$$\frac{\operatorname{MaxProj}(K\times L)}{\operatorname{vol}_{n+m}(K\times L)} = \left(\frac{\operatorname{MaxProj}(K)^2}{\operatorname{vol}_n(K)^2} + \frac{\operatorname{MaxProj}(L)^2}{\operatorname{vol}_m(L)^2}\right)^{\frac{1}{2}}.$$

Proof. Fix $z \in S^{n+m-1}$. By Cauchy's projection formula [102] that we recalled in (1.30), we have

$$\operatorname{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big) = \frac{1}{2}\int_{\partial(K\times L)} \big|\langle z, N_{K\times L}(w)\rangle\big|\,\mathrm{d}w,$$

where $N_{K\times L}(w)$ is the (almost-everywhere defined) unit outer normal to $\partial(K\times L)$ at $w \in \partial(K\times L)$. Now,

$$\partial(K\times L) = (\partial K\times L)\cup(K\times\partial L) \quad \text{and} \quad \operatorname{vol}_{n+m-1}\big((\partial K\times L)\cap(K\times\partial L)\big) = 0.$$

Consequently,

$$\begin{aligned}&\operatorname{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big)\\ &\quad= \frac{1}{2}\int_{\partial K\times L}\big|\langle z, N_{K\times L}(w)\rangle\big|\,\mathrm{d}w + \frac{1}{2}\int_{K\times\partial L}\big|\langle z, N_{K\times L}(w)\rangle\big|\,\mathrm{d}w.\end{aligned}$$

If we write each $w \in \mathbb{R}^n$ as $w = (w_1, w_2)$ where $w_1 \in \mathbb{R}^n$ and $w_2 \in \mathbb{R}^m$, then for almost every (with respect to the $(n+m-1)$-dimensional Hausdorff measure) $w \in \partial K \times L$ we have $N_{K\times L}(w) = (N_K(w_1), 0)$. Also, $N_{K\times L}(w) = (0, N_L(w_2))$ for almost every $w \in K \times \partial L$. We therefore have

$$\begin{aligned}
&\mathrm{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big)\\
&\quad = \frac{\mathrm{vol}_m(L)}{2}\int_{\partial K}\big|\langle z_1, N_K(x)\rangle\big|\,\mathrm{d}x + \frac{\mathrm{vol}_n(K)}{2}\int_{\partial L}\big|\langle z_2, N_L(y)\rangle\big|\,\mathrm{d}y\\
&\quad = \mathrm{vol}_m(L)\,\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z_1^\perp}K\big)\|z_1\|_{\ell_2^n} + \mathrm{vol}_n(K)\,\mathrm{vol}_{m-1}\big(\mathsf{Proj}_{z_2^\perp}L\big)\|z_2\|_{\ell_2^m},
\end{aligned}$$

where the last step is two applications of the Cauchy projection formula (in $\mathbb{R}^n$ and $\mathbb{R}^m$). Hence,

$$\begin{aligned}
&\frac{\mathrm{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big)}{\mathrm{vol}_{n+m}(K\times L)}\\
&\quad = \frac{\mathrm{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big)}{\mathrm{vol}_n(K)\,\mathrm{vol}_m(L)}\\
&\quad = \frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{z_1^\perp}K\big)}{\mathrm{vol}_n(K)}\|z_1\|_{\ell_2^n} + \frac{\mathrm{vol}_{m-1}\big(\mathsf{Proj}_{z_2^\perp}L\big)}{\mathrm{vol}_m(L)}\|z_2\|_{\ell_2^m}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{\mathrm{MaxProj}(K\times L)}{\mathrm{vol}_{n+m}(K\times L)}\\
&\quad = \max_{z\in S^{n+m-1}} \frac{\mathrm{vol}_{n+m-1}\big(\mathsf{Proj}_{z^\perp}(K\times L)\big)}{\mathrm{vol}_{n+m}(K\times L)}\\
&\quad = \max_{(u,v)\in S^1}\max_{x\in S^{n-1}}\max_{y\in S^{m-1}} \frac{\mathrm{vol}_{n+m-1}\big(\mathsf{Proj}_{(ux+vy)^\perp}(K\times L)\big)}{\mathrm{vol}_{n+m}(K\times L)}\\
&\quad = \max_{(u,v)\in S^1}\max_{x\in S^{n-1}}\max_{y\in S^{m-1}} \left(\frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp}K\big)}{\mathrm{vol}_n(K)}|u| + \frac{\mathrm{vol}_{m-1}\big(\mathsf{Proj}_{y^\perp}L\big)}{\mathrm{vol}_m(L)}|v|\right)\\
&\quad = \max_{(u,v)\in S^1}\left(\frac{\mathrm{MaxProj}(K)}{\mathrm{vol}_n(K)}|u| + \frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_m(L)}|v|\right)\\
&\quad = \left(\frac{\mathrm{MaxProj}(K)^2}{\mathrm{vol}_n(K)^2} + \frac{\mathrm{MaxProj}(L)^2}{\mathrm{vol}_m(L)^2}\right)^{\frac{1}{2}}. \qquad \blacksquare
\end{aligned}$$

We can now prove Theorem 24 in its full generality using the fact that we proved Theorem 48.

Proof of Theorem 24. Let $m \in \mathbb{N}$ satisfy $\max\{2, p\} \leqslant m \leqslant e^p$ (if $1 \leqslant p \leqslant 2$, then take $m = 2$, and if $p \geqslant 2$, then such an m exists because $e^p - p \geqslant e^2 - 2 > 5$). Write $n = km + r$ for some $k \in \mathbb{N}\cup\{0\}$ and $r \in \{0, \ldots, m-1\}$. If $r = 0$, then m divides n

and we can conclude by applying Theorem 48 as we did above (recall the paragraph immediately before Lemma 159). So, assume from now that $r \geqslant 1$.

By Theorem 48 there is a canonically positioned normed space $\mathbf{Y} = (\mathbb{R}^{km}, \|\cdot\|_{\mathbf{Y}})$ such that $\mathrm{iq}(B_{\mathbf{Y}}) \asymp \sqrt{km}$ and $\|\cdot\|_{\mathbf{Y}} \asymp \|\cdot\|_{\ell_p^{km}}$. Define $\mathbf{Y}_p^n = \mathbf{Y} \oplus_\infty \Omega_\beta^r$, where $\beta \asymp r$ and $\mathrm{iq}(\Omega_\beta^r) \asymp \sqrt{r}$; such β exists trivially if $r = 1$, and if $r \geqslant 2$, then its existence follows from an application of Lemma 158 (with the choices $n = 1$ and $p = m = r$).

Since $\beta \asymp r$, by (6.55) we have $\|\cdot\|_{\Omega_\beta^r} \asymp \|\cdot\|_{\ell_\infty^r}$. Also, $\|\cdot\|_{\ell_\infty^r} \asymp \|\cdot\|_{\ell_p^r}$ since $e^p \geqslant m > r$. Consequently, for every $(x, y) \in \mathbb{R}^{km} \times \mathbb{R}^r$ we have

$$\max\{\|x\|_{\mathbf{Y}}, \|y\|_{\Omega_\beta^r}\} \asymp \max\{\|x\|_{\ell_p^{km}}, \|y\|_{\ell_p^r}\} \asymp \left(\|x\|_{\ell_p^{km}}^p + \|y\|_{\ell_p^r}^p\right)^{\frac{1}{p}}.$$

Recalling the definition of $\mathbf{Y}_p^n$, this means that $\|\cdot\|_{\mathbf{Y}_p^n} \asymp \|\cdot\|_{\ell_p^n}$.

Since both $\mathbf{Y}$ and Ω_β^r are canonically positioned and hence in their minimum surface area positions,

$$\frac{\mathrm{MaxProj}(B_{\mathbf{Y}})}{\mathrm{vol}_{km}(B_{\mathbf{Y}})} \asymp \Big(\frac{\mathrm{iq}(B_{\mathbf{Y}})}{\sqrt{km}}\Big)\frac{1}{\mathrm{vol}_{km}(B_{\mathbf{Y}})^{\frac{1}{km}}} \asymp \frac{1}{\mathrm{vol}_{km}\big(B_{\ell_p^{km}}\big)^{\frac{1}{km}}} \asymp (km)^{\frac{1}{p}}$$

and

$$\frac{\mathrm{MaxProj}\big(B_{\Omega_\beta^r}\big)}{\mathrm{vol}_r\big(B_{\Omega_\beta^r}\big)} \asymp \left(\frac{\mathrm{iq}\big(\Omega_\beta^r\big)}{\sqrt{r}}\right)\frac{1}{\mathrm{vol}\big(\Omega_\beta^r\big)^{\frac{1}{r}}} \asymp \frac{1}{\mathrm{vol}\big(\ell_\infty^r\big)^{\frac{1}{r}}} \asymp 1 \asymp r^{\frac{1}{p}}.$$

Consequently, since $B_{\mathbf{Y}_p^n} = B_{\mathbf{Y}} \times B_{\Omega_\beta^r}$, by Lemma 159 we conclude that

$$\begin{aligned}\frac{\mathrm{MaxProj}\big(B_{\mathbf{Y}_p^n}\big)}{\mathrm{vol}_n\big(B_{\mathbf{Y}_p^n}\big)} &= \left(\frac{\mathrm{MaxProj}(B_{\mathbf{Y}})^2}{\mathrm{vol}_{km}(B_{\mathbf{Y}})^2} + \frac{\mathrm{MaxProj}\big(B_{\Omega_\beta^r}\big)^2}{\mathrm{vol}_r\big(B_{\Omega_\beta^r}\big)^2}\right)^{\frac{1}{2}} \\ &\asymp \left((km)^{\frac{2}{p}} + r^{\frac{2}{p}}\right)^{\frac{1}{2}} \asymp (km + r)^{\frac{1}{p}} = n^{\frac{1}{p}}.\end{aligned}$$ ■

The following lemma will be used in the proof of Lemma 158.

Lemma 160. *Suppose that* $m \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$ *and* $\beta > 0$ *satisfy* $\beta \leqslant \frac{m+r-2}{2}$. *Then*

$$\begin{aligned}\int_{\partial B_{\ell_1^m}} \left(e^{\beta|\tau_1|} - \sum_{k=r-1}^{\infty} \frac{\beta^k |\tau_1|^k}{k!}\right) \mathrm{d}\kappa_{\ell_1^m}(\tau) &= \int_{\partial B_{\ell_1^m}} \left(\sum_{k=r}^{\infty} \frac{\beta^k |\tau_1|^k}{k!}\right) \mathrm{d}\kappa_{\ell_1^m}(\tau) \\ &\asymp \frac{\beta^r (m-1)!}{(m+r-1)!}.\end{aligned} \tag{6.58}$$

Proof. Let $\mathsf{H}_1, \ldots, \mathsf{H}_m$ be independent random variables whose density at each $s \in \mathbb{R}$ is equal to $e^{-|s|}/2$. Then, $|\mathsf{H}_1|, \ldots, |H_m|$ are exponential random variables of rate 1, and therefore if we denote

$$\Gamma \stackrel{\mathrm{def}}{=} \sum_{i=1}^{m} |\mathsf{H}_i|,$$

then Γ has $\Gamma(m,1)$ distribution, i.e., its density at $s \geqslant 0$ equals $s^{m-1}e^{-s}/(m-1)!$; the proof of this standard probabilistic fact can be found in, e.g., [89]. By [266, 279] (or Lemma 156), the random vector $(\mathsf{H}_1,\ldots,\mathsf{H}_m)/\Gamma$ is distributed according to $\kappa_{\ell_1^m}$ and is independent of Γ. Thus, for every $k \in \mathbb{N}$,

$$\begin{aligned}\int_{\partial B_{\ell_1^m}} |\tau_1|^k \,\mathrm{d}\kappa_{\ell_1^m}(\tau) &= \mathbb{E}\left[\frac{|\mathsf{H}_1|^k}{\Gamma^k}\right] = \frac{\mathbb{E}\big[|\mathsf{H}_1|^k\big]}{\mathbb{E}\big[\Gamma^k\big]} \\ &= \frac{\int_0^\infty s^k e^{-s}\,\mathrm{d}s}{\frac{1}{(m-1)!}\int_0^\infty s^{k+m-1}e^{-s}\,\mathrm{d}s} \\ &= \frac{k!(m-1)!}{(k+m-1)!}.\end{aligned}$$

Consequently,

$$\begin{aligned}\int_{\partial B_{\ell_1^m}} \left(\sum_{k=r}^{\infty}\frac{\beta^k|\tau_1|^k}{k!}\right)\mathrm{d}\kappa_{\ell_1^m}(\tau) &= \frac{(m-1)!}{\beta^{m-1}}\sum_{k=r}^{\infty}\frac{\beta^{k+m-1}}{(k+m-1)!} \\ &= \frac{\beta^r(m-1)!}{(m+r-2)!}\int_0^1 e^{\beta t}(1-t)^{m+r-2}\,\mathrm{d}t, \end{aligned} \tag{6.59}$$

where the last step is the integral form of the remainder of the Taylor series of the exponential function.

It is mechanical to check that (6.58) holds for $m \in \{1,2\}$, so assume for the rest of the proof of Lemma 160 that $m \geqslant 3$. We then see from (6.59) that our goal (6.58) is equivalent to showing that

$$\int_0^1 e^{\beta t}(1-t)^{m+r-2}\,\mathrm{d}t \asymp \frac{1}{m+r}. \tag{6.60}$$

For the upper bound in (6.60), estimate the integrand using

$$(1-t)^{m+r-2} \leqslant e^{-(m+r-2)t}$$

to get

$$\begin{aligned}\int_0^1 e^{\beta t}(1-t)^{m+r-2}\,\mathrm{d}t &\leqslant \int_0^1 e^{-(m+r-2-\beta)t}\,\mathrm{d}t \\ &= \frac{1-e^{-(m+r-2-\beta)}}{m+r-2-\beta} \asymp \frac{1}{m+r},\end{aligned}$$

where we used $\beta < \frac{m+r-2}{2}$. For the lower bound in (6.60), since $(1-t)^{m+r-2} \gtrsim 1$ when $0 \leqslant t \leqslant \frac{1}{m+r-2}$,

$$\int_0^1 e^{\beta t}(1-t)^{m+r-2}\,\mathrm{d}t \geqslant \int_0^{\frac{1}{m+r-2}} e^{\beta t}(1-t)^{m+r-2}\,\mathrm{d}t$$
$$\gtrsim \int_0^{\frac{1}{m+r-2}} e^{\beta t}\,\mathrm{d}t = \frac{e^{\frac{\beta}{m+r-2}}-1}{\beta} \asymp \frac{1}{m+r},$$

where in the last step we used the assumption $\beta < \frac{m+r-2}{2}$ once more. ■

Proof of Lemma 158. By combining the case $g \equiv 1$ of (6.30) with (6.34), we see that

$$\mathrm{vol}_m\big(B_{\Omega_\beta^m}\big) = \frac{\beta^{m-1}2^m}{e^\beta m!} m \int_{\partial B_{\ell_1^m}} \big(e^{\beta|\tau_1|}-1\big)\,\mathrm{d}\kappa_{\ell_1^m}(\tau) \overset{(6.58)}{\asymp} \frac{(2\beta)^m}{e^\beta m!}. \tag{6.61}$$

Since we are assuming in Lemma 158 that $\beta \lesssim m$, in combination with (6.4) we get from (6.61) that

$$\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^m)}\big)^{\frac{1}{nm}} \asymp \frac{\beta}{n^{\frac{1}{p}}m}, \tag{6.62}$$

At the same time, by applying Cauchy–Schwarz to the identity (6.43) of Lemma 157 we have

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\Omega_\beta^m)}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^m)}\big)}$$
$$\leqslant \frac{p\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)}\bigg(n\big(\mathbb{E}[\mathsf{R}_1]\big)\int_{\partial B_{\Omega_\beta^m}} \big\|\nabla\|\cdot\|_{\Omega_\beta^m}(\theta)\big\|_{\ell_2^m}^2\,\mathrm{d}\kappa_{\Omega_\beta^m}(\theta)\bigg)^{\frac{1}{2}}$$
$$\asymp n^{\frac{1}{p}+\frac{1}{2}} m\bigg(\int_{\partial B_{\Omega_\beta^m}} \big\|\nabla\|\cdot\|_{\Omega_\beta^m}(\theta)\big\|_{\ell_2^m}^2\,\mathrm{d}\kappa_{\Omega_\beta^m}(\theta)\bigg)^{\frac{1}{2}}, \tag{6.63}$$

where the random variable R_1 is as in Lemma 157, i.e., its density is in (6.42), and the last step is an application the evaluation (6.52) of its moments and Stirling's formula, using the assumption $1 \leqslant p \leqslant m$.

Recalling (6.31) and (6.32), even though $\|\cdot\|_{\Omega_\beta^m}$ is defined implicitly by (6.28), we can compute $\nabla\|\cdot\|_{\Omega_\beta^m}(\theta)$ for almost every $\theta \in \partial B_{\Omega_\beta^m}$ as the unique vector $v \in \mathbb{R}^m$ that is normal to $\partial B_{\Omega_\beta^m}$ and satisfies $\langle v, \theta\rangle = 1$. Indeed, since $\partial\Omega_\beta^m$ is parameterized as the zero set of the function $\Psi_\beta : \mathbb{R}^n \to \mathbb{R}^n$ that is given by

$$\forall x \in \mathbb{R}^n, \quad \Psi_\beta(x) \stackrel{\mathrm{def}}{=} 1 - \sum_{i=1}^m \psi_\beta(|x_i|),$$

the following vector is normal to $\partial B_{\Omega_\beta^m}$ for almost every $\theta \in \partial B_{\Omega_\beta^m}$:

$$v_\beta(\theta) \stackrel{\text{def}}{=} \nabla \Psi_\beta(\theta) = -\big(\psi_\beta'(|\theta_1|)\,\mathrm{sign}(\theta_1), \ldots, \psi_\beta'(|\theta_m|)\,\mathrm{sign}(\theta_m)\big).$$

So, $\nabla\|\cdot\|_{\Omega_\beta^m}(\theta) = \lambda_\beta(\theta) v_\beta(\theta)$ for almost every $\theta \in \partial B_{\Omega_\beta^m}$, where $\lambda_\beta(\theta) \in \mathbb{R}$ is such that $\langle \lambda_\beta(\theta) v_\beta(\theta), \theta\rangle = 1$, i.e., $\lambda_\beta(\theta) = -1/\langle v_\beta(\theta), \theta\rangle$. This shows that for almost every $\theta \in \partial B_{\Omega_\beta^m}$,

$$\begin{aligned} \nabla\|\cdot\|_{\Omega_\beta^m}(\theta) &= \frac{1}{\sum_{i=1}^m |\theta_i|\psi_\beta'(|\theta_i|)}\big(\psi_\beta'(|\theta_1|)\,\mathrm{sign}(\theta_1), \ldots, \psi_\beta'(|\theta_m|)\,\mathrm{sign}(\theta_m)\big) \\ &= \frac{1}{\sum_{i=1}^m \frac{|\theta_i|}{1-|\theta_i|}}\left(\frac{\mathrm{sign}(\theta_1)}{1-|\theta_1|}, \ldots, \frac{\mathrm{sign}(\theta_m)}{1-|\theta_m|}\right), \end{aligned} \tag{6.64}$$

where the first equality in (6.64) holds for any ψ_β that satisfies the conditions of Lemma 154, and for the second equality in (6.64) recall the definition (6.32) of the specific ψ_β that we are using here. Therefore,

$$\begin{aligned} \int_{\partial B_{\Omega_\beta^m}} \big\|\nabla\|\cdot\|_{\Omega_\beta^m}(\theta)\big\|_{\ell_2^m}^2 \,\mathrm{d}\kappa_{\Omega_\beta^m}(\theta) &= \frac{\int_{\partial B_{\ell_1^m}} \frac{\sum_{i=1}^m e^{2\beta|\tau_i|}}{\sum_{i=1}^m (e^{\beta|\tau_i|}-1)}\,\mathrm{d}\kappa_{\ell_1^m}(\tau)}{m\int_{\partial B_{\ell_1^m}}\big(e^{\beta|\tau_1|}-1\big)\,\mathrm{d}\kappa_{\ell_1^m}(\tau)} \\ &\leqslant \frac{\int_{\partial B_{\ell_1^m}} \frac{\sum_{i=1}^m e^{2\beta|\tau_i|}}{\beta\sum_{i=1}^m |\tau_i|}\,\mathrm{d}\kappa_{\ell_1^m}(\tau)}{m\int_{\partial B_{\ell_1^m}}\big(e^{\beta|\tau_1|}-1\big)\,\mathrm{d}\kappa_{\ell_1^m}(\tau)} \\ &= \frac{\int_{\partial B_{\ell_1^m}} e^{2\beta|\tau_1|}\,\mathrm{d}\kappa_{\ell_1^m}(\tau)}{\beta\int_{\partial B_{\ell_1^m}}\big(e^{\beta|\tau_1|}-1\big)\,\mathrm{d}\kappa_{\ell_1^m}(\tau)} \\ &\asymp \frac{m}{\beta^2}, \end{aligned} \tag{6.65}$$

where the first step of (6.65) is a substitution of (6.64) into (6.33) while using (6.34) and that $\psi_\beta^{-1}(t) = 1 - e^{-\beta t}$ for every $t \geqslant 0$, the second step of (6.65) uses the inequality $e^t \geqslant t+1$ which holds for any $t \in \mathbb{R}$, and the final step of (6.65) is an application of Lemma 160. Now, a combination of (6.63) and (6.65) gives

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\Omega_\beta^m)}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^m)}\big)} \lesssim \frac{n^{\frac{1}{p}+\frac{1}{2}} m^{\frac{3}{2}}}{\beta}. \tag{6.66}$$

By combining (6.62) and (6.66) we conclude that

$$\mathrm{iq}\big(B_{\ell_p^n(\Omega_\beta^m)}\big) = \frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\Omega_\beta^m)}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^m)}\big)}\,\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^n)}\big)^{\frac{1}{nm}} \lesssim \sqrt{nm}.$$

The reverse inequality, namely $\mathrm{iq}(B_{\ell_p^n(\Omega_\beta^m)}) \gtrsim \sqrt{nm}$, follows from the isoperimetric theorem (1.12), so the proof of Lemma 158 is complete. Note that this also shows that all of the inequalities that we derived in the above proof of Lemma 158 are in fact asymptotic equivalences. This holds in particular for (6.66), i.e.,

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\Omega_\beta^m)}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\Omega_\beta^m)}\big)} \asymp \frac{n^{\frac{1}{p}+\frac{1}{2}}m^{\frac{3}{2}}}{\beta}. \qquad \blacksquare$$

The following asymptotic evaluation of the surface area of the sphere of $\ell_p^n(\ell_q^m)$ in the entire range of possible values of $p,q \geqslant 1$ and $m,n \in \mathbb{N}$ is an application of Lemma 157; by (6.7) it is equivalent to (1.82).

Theorem 161. *For every $n,m \in \mathbb{N}$ and $p,q \in [1,\infty]$ we have*

$$\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\ell_q^m)}\big)$$
$$\asymp \frac{2^{nm}\Gamma\big(1+\frac{1}{q}\big)^{nm}\Gamma\big(1+\frac{m}{p}\big)^{n}}{\Gamma\big(1+\frac{m}{q}\big)^{n}\Gamma\big(1+\frac{nm}{p}\big)} \cdot \begin{cases} n^{1+\frac{1}{p}}m^{1+\frac{1}{q}} & m \leqslant \min\{\frac{p}{n},q\}, \\ \sqrt{q}n^{1+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & q \leqslant m \leqslant \frac{p}{n}, \\ \sqrt{p}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & \frac{p}{n} \leqslant m \leqslant \min\{p,q\}, \\ \sqrt{pq}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{q}} & \max\{\frac{p}{n},q\} \leqslant m \leqslant p, \\ n^{\frac{1}{2}+\frac{1}{p}}m^{1+\frac{1}{q}} & p \leqslant m \leqslant q, \\ \sqrt{q}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & m \geqslant \max\{p,q\}. \end{cases}$$

Proof. By continuity we may assume that $p,q \in (1,\infty)$. Suppose that G is a symmetric real-valued random variable whose density at each $s \in \mathbb{R}$ is equal to

$$\frac{1}{2\Gamma\big(1+\frac{1}{q}\big)}e^{-|s|^q}. \tag{6.67}$$

Let $\mathsf{G}_1,\ldots,\mathsf{G}_m$ be independent copies of G. Set $\mathsf{U} \stackrel{\mathrm{def}}{=} (\mathsf{G}_1,\ldots,\mathsf{G}_m) \in \mathbb{R}^m$. By the probabilistic representation of the cone measure on $\partial B_{\ell_q^m}$ in [266, 279] (or Lemma 156), the random vector $\mathsf{U}/\|\mathsf{U}\|_{\ell_q^m}$ is distributed according to the cone measure on $\partial B_{\ell_q^m}$, and moreover it is independent of $\|\mathsf{U}\|_{\ell_q^m}$.

Consider the following random variable:

$$\mathsf{N} \stackrel{\mathrm{def}}{=} \left\|\nabla\|\cdot\|_{\ell_q^m}\left(\frac{\mathsf{U}}{\|\mathsf{U}\|_{\ell_q^m}}\right)\right\|_{\ell_2^m}^2 = \frac{1}{\|\mathsf{U}\|_{\ell_q^m}^{2q-2}}\sum_{j=1}^m |\mathsf{G}_j|^{2q-2} = \frac{\|\mathsf{U}\|_{\ell_{2q-2}^m}^{2q-2}}{\|\mathsf{U}\|_{\ell_q^m}^{2q-2}}. \tag{6.68}$$

If we let $\mathsf{N}_1,\ldots,\mathsf{N}_n,\mathsf{R}_1,\ldots,\mathsf{R}_n$ be independent random variables such that $\mathsf{N}_1,\ldots,\mathsf{N}_n$ have the same distribution as N, and $\mathsf{R}_1,\ldots,\mathsf{R}_n$ are as in Lemma 157, then Lemma 157 gives that

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\ell_q^m)}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\ell_q^m)}\big)} = \frac{p\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)}\mathbb{E}[\mathsf{Z}] \asymp pn^{\frac{1}{p}}m^{\frac{1}{p}}\mathbb{E}[\mathsf{Z}], \tag{6.69}$$

where for (6.69) we introduce the following notation:

$$\mathsf{Z} \stackrel{\mathrm{def}}{=} \left(\sum_{i=1}^{n} \mathsf{R}_i \mathsf{N}_i \right)^{\frac{1}{2}}. \tag{6.70}$$

Let R be a random variable that takes values in $[0, \infty)$ whose density at each $t \in (0, \infty)$ is given by (6.42), i.e., $\mathsf{R}_1, \ldots, \mathsf{R}_n$ are independent copies of R. We computed the moments of R in (6.52) and by Stirling's formula this gives the following asymptotic evaluations:

$$\mathbb{E}\big[\mathsf{R}^{\frac{1}{2}}\big] \asymp \frac{m^{1-\frac{1}{p}}}{p}, \tag{6.71}$$

$$\mathbb{E}[\mathsf{R}] \asymp \max\left\{\frac{m}{p}, 1\right\} \frac{m^{1-\frac{2}{p}}}{p}, \tag{6.72}$$

$$\mathbb{E}\big[\mathsf{R}^2\big] \asymp \max\left\{\frac{m^3}{p^3}, 1\right\} \frac{m^{1-\frac{4}{p}}}{p}. \tag{6.73}$$

We also need an analogous asymptotic evaluation of moments of the random variable N in (6.68). Observe that the random variables N and $\|\mathsf{U}\|_{\ell_q^m}$ are independent, since $\mathsf{U}/\|\mathsf{U}\|_{\ell_q^m}$ and $\|\mathsf{U}\|_{\ell_q^m}$ are independent and N is a function $\mathsf{U}/\|\mathsf{U}\|_{\ell_q^m}$. Consequently, for every $\beta > 0$ we have

$$\mathbb{E}\big[\|\mathsf{U}\|_{\ell_q^m}^{(2q-2)\beta}\big]\mathbb{E}\big[\mathsf{N}^\beta\big] = \mathbb{E}\big[\|\mathsf{U}\|_{\ell_q^m}^{(2q-2)\beta}\mathsf{N}^\beta\big] \stackrel{(6.68)}{=} \mathbb{E}\big[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{(2q-2)\beta}\big]. \tag{6.74}$$

Since (e.g., by Lemma 156) the density of $\|\mathsf{U}\|_{\ell_q^m}$ at $s \in (0, \infty)$ is proportional to $s^{m-1}e^{-s^q}$, we can compute analogously to (6.48) that

$$\mathbb{E}\big[\|\mathsf{U}\|_{\ell_q^m}^{(2q-2)\beta}\big] = \frac{\int_0^\infty s^{m-1+(2q-2)\beta} e^{-s^q}\,\mathrm{d}s}{\int_0^\infty r^{m-1}e^{-r^q}\,\mathrm{d}r} = \frac{\Gamma\big(2\beta + \frac{m-2\beta}{q}\big)}{\Gamma\big(\frac{m}{q}\big)}.$$

Therefore, (6.74) implies that

$$\mathbb{E}\big[\mathsf{N}^\beta\big] = \frac{\Gamma\big(\frac{m}{q}\big)}{\Gamma\big(2\beta + \frac{m-2\beta}{q}\big)}\mathbb{E}\big[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{(2q-2)\beta}\big].$$

By considering each of the values $\beta \in \{\frac{1}{2}, 1, 2\}$ in this identity and using Stirling's formula, we get the following asymptotic evaluations of moments of N in terms of moments of $\|\mathsf{U}\|_{\ell_{2q-2}^m}$:

$$\mathbb{E}\big[\mathsf{N}^{\frac{1}{2}}\big] \asymp \frac{q}{m^{1-\frac{1}{q}}}\mathbb{E}\big[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\big], \tag{6.75}$$

$$\mathbb{E}[\mathsf{N}] \asymp \min\left\{\frac{q}{m}, 1\right\} \frac{q}{m^{1-\frac{2}{q}}} \mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{2q-2}\right], \tag{6.76}$$

$$\mathbb{E}\left[\mathsf{N}^2\right] \asymp \min\left\{\frac{q^3}{m^3}, 1\right\} \frac{q}{m^{1-\frac{4}{q}}} \mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{4q-4}\right]. \tag{6.77}$$

Due to (6.75), (6.76), (6.77), we will next evaluate the corresponding moments of $\|\mathsf{U}\|_{\ell_{2q-2}^m}$. Recalling the density (6.67) of G, for every $\beta > -1/(2q-2)$ we have

$$\mathbb{E}\left[|\mathsf{G}|^{(2q-2)\beta}\right] = \frac{1}{\Gamma\left(1+\frac{1}{q}\right)} \int_0^\infty s^{(2q-2)\beta} e^{-s^q}\, ds = \frac{\Gamma\left(\frac{2q-2}{q}\beta + \frac{1}{q}\right)}{q\Gamma\left(1+\frac{1}{q}\right)}.$$

Hence,

$$\mathbb{E}\left[|\mathsf{G}|^{q-1}\right] \asymp \mathbb{E}\left[|\mathsf{G}|^{2q-2}\right] \asymp \mathbb{E}\left[|\mathsf{G}|^{4q-4}\right] \asymp \frac{1}{q}. \tag{6.78}$$

We therefore have

$$\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{2q-2}\right] = m\mathbb{E}\left[|\mathsf{G}|^{2q-2}\right] \overset{(6.78)}{\asymp} \frac{m}{q} \tag{6.79}$$

and

$$\begin{aligned}
\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{4q-4}\right] &= \mathbb{E}\left[\left(\sum_{j=1}^m |\mathsf{G}_j|^{2q-2}\right)^2\right] \\
&= m\mathbb{E}\left[|\mathsf{G}|^{4q-4}\right] + m(m-1)\left(\mathbb{E}\left[|\mathsf{G}|^{2q-2}\right]\right)^2 \\
&\overset{(6.78)}{\asymp} \max\left\{\frac{m}{q}, 1\right\} \frac{m}{q}.
\end{aligned} \tag{6.80}$$

Consequently, using Hölder's inequality we get the following estimate:

$$\begin{aligned}
\frac{m}{q} &\overset{(6.79)}{\asymp} \mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{2q-2}\right] \\
&= \mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{\frac{2}{3}(q-1)} \|\mathsf{U}\|_{\ell_{2q-2}^m}^{\frac{1}{3}(4q-4)}\right] \\
&\leqslant \left(\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\right]\right)^{\frac{2}{3}} \left(\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{4q-4}\right]\right)^{\frac{1}{3}} \\
&\overset{(6.80)}{\asymp} \left(\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\right]\right)^{\frac{2}{3}} \left(\max\left\{\frac{m}{q}, 1\right\} \frac{m}{q}\right)^{\frac{1}{3}}.
\end{aligned} \tag{6.81}$$

This simplifies to give

$$\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\right] \gtrsim \min\left\{\sqrt{\frac{m}{q}}, \frac{m}{q}\right\}. \tag{6.82}$$

At the same time, by Cauchy–Schwarz,

$$\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\right] \leqslant \left(\mathbb{E}\left[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{2q-2}\right]\right)^{\frac{1}{2}} \overset{(6.79)}{\asymp} \sqrt{\frac{m}{q}}. \tag{6.83}$$

Also, by the subadditivity of the square root on $[0,\infty)$,

$$\mathbb{E}\Big[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\Big] = \mathbb{E}\left[\left(\sum_{j=1}^m |\mathsf{G}_j|^{2q-2}\right)^{\frac{1}{2}}\right] \leqslant \mathbb{E}\left[\sum_{j=1}^m |\mathsf{G}_j|^{q-1}\right]$$
$$= m\mathbb{E}\big[|\mathsf{G}|^{q-1}\big] \overset{(6.78)}{\asymp} \frac{m}{q}. \tag{6.84}$$

By combining (6.83) and (6.84) we see that (6.82) is in fact sharp, i.e.,

$$\mathbb{E}\Big[\|\mathsf{U}\|_{\ell_{2q-2}^m}^{q-1}\Big] \asymp \min\left\{\sqrt{\frac{m}{q}}, \frac{m}{q}\right\}. \tag{6.85}$$

By substituting (6.85) into (6.75), and correspondingly (6.79) into (6.76) and (6.80) into (6.77), we get the following asymptotic identities:

$$\mathbb{E}\big[\mathsf{N}^{\frac{1}{2}}\big] \asymp \min\left\{\sqrt{\frac{q}{m}}, 1\right\} m^{\frac{1}{q}}, \tag{6.86}$$

$$\mathbb{E}[\mathsf{N}] \asymp \min\left\{\frac{q}{m}, 1\right\} m^{\frac{2}{q}}, \tag{6.87}$$

$$\mathbb{E}\big[\mathsf{N}^2\big] \asymp \min\left\{\frac{q^2}{m^2}, 1\right\} m^{\frac{4}{q}}. \tag{6.88}$$

By combining (6.72) and (6.87) we see that

$$\mathbb{E}\big[\mathsf{Z}^2\big] = n\big(\mathbb{E}[\mathsf{R}]\big)\big(\mathbb{E}[\mathsf{N}]\big) \asymp \frac{\max\{m,p\}\min\{q,m\}}{p^2} n m^{\frac{2}{q}-\frac{2}{p}}.$$

Using Cauchy–Schwarz, this implies the following upper bound on the final term in (6.69):

$$pn^{\frac{1}{p}} m^{\frac{1}{p}} \mathbb{E}[\mathsf{Z}] \leqslant pn^{\frac{1}{p}} m^{\frac{1}{p}} \big(\mathbb{E}\big[\mathsf{Z}^2\big]\big)^{\frac{1}{2}}$$
$$\asymp n^{\frac{1}{2}+\frac{1}{p}} m^{\frac{1}{q}} \sqrt{\max\{m,p\}\min\{m,q\}}. \tag{6.89}$$

Also, recalling (6.70) and using the subadditivity of the square root on $[0,\infty)$ in combination with (6.71) and (6.86), we have the following additional upper bound on the final term in (6.69):

$$pn^{\frac{1}{p}} m^{\frac{1}{p}} \mathbb{E}[\mathsf{Z}] \leqslant pn^{\frac{1}{p}} m^{\frac{1}{p}} \mathbb{E}\left[\sum_{i=1}^n \mathsf{R}_i^{\frac{1}{2}} \mathsf{N}_i^{\frac{1}{2}}\right]$$
$$= pn^{1+\frac{1}{p}} m^{\frac{1}{p}} \big(\mathbb{E}\big[\mathsf{R}^{\frac{1}{2}}\big]\big)\big(\mathbb{E}\big[\mathsf{N}^{\frac{1}{2}}\big]\big)$$
$$\asymp n^{1+\frac{1}{p}} m^{\frac{1}{2}+\frac{1}{q}} \sqrt{\min\{m,q\}}. \tag{6.90}$$

It follows from (6.89) and (6.90) that

$$pn^{\frac{1}{p}}m^{\frac{1}{p}}\mathbb{E}[\mathsf{Z}] \lesssim n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{q}}\sqrt{\min\{m,q\}}\min\left\{\sqrt{nm},\sqrt{\max\{m,p\}}\right\}$$
$$= \begin{cases} n^{1+\frac{1}{p}}m^{1+\frac{1}{q}} & m \leqslant \min\{\frac{p}{n},q\},\\ \sqrt{q}n^{1+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & q \leqslant m \leqslant \frac{p}{n},\\ \sqrt{p}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & \frac{p}{n} \leqslant m \leqslant \min\{p,q\},\\ \sqrt{pq}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{q}} & \max\{\frac{p}{n},q\} \leqslant m \leqslant p,\\ n^{\frac{1}{2}+\frac{1}{p}}m^{1+\frac{1}{q}} & p \leqslant m \leqslant q,\\ \sqrt{q}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & m \geqslant \max\{p,q\}. \end{cases} \tag{6.91}$$

We will next prove that (6.91) is optimal in all of the six ranges that appear in (6.91); by (6.69) and (6.6), this will complete the proof of Theorem 161. Recalling (6.70) and using (6.72), (6.73), (6.87), (6.88), the fourth moment of Z can be evaluated (up to universal constant factors) as follows:

$$\begin{aligned}\mathbb{E}\big[\mathsf{Z}^4\big] &= \mathbb{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{R}_i\mathsf{R}_j\mathsf{N}_i\mathsf{N}_j\right]\\ &= n\big(\mathbb{E}\big[\mathsf{R}^2\big]\big)\big(\mathbb{E}\big[\mathsf{N}^2\big]\big) + n(n-1)\big(\mathbb{E}[\mathsf{R}]\big)^2\big(\mathbb{E}[\mathsf{N}]\big)^2\\ &\asymp \frac{(\max\{m,p\})^3(\min\{m,q\})^2}{p^4}nm^{\frac{4}{q}-\frac{4}{p}-1}\\ &\quad+\frac{(\max\{m,p\}\min\{m,q\})^2}{p^4}n^2m^{\frac{4}{q}-\frac{4}{p}}\\ &\asymp \frac{(\max\{m,p\}\min\{m,q\})^2\max\{nm,p\}}{p^4}nm^{\frac{4}{q}-\frac{4}{p}-1}.\end{aligned} \tag{6.92}$$

By using Hölder's inequality similarly to (6.81), we conclude that

$$\begin{aligned}pn^{\frac{1}{p}}m^{\frac{1}{p}}\mathbb{E}[\mathsf{Z}] &\geqslant pn^{\frac{1}{p}}m^{\frac{1}{p}}\frac{\big(\mathbb{E}\big[\mathsf{Z}^2\big]\big)^{\frac{3}{2}}}{\big(\mathbb{E}\big[\mathsf{Z}^4\big]\big)^{\frac{1}{2}}}\\ &\overset{(6.89)\wedge(6.92)}{\asymp} n^{1+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}}\frac{\sqrt{\max\{m,p\}\min\{m,q\}}}{\sqrt{\max\{nm,p\}}}\\ &= \begin{cases} n^{1+\frac{1}{p}}m^{1+\frac{1}{q}} & m \leqslant \min\{\frac{p}{n},q\},\\ \sqrt{q}n^{1+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & q \leqslant m \leqslant \frac{p}{n},\\ \sqrt{p}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & \frac{p}{n} \leqslant m \leqslant \min\{p,q\},\\ \sqrt{pq}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{q}} & \max\{\frac{p}{n},q\} \leqslant m \leqslant p,\\ n^{\frac{1}{2}+\frac{1}{p}}m^{1+\frac{1}{q}} & p \leqslant m \leqslant q,\\ \sqrt{q}n^{\frac{1}{2}+\frac{1}{p}}m^{\frac{1}{2}+\frac{1}{q}} & m \geqslant \max\{p,q\}. \end{cases}\end{aligned}$$

■

Lemma 162 below applies Theorem 161 iteratively to obtain an upper bound on the surface area of the unit sphere of nested ℓ_p norms on k-tensors (the case $k=2$ corresponds to n by m matrices equipped with the $\ell_p^n(\ell_q^m)$ norm). The second part of Lemma 162, namely the conclusion (6.94) below, is an implementation of the approach towards Conjecture 9 for the hypercube that we described in Remark 56.

Lemma 162. *Suppose that $k, n_1, \ldots, n_k \in \mathbb{N}$ and $p_1, \ldots, p_k \in [1,\infty]$ are such that $n_1 \geqslant \max\{3, p_1 - 2\}$ and $n_1 n_2 \cdots n_{j-1} \geqslant p_j - 2$ for every $j \in \{2, \ldots, k\}$. Define normed spaces $\mathbf{Y}_0, \mathbf{Y}_1, \ldots, \mathbf{Y}_k$ by setting $\mathbf{Y}_0 = \mathbb{R}$ and inductively $\mathbf{Y}_j = \ell_{p_j}^{n_j}(\mathbf{Y}_{j-1})$ for $j \in \{1, \ldots, k\}$. Then,*

$$\frac{\mathrm{vol}_{n_1\cdots n_k - 1}\big(\partial B_{\mathbf{Y}_k}\big)}{\mathrm{vol}_{n_1\cdots n_k}\big(B_{\mathbf{Y}_k}\big)} \leqslant e^{O(k)} \sqrt{p_1} \prod_{j=1}^{k} n_j^{\frac{1}{2}+\frac{1}{p_j}}. \tag{6.93}$$

Hence, using the natural identification of the vector space that underlies $\mathbf{Y}_k$ with $\mathbb{R}^{\dim(\mathbf{Y}_k)} = \mathbb{R}^{n_1 n_2 \cdots n_k}$, if in addition we have $n_1 = O(1)$ and $p_j = \log n_j$ for every $j \in \{1, \ldots, k\}$, then

$$B_{\mathbf{Y}_k} \subseteq B_{\ell_\infty^{\dim(\mathbf{Y}_k)}} \subseteq e^{O(k)} B_{\mathbf{Y}_k} \quad \textit{and} \quad \frac{\mathrm{MaxProj}\big(B_{\mathbf{Y}_k}\big)}{\mathrm{vol}_{\dim(\mathbf{Y}_k)}\big(B_{\mathbf{Y}_k}\big)} \leqslant e^{O(k)}, \tag{6.94}$$

where we recall the notation (1.53).

Proof. Suppose that $n, m \in \mathbb{N}$ and $p \in (1,\infty)$. By applying Cauchy–Schwarz to the right-hand side of (6.43) while using the case $\alpha = 1$ of (6.52), we see that for every normed space $\mathbf{X} = (\mathbb{R}^m, \|\cdot\|_{\mathbf{X}})$ we have

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} \leqslant \frac{p\Gamma\big(1+\frac{nm}{p}\big)}{\Gamma\big(1+\frac{nm-1}{p}\big)} \left(\frac{n\Gamma\big(\frac{m+2p-2}{p}\big)}{\Gamma\big(\frac{m}{p}\big)} \int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}\big\|_{\ell_2^m}^2 \,\mathrm{d}\kappa_{\mathbf{X}}\right)^{\frac{1}{2}}. \tag{6.95}$$

If also $m \geqslant \max\{3, p-2\}$, then by Stirling's formula (6.95) gives the following estimate:

$$\frac{\mathrm{vol}_{nm-1}\big(\partial B_{\ell_p^n(\mathbf{X})}\big)}{\mathrm{vol}_{nm}\big(B_{\ell_p^n(\mathbf{X})}\big)} \lesssim n^{\frac{1}{2}+\frac{1}{p}} m \left(\int_{\partial B_{\mathbf{X}}} \big\|\nabla\|\cdot\|_{\mathbf{X}}\big\|_{\ell_2^m}^2 \,\mathrm{d}\kappa_{\mathbf{X}}\right)^{\frac{1}{2}}. \tag{6.96}$$

By continuity we may assume that $p_1, \ldots, p_k \in (1,\infty)$. Denote $d_0 = 1$ and for $j \in \{1, \ldots, k\}$ denote $d_j = \dim(\mathbf{Y}_j) = n_1 n_2 \cdots n_j$. We will naturally identify $\mathbf{Y}_j$ with $(\mathbb{R}^{d_j}, \|\cdot\|_{\mathbf{Y}_j})$. As $\mathbf{Y}_k = \ell_{p_k}^{n_k}(\mathbf{Y}_{k-1})$, we deduce from (6.96) that

$$\frac{\mathrm{vol}_{d_k-1}\big(\partial B_{\mathbf{Y}_k}\big)}{\mathrm{vol}_{d_k}\big(B_{\mathbf{Y}_k}\big)} \lesssim n_k^{\frac{1}{2}+\frac{1}{p_k}} \left(\prod_{j=1}^{k-1} n_j\right) \left(\int_{\partial B_{\mathbf{Y}_{k-1}}} \big\|\nabla\|\cdot\|_{\mathbf{Y}_{k-1}}\big\|_{\ell_2^{d_{k-1}}}^2 \,\mathrm{d}\kappa_{\mathbf{Y}_{k-1}}\right)^{\frac{1}{2}}. \tag{6.97}$$

At the same time, by (6.44) for every $j \in \{1, \ldots, k\}$ we have

$$\int_{\partial B_{\mathbf{Y}_j}} \|\nabla\| \cdot \|_{\mathbf{Y}_j}\|^2_{\ell_2^{d_j}} \, \mathrm{d}\kappa_{\mathbf{Y}_j} = \frac{n_j \Gamma\left(\frac{d_j}{p_j}\right)\Gamma\left(\frac{d_{j-1}+2p_j-2}{p_j}\right)}{\Gamma\left(\frac{d_{j-1}}{p_j}\right)\Gamma\left(\frac{d_j+2p_j-2}{p_j}\right)} \int_{\partial B_{\mathbf{Y}_{j-1}}} \|\nabla\| \cdot \|_{\mathbf{Y}_{j-1}}\|^2_{\ell_2^{d_{j-1}}} \, \mathrm{d}\kappa_{\mathbf{Y}_{j-1}}. \quad (6.98)$$

If also $j \geqslant 2$, then $d_{j-1} \geqslant n_1 \geqslant 3$ and by assumption $d_{j-1} \geqslant p_j - 2$, so by Stirling's formula (6.98) gives that for every $j \in \{2, \ldots, k\}$ we have

$$\int_{\partial B_{\mathbf{Y}_j}} \|\nabla\| \cdot \|_{\mathbf{Y}_j}\|^2_{\ell_2^{d_j}} \, \mathrm{d}\kappa_{\mathbf{Y}_j} \asymp n_j^{\frac{2}{p_j}-1} \int_{\partial B_{\mathbf{Y}_{j-1}}} \|\nabla\| \cdot \|_{\mathbf{Y}_{j-1}}\|^2_{\ell_2^{d_{j-1}}} \, \mathrm{d}\kappa_{\mathbf{Y}_{j-1}}. \quad (6.99)$$

When $j = 1$ we have $d_0 = 1$ and $n_1 \geqslant \max\{3, p_1 - 2\}$, and therefore by Stirling's formula (6.98) gives

$$\int_{\partial B_{\mathbf{Y}_1}} \|\nabla\| \cdot \|_{\mathbf{Y}_1}\|^2_{\ell_2^{d_1}} \, \mathrm{d}\kappa_{\mathbf{Y}_1} \asymp p_1 n_1^{\frac{2}{p_1}-1}. \quad (6.100)$$

Hence, by applying (6.99) iteratively in combination with the base case (6.100), we conclude that

$$\int_{\partial B_{\mathbf{Y}_{k-1}}} \|\nabla\| \cdot \|_{\mathbf{Y}_{k-1}}\|^2_{\ell_2^{d_{k-1}}} \, \mathrm{d}\kappa_{\mathbf{Y}_{k-1}} \leqslant e^{O(k)} p_1 \prod_{j=1}^{k-1} n_j^{\frac{2}{p_j}-1}. \quad (6.101)$$

A substitution of (6.101) into (6.97) yields the desired estimate (6.93).

To deduce the conclusion (6.94), note that for every $j \in \{1, \ldots, k\}$ we have the point-wise bounds

$$\|\cdot\|_{\ell_\infty^{n_j}(\mathbf{Y}_{j-1})} \leqslant \|\cdot\|_{\mathbf{Y}_j} = \|\cdot\|_{\ell_{p_j}^{n_j}(\mathbf{Y}_{j-1})} \leqslant n_j^{\frac{1}{p_j}} \|\cdot\|_{\ell_\infty^{n_j}(\mathbf{Y}_{j-1})}.$$

It follows by induction that

$$\|\cdot\|_{\ell_\infty^{d_k}} \leqslant \|\cdot\|_{\mathbf{Y}_k} \leqslant \left(\prod_{j=1}^{k} n_j^{\frac{1}{p_j}}\right) \|\cdot\|_{\ell_\infty^{d_k}} = e^{O(k)} \|\cdot\|_{\ell_\infty^{d_k}},$$

where the final step holds if $p_j = \log n_j$ for every $j \in \{1, \ldots, k\}$. This implies the inclusions in (6.94). Furthermore, $\mathbf{Y}_k$ belongs to the class of spaces from Example 40. Hence $\mathbf{Y}_k$ is canonically positioned and by the discussion in Section 1.6.2 know that $B_{\mathbf{Y}'}$ is in its minimum surface area position. Therefore,

$$\frac{\mathrm{MaxProj}\left(B_{\mathbf{Y}_k}\right)}{\mathrm{vol}_{d_k}\left(B_{\mathbf{Y}_k}\right)} \asymp \frac{\mathrm{vol}_{d_k-1}\left(\partial B_{\mathbf{Y}_k}\right)}{\mathrm{vol}_{d_k}\left(B_{\mathbf{Y}_k}\right)\sqrt{d_k}} \leqslant e^{O(k)} \sqrt{p_1} \prod_{j=1}^{k} n_j^{\frac{1}{p_j}} \asymp e^{O(k)},$$

where the first step uses [104, Proposition 3.1], the second step is (6.93), and the final step holds because $p_1 = O(1)$ and $p_j = \log n_j$. This completes the proof of (6.94). ∎

The following technical lemma replaces a more ad-hoc argument that we previously had to deduce Proposition 164 below from Lemma 162; it is due to Noga Alon and we thank him for allowing us to include it here. This lemma shows that the set of super-lacunary products $n_1 n_2 \cdots n_k$ that can serve as dimensions of the space $\mathbf{Y}_k$ in Lemma 162 for which (6.94) holds is quite dense in $\mathbb{N}$.

Lemma 163. *For every integer $n \geqslant 3$ there are $k, m \in \mathbb{N} \cup \{0\}$ and integers $n_1 < n_2 < \cdots < n_k$ that satisfy*

- $n = n_1 n_2 \cdots n_k + m$,
- $n_1 \in \{6,7\}$ *and* $n_{i+1} \lesssim 2^{n_i} \lesssim n_{i+1}^3$ *for every* $i \in \{1, \ldots, k-1\}$,
- $m \lesssim (\log n)^{1+o(1)}$.

Prior to proving Lemma 163, we will make some preparatory (mechanical) observations for ease of later reference. Note first that the conclusion $n_{i+1} \lesssim 2^{n_i} \lesssim n_{i+1}^3$ of Lemma 163 can be rewritten as

$$\forall i \in \{1, \ldots, k-1\}, \quad \log_2 n_{i+1} \lesssim n_i \lesssim \log_{\sqrt[3]{2}} n_{i+1}.$$

It follows by induction that

$$\forall i \in \{1, \ldots, k\}, \quad \log_2^{[k-i]} n_k \lesssim n_i \lesssim \log_{\sqrt[3]{2}}^{[k-i]} n_k, \tag{6.102}$$

where, as we also did in (1.131), we denote the iterates of a function $\varphi : (0,\infty) \to \mathbb{R}$ by $\varphi^{[j]} = \varphi \circ \varphi^{[j-1]} : (\varphi^{[j-1]})^{-1}(0,\infty) \to \mathbb{R}$ for each $j \in \mathbb{N}$, with the convention $\varphi^{[0]}(x) = x$ for every $x \in (0,\infty)$. Since $n_1 \in \{6,7\}$, it follows from (6.102) that

$$k \asymp \log^* n_k \lesssim \log^* n. \tag{6.103}$$

Consequently,

$$\begin{aligned} n_k \log n_k \asymp n_k n_{k-1} \lesssim \prod_{i=1}^{k} n_k \lesssim n = m + \prod_{i=1}^{k} n_k \lesssim (\log n)^{1+o(1)} + \prod_{i=1}^{k} \log_{\sqrt[3]{2}}^{[k-i]} n_k \\ \lesssim (\log n)^2 + n_k (\log n_k)(\log\log n_k)^{O(\log^* n_k)} \lesssim (\log n)^2 + n_k (\log n_k)^2. \end{aligned}$$

This implies the following (quite crude) bounds on n_k:

$$\frac{n}{(\log n)^2} \lesssim n_k \lesssim \frac{n}{\log n}. \tag{6.104}$$

Note in particular that thanks to (6.104) we know that (6.103) can be improved to $k \asymp \log^* n$.

Proof of Lemma 163. Let $\mathbb{M} \subseteq \mathbb{N}$ be the set of all those $x \in \mathbb{N}$ that can be written as $x = n_1 n_2 \cdots n_k$ for some $k, n_1, \ldots, n_k \in \mathbb{N}$ that satisfy $n_k > n_{k-1} > \cdots > n_1 \in \{6, 7\}$ and

$$\forall i \in \{1, \ldots, k-1\}, \quad n_{i+1} \leqslant 2^{n_i} \leqslant n_{i+1}^3. \tag{6.105}$$

The goal of Lemma 163 is to show that there exists $x \in \mathbb{M}$ such that

$$n - (\log n)^{1+o(1)} \leqslant x \leqslant n. \tag{6.106}$$

By adjusting the $o(1)$ term, we may assume that n is sufficiently large, say, $n \geqslant n(0)$ for some fixed $n(0) \in \mathbb{N}$ that will be determined later. We will then find $x \in \mathbb{M}$ with a representation $x = n_1 n_2 \cdots n_k$ as above and

$$n - n_1 n_2 \cdots n_{k-1} \leqslant x \leqslant n. \tag{6.107}$$

This would imply the desired bound (6.106) because

$$\prod_{i=1}^{k-1} n_i \overset{(6.102)}{\leqslant} \prod_{i=1}^{k-1} \log_{\sqrt[3]{2}}^{[k-i]} n_k \overset{(6.103)}{\lesssim} (\log n_k)^{1+o(1)} \overset{(6.104)}{\lesssim} (\log n)^{1+o(1)}.$$

We will first construct $\{y_i\}_{i=1}^{\infty} \subseteq \mathbb{M}$ such that $y_1 = 7$ and $y_i < y_{i+1} < 12 y_i$ for every $i \in \mathbb{N}$. Furthermore, for each $i \in \mathbb{N}$ there are $k, n_1, \ldots, n_k \in \mathbb{N}$ with $y_i = n_1 n_2 \cdots n_k$ such that $n_k > n_{k-1} > \cdots > n_1 \in \{6, 7\}$ and

$$\forall j \in \{1, \ldots, k-1\}, \quad n_{j+1}^2 \leqslant 2^{n_j} \leqslant 2 n_{j+1}^2, \tag{6.108}$$

which is a more stringent requirement than (6.105). Note in passing that (6.108) implies the (crude) bound

$$\prod_{j=1}^{k} \left(1 + \frac{1}{n_j}\right) \leqslant 2. \tag{6.109}$$

To verify (6.109), note that since $\{n_j\}_{j=1}^{k}$ is strictly increasing and the second inequality in (6.108) holds, it is mechanical to check that $n_1 \geqslant 6$, $n_2 \geqslant 7$, $n_3 \geqslant 8$, $n_4 \geqslant 12$ and $n_{j+1} \geqslant 3 n_j$ for every $j \in \{4, 5, \ldots, k-1\}$. So,

$$\begin{aligned} \prod_{j=1}^{k} \left(1 + \frac{1}{n_j}\right) &\leqslant \left(1 + \frac{1}{6}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{8}\right) e^{\sum_{s=0}^{\infty} \frac{1}{12 \cdot 3^s}} \\ &= \left(1 + \frac{1}{6}\right)\left(1 + \frac{1}{7}\right)\left(1 + \frac{1}{8}\right) e^{\frac{1}{8}} \leqslant 2. \end{aligned}$$

Suppose that y_i has been defined with a representation $y_i = n_1 n_2 \cdots n_k$ that fulfils the above requirements. Define $m_0, m_1, \ldots, m_k \in \mathbb{N}$ with $m_0 = 6$, $m_k = n_k + 1$ and $m_j \in \{n_j, n_j + 1\}$ for all $j \in \{1, \ldots, k-1\}$ by induction as follows. Assuming that

m_{j+1} has already been constructed for some $j \in \{1, \ldots, k-1\}$, let

$$m_j \stackrel{\text{def}}{=} \begin{cases} n_j & \text{if } m_{j+1}^2 \leqslant 2^{n_j}, \\ n_j + 1 & \text{if } m_{j+1}^2 > 2^{n_j}. \end{cases} \tag{6.110}$$

Definition (6.110) implies that $m_j < m_{j+1}$. Indeed, $n_j < n_{j+1}$ so if $m_j = n_j$, then $n_j < n_{j+1} \leqslant m_{j+1}$ since $m_{j+1} \geqslant n_{j+1}$ by the induction hypothesis. On the other hand, if $m_j = n_j + 1$, then since the first inequality in (6.108) holds, the definition (6.110) necessitates that $m_{j+1} = n_j + 1$, so $m_j < m_{j+1}$ in this case as well.

Next, Definition (6.110) also ensures that the requirement (6.108) is inherited by $\{m_j\}_{j=1}^k$, i.e.,

$$\forall\, j \in \{1, \ldots, k-1\}, \quad m_{j+1}^2 \leqslant 2^{m_j} \leqslant 2m_{j+1}^2. \tag{6.111}$$

Indeed, if $m_j = n_j$, then $m_{j+1}^2 \leqslant 2^{n_j} = 2^{m_j}$ by (6.110), i.e., the first inequality in (6.111) holds, and the second inequality in (6.111) holds because $m_{j+1} \geqslant n_{j+1}$ and (6.108) holds. On the other hand, if $m_j = n_j + 1$, then by (6.110) we have $m_{j+1} = n_j + 1$ and $m_{j+1}^2 > 2^{n_j}$, which directly gives the second inequality in (6.111), and in combination with (6.108) we also get the first inequality in (6.111) because

$$\frac{m_{j+1}}{2^{m_j}} = \frac{(n_j+1)^2}{2^{n_j+1}} \stackrel{(6.108)}{\leqslant} \frac{(n_j+1)^2}{2n_j^2} \leqslant 1,$$

where the final step uses $n_j \geqslant 6$, though $n_j \geqslant 1/(\sqrt{2}-1) = 2.414\ldots$ is all that is needed for this purpose.

If the above construction produces $m_1 \in \{6, 7\}$, then define $y_{i+1} = m_1 m_2 \cdots m_k$. Otherwise necessarily $m_1 = n_1 + 1 = 8$, so (6.111) holds also when $j = 0$ (recall that $m_0 = 6$, hence $m_1^2 = 2^6 = 2^{m_0}$), so we can define $y_{i+1} = m_0 m_1 \cdots m_k$ and thanks to (6.111) in both cases y_{i+1} has the desired form. Moreover,

$$\frac{y_{i+1}}{y_i} \leqslant 6 \prod_{j=1}^{k} \left(1 + \frac{1}{n_j}\right) \stackrel{(6.109)}{\leqslant} 12.$$

This completes the inductive construction of the desired sequence $\{y_i\}_{i=1}^\infty \subseteq \mathbb{M}$.

With the sequence $\{y_i\}_{i=1}^\infty \subseteq \mathbb{M}$ at hand, will next explain how to obtain for each integer $n \geqslant n(0)$, where $n(0) \in \mathbb{N}$ is a sufficiently large universal constant that is yet to be determined, an element $x \in \mathbb{M}$ that approximates n as in (6.107). Let $i \in \mathbb{N}$ be such that $y_i \leqslant n \leqslant y_{i+1}$ and denote $y = y_i$. Thus, there are $k, n_1, \ldots, n_k \in \mathbb{N}$ for which $y = n_1 n_2 \cdots n_k$ such that $n_k > n_{k-1} > \cdots > n_1 \in \{6, 7\}$ and (6.108) holds.

If $y \geqslant n - n_1 n_2 \cdots n_{k-1}$, then $x = y$ has the desired approximation property, so suppose from now that $y < n - n_1 n_2 \cdots n_{k-1}$, or equivalently

$$\frac{n}{n_1 n_2 \cdots n_{k-1}} > \frac{y}{n_1 n_2 \cdots n_{k-1}} + 1 = n_k + 1.$$

Hence, if we define

$$n'_k \stackrel{\text{def}}{=} \left\lfloor \frac{n}{n_1 n_2, \ldots, n_{k-1}} \right\rfloor \quad \text{and} \quad x = n_1 n_2 \cdots n_{k-1} n'_k,$$

then $n'_k \geqslant n_k + 1 \gtrsim n/(\log n)^2$, where we used (6.104). Consequently, recalling (6.102), there is a universal constant $n(0) \in \mathbb{N}$ such that if $n \geqslant n(0)$, then $n'_k > \max\{144, n_{k-1}\}$. Thus, the sequence $n_1, n_2, \ldots, n_{k-1}, n'_k$ is still increasing. Since by design x satisfies (6.107), it remains to check that $x \in \mathbb{M}$, i.e., that (6.105) holds. Since $n_1, \ldots, n_k$ are assumed to satisfy the more stringent requirement (6.108), we only need to check that

$$n'_k \leqslant 2^{n_{k-1}} \leqslant (n'_k)^3. \tag{6.112}$$

The second inequality in (6.112) is valid since (6.108) holds and $n'_k > n_k$. To justify the first inequality in (6.112), observe that $y \leqslant n \leqslant 12y$, as $y_{i+1} \leqslant 12 y_i$. Consequently,

$$n'_k \leqslant n/(n_1 n_2 \cdots n_{k-1}) \leqslant 12y/(n_1 n_2 \cdots n_{k-1}) = 12 n_k.$$

Therefore,

$$2^{n_{k-1}} \overset{(6.108)}{\geqslant} n_k^2 \geqslant \left(\frac{n'_k}{12}\right)^2 > n'_k,$$

where the last step uses the fact that $n'_k > 144$. ∎

We are now ready to extend the conclusion (6.94) of Lemma 162 to all dimensions $n \in \mathbb{N}$. Namely, we will prove the following proposition, which comes very close to proving Conjecture 9 for the hypercube $[-1,1]^n$ via a route that differs from the way by which we proved Theorem 24.

Proposition 164. *For any $n \in \mathbb{N}$ there is a normed space $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ that for every $x \in \mathbb{R}^n \smallsetminus \{0\}$ we have*

$$\|x\|_{\ell_\infty^n} \leqslant \|x\|_{\mathbf{Y}} \leqslant e^{O(\log^* n)} \|x\|_{\ell_\infty^n} \quad \textit{and} \quad \frac{\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp} B_{\mathbf{Y}}\big)}{\mathrm{vol}_n(B_{\mathbf{Y}})} \leqslant e^{O(\log^* n)}.$$

Furthermore, $\mathbf{Y}$ can be taken to be an ℓ_∞ direct sum of nested ℓ_p spaces as in Lemma 162.

Proof. Let $\mathbb{M} \subseteq \mathbb{N}$ be the set of integers from the proof of Lemma 163, namely $m \in \mathbb{M}$ if and only if there are integers $n_k > n_{k-1} > \cdots > n_1 \in \{6,7\}$ that satisfy (6.105) such that $m = n_1 n_2 \cdots n_k$. By Lemma 162, there exists $C > 1$ such that for every $m \in \mathbb{M}$ there is a normed space $\mathbf{Y}^m = (\mathbb{R}^m, \|\cdot\|_{\mathbf{Y}^m})$ that satisfies

$$\|\cdot\|_{\ell_\infty^m} \leqslant \|\cdot\|_{\mathbf{Y}^m} \leqslant e^{C\log^* m} \|\cdot\|_{\ell_\infty^m} \quad \text{and} \quad \frac{\mathrm{MaxProj}\big(B_{\mathbf{Y}^m}\big)}{\mathrm{vol}_n\big(B_{\mathbf{Y}^m}\big)} \leqslant e^{C\log^* m}.$$

By applying Lemma 163 iteratively write $n = m_1 + \cdots + m_{s+1}$ for $m_1, \ldots, m_s \in \mathbb{M}$ and $m_{s+1} \in \{1, 2\}$ that satisfy $m_{i+1} \leqslant (\log m_i)^c$ for every $i \in \{1, \ldots, s\}$, where $c > 1$ is a universal constant. Denote $\mathbf{Y}^{m_{s+1}} = \ell_\infty^{m_{s+1}}$ and consider the ℓ_∞ direct sum

$$\mathbf{Y} \stackrel{\mathrm{def}}{=} \mathbf{Y}^{m_1} \oplus_\infty \mathbf{Y}^{m_2} \oplus_\infty \cdots \oplus_\infty \mathbf{Y}^{m_{s+1}} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}}).$$

Then $\|\cdot\|_{\ell_\infty^n} \leqslant \|\cdot\|_{\mathbf{Y}} \leqslant \max_{i \in \{1,\ldots,s+1\}} e^{C \log^* m_i} \|\cdot\|_{\ell_\infty^{m_i}} \leqslant e^{C \log^* n} \|\cdot\|_{\ell_\infty^n}$. We claim that

$$\frac{\mathrm{MaxProj}(B_{\mathbf{Y}})}{\mathrm{vol}_n(B_{\mathbf{Y}})} \leqslant e^{O(\log^* n)}.$$

Since $B_{\mathbf{Y}} = B_{\mathbf{Y}^{m_1}} \times B_{\mathbf{Y}^{m_2}} \times \cdots \times B_{\mathbf{Y}^{m_{s+1}}}$, by an inductive application of Lemma 159 we have

$$\frac{\mathrm{MaxProj}\big(B_{\mathbf{Y}}\big)}{\mathrm{vol}_n\big(B_{\mathbf{Y}}\big)} \leqslant \left(\sum_{i=1}^{s+1} \frac{\mathrm{MaxProj}\big(B_{\mathbf{Y}^{m_i}}\big)^2}{\mathrm{vol}_{m_i}\big(B_{\mathbf{Y}^{m_i}}\big)^2}\right)^{\frac{1}{2}} \leqslant \left(\sum_{i=1}^{s+1} e^{2C \log^* m_i}\right)^{\frac{1}{2}} \lesssim e^{C \log^* n},$$

where the first step uses Lemma 159, the penultimate step is our assumption on $\mathbf{Y}^{m_i}$, and the final step has the following justification. Recall that $m_{i+1} \leqslant (\log m_i)^c$ for every $i \in \{1, \ldots, s\}$, where $c > 1$ is a universal constant. So, $m_{i+2} \leqslant c^c (\log\log m_i)^c$ for every $i \in \{1, \ldots, s-1\}$. Fix $n_0 \in \mathbb{N}$ such that $c^c(\log\log n)^c \leqslant \log n$ for every $n \geqslant n_0$. Then, $m_{i+2} \leqslant \log m_i$ if $m_i \geqslant n_0$, hence $\log^* m_{i+2} \leqslant \log^* m_i - 1$. Let i_0 be the largest $i \in \{1, \ldots, s+1\}$ for which $m_i < n_0$. Then,

$$\log^* m_{2i} \leqslant \log^* m_2 - i \leqslant \log^* n - i$$

and $\log^* m_{2j+1} \leqslant \log^* m_1 - j \leqslant \log^* n - j$ if $2i, 2j+1 \in \{1, \ldots, i_0 - 1\}$. We also have $|\{i_0, \ldots, s+1\}| = O(1)$. Consequently,

$$\sum_{i=1}^{s+1} e^{2C \log^* m_i} \leqslant e^{2C \log^* n} \sum_{k=0}^{\infty} e^{-2Ck} + O(1) \lesssim e^{2C \log^* n}. \qquad \blacksquare$$

Remark 165. A straightforward way to attempt to compute the surface area of the unit sphere of a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is to fix a direction $z \in S^{n-1}$ and consider $\partial B_{\mathbf{X}}$ as the union of the two graphs of the functions $\Psi_z^{\mathbf{X}}, \psi_z^{\mathbf{X}} : \mathsf{Proj}_{z^\perp}(B_{\mathbf{X}}) \to \mathbb{R}$ that are defined by setting $\Psi_z^{\mathbf{X}}(x)$ and $\psi_z^{\mathbf{X}}(x)$ for each $x \in \mathsf{Proj}_{z^\perp}(B_{\mathbf{X}})$ to be, respectively, the largest and smallest $s \in \mathbb{R}$ for which $x + sz \in \partial B_{\mathbf{X}}$. We then have

$$\mathrm{vol}_{n-1}(\partial B_{\mathbf{X}}) = \int_{\mathsf{Proj}_{z^\perp}(B_{\mathbf{X}})} \sqrt{1 + \|\nabla \Psi_z^{\mathbf{X}}(x)\|_{\ell_2^n}^2}\, \mathrm{d}x + \int_{\mathsf{Proj}_{z^\perp}(B_{\mathbf{X}})} \sqrt{1 + \|\nabla \psi_z^{\mathbf{X}}(x)\|_{\ell_2^n}^2}\, \mathrm{d}x. \qquad (6.113)$$

When $\mathbf{X} = \ell_p^n$ for some $p \in (1, \infty)$ and $z = e_n$,

$$\forall x \in \mathsf{Proj}_{e_n^\perp}(B_{\ell_p^n}) = B_{\ell_p^{n-1}}, \quad \Psi_{e_n}^{\ell_p^n}(x) = -\psi_{e_n}^{\ell_p^n}(x) = \left(1 - \|x\|_{\ell_p^{n-1}}^p\right)^{\frac{1}{p}}.$$

Therefore, (6.113) becomes

$$\frac{\mathrm{vol}_{n-1}(\partial B_{\ell_p^n})}{\mathrm{vol}_{n-1}(B_{\ell_p^{n-1}})} = 2 \fint_{B_{\ell_p^{n-1}}} \left(1 + (1 - \|x\|_{\ell_p^{n-1}}^p)^{-\frac{2(p-1)}{p}} \sum_{i=1}^{n-1} |x_i|^{2(p-1)}\right)^{\frac{1}{2}} \mathrm{d}x.$$

By [31], a point chosen from the normalized volume measure on $B_{\ell_p^{n-1}}$ is equidistributed with

$$\left(|\mathsf{G}_1|^p + \cdots + |\mathsf{G}_{n-1}|^p + \mathsf{Z}\right)^{-\frac{1}{p}}(\mathsf{G}_1, \ldots, \mathsf{G}_{n-1}) \in \mathbb{R}^{n-1},$$

where $\mathsf{G}_1, \ldots, \mathsf{G}_{n-1}, \mathsf{Z}$ are independent random variables, the density of $\mathsf{G}_1, \ldots, \mathsf{G}_{n-1}$ at $s \in \mathbb{R}$ is equal to $2\Gamma(1 + 1/p)^{-1} e^{-|s|^p}$ and the density of Z at $t \in [0, \infty)$ is equal to e^{-t}. Consequently,

$$\frac{\mathrm{vol}_{n-1}(\partial B_{\ell_p^n})}{\mathrm{vol}_{n-1}(B_{\ell_p^{n-1}})} = 2\mathbb{E}\left[\left(1 + \mathsf{Z}^{-\frac{2(p-1)}{p}} \sum_{i=1}^{n-1} |\mathsf{G}_i|^{2(p-1)}\right)^{\frac{1}{2}}\right]. \tag{6.114}$$

Optimal estimates on moments such as the right-hand side of (6.114) were derived (in greater generality) in [225], using which one can quickly get asymptotically sharp bounds on the left-hand side of (6.114). It is possible to implement this approach to get an alternative treatment of $\ell_p^n(\ell_q^m)$, though it is significantly more involved than the different way by which we proceeded above, and it becomes much more tedious and technically intricate when one aims to treat hierarchically nested ℓ_p norms as we did in Lemma 162. Nevertheless, an advantage of (6.113) is that it applies to normed spaces that do not have a product structure as in Lemma 157, which is helpful in other settings that we will study elsewhere.

6.2 Negatively correlated normed spaces

Our goal here is to further elucidate the role of symmetries in the context of the discussion in Section 1.6.2. Fix $n \in \mathbb{N}$ and $\gamma \geqslant 1$. Say that a normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is *γ-negatively correlated* if the standard scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ is invariant under its isometry group $\mathsf{Isom}(\mathbf{X})$, i.e., $\mathsf{Isom}(\mathbf{X}) \leqslant \mathsf{O}_n$, and there exists a Borel probability measure μ on $\mathsf{Isom}(\mathbf{X})$ such that

$$\forall x, y \in \mathbb{R}^n, \quad \int_{\mathsf{Isom}(\mathbf{X})} |\langle Ux, y \rangle| \,\mathrm{d}\mu(U) \leqslant \frac{\gamma}{\sqrt{n}} \|x\|_{\ell_2^n} \|y\|_{\ell_2^n}. \tag{6.115}$$

We were inspired to formulate this notion by the proof of [286, Theorem 1.1]. It is tailored for the purpose of bounding volumes of hyperplane projections of $B_{\mathbf{X}}$ from above in terms of the surface area of $\partial B_{\mathbf{X}}$, as exhibited by the following lemma which generalizes the reasoning in [286].

Lemma 166. *Fix $n \in \mathbb{N}$ and $\gamma \geqslant 1$. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is γ-negatively correlated, then*

$$\mathrm{MaxProj}(B_{\mathbf{X}}) \leqslant \frac{\gamma}{2\sqrt{n}} \mathrm{vol}_{n-1}(\partial B_{\mathbf{X}}).$$

Proof. Recall that for every $y \in \partial B_{\mathbf{X}}$ at which $\partial B_{\mathbf{X}}$ is smooth we denote the unit outer normal to $\partial B_{\mathbf{X}}$ at y by $N_{\mathbf{X}}(y) \in S^{n-1}$. By the Cauchy projection formula (1.30) for every $x \in S^{n-1}$ we have

$$\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp}(B_{\mathbf{X}})\big) = \frac{1}{2}\int_{\partial B_{\mathbf{X}}} |\langle x, N_{\mathbf{X}}(y)\rangle| \,\mathrm{d}y.$$

Since every $U \in \mathsf{Isom}(\mathbf{X})$ is an orthogonal transformation and $N_{\mathbf{X}} \circ U^* = U^* \circ N_{\mathbf{X}}$ almost surely on $\partial B_{\mathbf{X}}$,

$$\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp}(B_{\mathbf{X}})\big) = \frac{1}{2}\int_{\partial B_{\mathbf{X}}} |\langle Ux, N_{\mathbf{X}}(y)\rangle| \,\mathrm{d}y.$$

By integrating this identity with respect to μ, we therefore conclude that

$$\begin{aligned}\mathrm{vol}_{n-1}\big(\mathsf{Proj}_{x^\perp}(B_{\mathbf{X}})\big) &= \frac{1}{2}\int_{\partial B_{\mathbf{X}}} \bigg(\int_{\mathsf{Isom}(\mathbf{X})} |\langle Ux, N_{\mathbf{X}}(y)\rangle| \,\mathrm{d}\mu(U)\bigg) \mathrm{d}y \\ &\leqslant \frac{\gamma}{2\sqrt{n}} \mathrm{vol}_{n-1}(\partial B_{\mathbf{X}}),\end{aligned}$$

where we used (6.115) and the fact that $\|x\|_{\ell_2^n} = 1$ and $\|N_{\mathbf{X}}(y)\|_{\ell_2^n} = 1$ for almost every $y \in \partial B_{\mathbf{X}}$. ■

By substituting Lemma 166 into Theorem 76 and using (1.96), we get the following corollary.

Corollary 167. *Fix $n \in \mathbb{N}$ and $\gamma \geqslant 1$. If $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ is γ-negatively correlated, then*

$$\mathsf{e}(\mathbf{X}) \lesssim \mathsf{SEP}(\mathbf{X}) \leqslant 2\gamma \frac{\mathrm{vol}_{n-1}(\partial B_{\mathbf{X}})\, \mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})\sqrt{n}}.$$

Corollary 167 generalizes Corollary 45 since any canonically positioned normed space is 1-negatively correlated. Indeed, suppose that

$$\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$$

is canonically positioned. Recall that in Section 1.6.2 we denoted the Haar probability measure on $\mathsf{Isom}(\mathbf{X})$ by $\hbar_{\mathbf{X}}$. Fix $x, y \in \mathbb{R}^n$. The distribution of the random vector

Ux when U is distributed according to $\hbar_{\mathbf{X}}$ is $\mathsf{Isom}(\mathbf{X})$-invariant, and therefore it is isotropic. Hence,

$$\int_{\mathsf{Isom}(\mathbf{X})} |\langle Ux, y\rangle| \, \mathrm{d}\hbar_{\mathbf{X}}(U) \leqslant \left(\int_{\mathsf{Isom}(\mathbf{X})} \langle Ux, y\rangle^2 \, \mathrm{d}\hbar_{\mathbf{X}}(U)\right)^{\frac{1}{2}}$$
$$\stackrel{(1.69)}{=} \frac{\|y\|_{\ell_2^n}}{\sqrt{n}} \left(\int_{\mathsf{Isom}(\mathbf{X})} \|Ux\|_{\ell_2^n}^2 \, \mathrm{d}\hbar_{\mathbf{X}}(U)\right)^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{n}} \|x\|_{\ell_2^n} \|y\|_{\ell_2^n},$$

where the final step uses the fact that each $U \in \mathsf{Isom}(\mathbf{X})$ is an orthogonal transformation.

One way to achieve (6.115), which is close in spirit to the considerations in [286], is when there are $\Gamma \subseteq \{-1, 1\}^n$ and $\Pi \subseteq S_n$ such that $U_{\varepsilon,\pi} \in \mathsf{Isom}(\mathbf{X})$ for every $(\varepsilon, \pi) \in \Gamma \times \Pi$, where $U_{\varepsilon,\pi} \in \mathsf{GL}_n(\mathbb{R})$ is given by

$$\forall x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad U_{\varepsilon,\pi} x \stackrel{\mathrm{def}}{=} \big(\varepsilon_1 x_{\pi(1)}, \ldots \varepsilon_n x_{\pi(n)}\big),$$

and also there are $\alpha, \beta > 0$ such that

$$\forall w \in \mathbb{R}^n, \quad \frac{1}{|\Gamma|} \sum_{\varepsilon \in \Gamma} |\langle \varepsilon, w\rangle| \leqslant \alpha \|w\|_{\ell_2^n} \tag{6.116}$$

and

$$\forall i, j \in \{1, \ldots, n\}, \quad |\{\pi \in \Pi : \pi(i) = j\}| \leqslant \beta \frac{|\Pi|}{n}. \tag{6.117}$$

Under these assumptions, $\mathbf{X}$ is γ-negatively correlated with $\gamma = \alpha\sqrt{\beta}$. Indeed, we can take μ in (6.115) to be the uniform distribution over the finite set

$$\{U_{\varepsilon,\pi} : (\varepsilon, \pi) \in \Gamma \times \Pi\} \subseteq \mathsf{Isom}(\mathbf{X}),$$

since every $x, y \in \mathbb{R}^n$ satisfy

$$\frac{1}{|\Gamma \times \Pi|} \sum_{(\varepsilon,\pi)\in\Gamma\times\Pi} |\langle U_{\varepsilon,\pi} x, y\rangle| \stackrel{(6.116)}{\leqslant} \frac{1}{|\Pi|} \sum_{\pi\in\Pi} \alpha \left(\sum_{i=1}^n (x_{\pi(i)} y_i)^2\right)^{\frac{1}{2}}$$
$$\leqslant \alpha \left(\sum_{i=1}^n \left(\frac{1}{|\Pi|} \sum_{\pi\in\Pi} x_{\pi(i)}^2\right) y_i^2\right)^{\frac{1}{2}}$$
$$= \alpha \left(\sum_{i=1}^n \left(\frac{1}{|\Pi|} \sum_{j=1}^n |\{\pi \in \Pi : \pi(i) = j\}| x_j^2\right) y_i^2\right)^{\frac{1}{2}}$$
$$\stackrel{(6.117)}{\leqslant} \frac{\alpha\sqrt{\beta}}{\sqrt{n}} \|x\|_{\ell_2^n} \|y\|_{\ell_2^n}.$$

Condition (6.116) can be viewed as a negative correlation property of the coordinates of sign vectors that are chosen uniformly from Γ. Condition (6.117) roughly means that for each $i \in \{1,\ldots,n\}$ the sets $\{\pi \in \Pi : \pi(i) = 1\},\ldots,\{\pi \in \Pi : \pi(i) = n\}$ form an approximately equitable partition of Π. This holds with $\beta = 1$ if Π is a transitive subgroup of S_n. One could formulate weaker conditions that ensure the validity of the conclusion of Lemma 166 (e.g., considering bi-Lipschitz automorphisms of $\mathbf{X}$ rather than isometries of $\mathbf{X}$), and hence also the conclusion of Corollary 167, though we will not pursue this here as we expect that in concrete cases such issues should be easy to handle.

6.3 Volume ratio computations

Here we will present asymptotic evaluations of volume ratios of some normed spaces, for the purpose of plugging them into results that we stated in the Introdcution. Due to the large amount of knowledge on this topic that is available in the literature, we will only give a flavor of such applications. The main reference for the contents of this section is the valuable work [285].

We will start by examining the iteratively nested ℓ_p products $\{\mathbf{X}_k\}_{k=0}^{\infty}$ that we considered in Corollary 153, in the special case when the initial space $\mathbf{X} = \mathbf{X}_0$ is a canonically positioned normed space for which Conjecture 49 holds. Thus, we are fixing $\{n_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ and $\{p_k\}_{k=1}^{\infty} \subseteq [1,\infty]$, and assuming that

$$\mathbf{X} = (\mathbb{R}^{n_0}, \|\cdot\|_{\mathbf{X}})$$

is a canonically positioned normed space satisfying Conjecture 49, i.e., (6.16) holds with $\alpha = O(1)$; the case $\mathbf{X} = \mathbb{R}$ is sufficiently rich for our present illustrative purposes, but one can also take $\mathbf{X} = \mathbf{E}$ to be any symmetric space, per Lemma 54. By Corollary 153 and Corollary 79, if we define inductively

$$\forall k \in \mathbb{N}, \quad \mathbf{X}_{k+1} = \ell_{p_k}^{n_k}(\mathbf{X}_k), \quad \text{where } \mathbf{X}_0 = \mathbf{X},$$

then, because $\{\mathbf{X}_k\}_{k=1}^{\infty}$ are canonically positioned (they belong to the class of spaces in Example 40),

$$\forall m \in \mathbb{N}, \quad \mathsf{SEP}(\mathbf{X}_m) \asymp \mathrm{evr}(\mathbf{X}_m)\sqrt{\dim(\mathbf{X}_m)} = \mathrm{evr}(\mathbf{X}_m)\sqrt{n_0\cdots n_m}. \tag{6.118}$$

Let $\{\mathbf{H}_k\}_{k=0}^{\infty}$ be the sequence of Euclidean spaces that arise from the above construction with the same $\{n_k\}_{k=0}^{\infty} \subseteq \mathbb{N}$ but with $p_k = 2$ for all $k \in \mathbb{N}$ and $\mathbf{X} = \ell_2^{n_0}$. Thus, for each $m \in \mathbb{N}$ the Euclidean space $\mathbf{H}_m$ can be identified naturally with $\ell_2^{n_0\cdots n_m}$. Under this identification, by a straightforward inductive application of Hölder's inequality and the fact that the ℓ_p norm deceases with p, the Löwner ellipsoid of

$\mathbf{X}_m$ satisfies[1]

$$\mathcal{L}_{\mathbf{X}_m} \subseteq \left(\prod_{k=1}^{m} n_k^{\max\{\frac{1}{2}-\frac{1}{p_k},0\}} \right) (\mathcal{L}_{\mathbf{X}})^{n_1\cdots n_m}.$$

Also, by Lemma 150 we have

$$\mathrm{vol}_{n_0\cdots n_m}(B_{\mathbf{X}_m})^{\frac{1}{n_0\cdots n_k}} \asymp \frac{\mathrm{vol}_{n_0}(B_{\mathbf{X}})^{\frac{1}{n_0}}}{\prod_{k=1}^{m} n_k^{\frac{1}{p_k}}}.$$

These facts combine to give the following consequence of (6.118):

$$\mathsf{SEP}(\mathbf{X}_m) \asymp \mathrm{evr}(\mathbf{X}) \prod_{k=1}^{m} n_k^{\max\{\frac{1}{2},\frac{1}{p_k}\}}.$$

In particular, when we take $\mathbf{X} = \mathbb{R}$ and consider only two steps of the above iteration, we get the following asymptotic evaluation of the separation modulus of the $\ell_p^n(\ell_q^m)$ norm the space of n-by-m matrices $M_{n\times m}(\mathbb{R})$ for any $n,m \in \mathbb{N}$ and $p,q \geqslant 1$; the case of square matrices was stated in the Introduction as (1.5):

$$\mathsf{SEP}\big(\ell_p^n(\ell_q^m)\big) \asymp n^{\max\{\frac{1}{p},\frac{1}{2}\}} m^{\max\{\frac{1}{q},\frac{1}{2}\}} = \max\Big\{\sqrt{nm}, m^{\frac{1}{q}}\sqrt{n}, n^{\frac{1}{p}}\sqrt{m}, n^{\frac{1}{p}} m^{\frac{1}{q}}\Big\}.$$

Next, fix an integer $n \geqslant 2$ and let $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ be an unconditional normed space. Given $q \in [2,\infty]$ and $\Lambda \geqslant 1$, one says (see, e.g., [182, Definition 1.f.4]) that $\mathbf{E}$ satisfies a lower q-estimate with constant Λ if for every $\{u_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ with pairwise disjoint supports we have

$$\left(\sum_{k=1}^{\infty} \|u_k\|_{\mathbf{E}}^q \right)^{\frac{1}{q}} \leqslant \Lambda \left\| \sum_{k=1}^{\infty} u_k \right\|_{\mathbf{E}}. \tag{6.119}$$

Note that by (6.14) this always holds with $\Lambda = 1$ if $q = \infty$.

In concrete cases it is often mechanical to evaluate up to universal constant factors the minimum radius of a Euclidean ball that circumscribes $B_{\mathbf{X}}$, but it is always within a $O(\sqrt{\log n})$ factor of the expression

$$R_{\mathbf{E}} \stackrel{\mathrm{def}}{=} \max_{\varnothing \neq S \subseteq \{1,\ldots,n\}} \left(\frac{\sqrt{|S|}}{\|\sum_{i\in S} e_i\|_{\mathbf{E}}} \right). \tag{6.120}$$

More precisely, if $\mathbf{E}$ satisfies a lower q-estimate with constant Λ, then

$$R_{\mathbf{E}} \leqslant \mathrm{outradius}_{\ell_2^n}(B_{\mathbf{X}}) \lesssim \Lambda (\log n)^{\frac{1}{2}-\frac{1}{q}} R_{\mathbf{E}}. \tag{6.121}$$

[1] As $\mathbf{X}_m$ is canonically positioned, this holds as an equality, but for the present purposes we just need the stated inclusion.

The first inequality in (6.121) is immediate because $\|\sum_{i\in S} e_i\|_{\mathbf{E}}^{-1}\sum_{i\in S} e_i \in B_{\mathbf{E}}$ if $\varnothing \neq S \subseteq \{1,\ldots,n\}$. For a quick justification of the second inequality in (6.121), note that by homogeneity we may assume without loss of generality that $\|e_i\|_{\mathbf{E}} \geqslant 1$ for every $i \in \mathbb{N}$. Therefore, using (6.14) we see that if $x = (x_1,\ldots,x_n) \in B_{\mathbf{E}}$, then $\max_{i\in\{1,\ldots,n\}} |x_i| \leqslant 1$. Consequently, if we fix $x \in B_{\mathbf{E}}$ and denote for each $k \in \mathbb{N}$,

$$S_k = S_k(x) \stackrel{\text{def}}{=} \left\{ i \in \{1,\ldots,n\} : \frac{1}{2^k} < |x_i| \leqslant \frac{1}{2^{k-1}} \right\}, \tag{6.122}$$

then the sets $\{S_k\}_{k=1}^{\infty}$ are a partition of $\{1,\ldots,n\}$ and in particular $\sum_{k=1}^{\infty}|S_k| = n$. Next,

$$\begin{aligned} \Lambda R_{\mathbf{E}} \geqslant \Lambda R_{\mathbf{E}}\|x\|_{\mathbf{E}} &\geqslant R_{\mathbf{E}}\left(\sum_{k=1}^{\infty}\Big\|\sum_{i\in S_k} x_i e_i\Big\|_{\mathbf{E}}^q\right)^{\frac{1}{q}} \\ &\geqslant \left(\sum_{k=1}^{\infty} R_{\mathbf{E}}^q \Big\|\sum_{i\in S_k}\frac{1}{2^k}e_i\Big\|_{\mathbf{E}}^q\right)^{\frac{1}{q}} \geqslant \left(\sum_{k=1}^{\infty}\frac{|S_k|^{\frac{q}{2}}}{2^{kq}}\right)^{\frac{1}{q}}. \end{aligned} \tag{6.123}$$

The second step of (6.123) uses (6.119), the penultimate step of (6.123) uses (6.14) and (6.122), and the final step of (6.123) uses (6.120). Now, for every $0 < \theta < 1$ we have

$$\begin{aligned} \|x\|_{\ell_2^n} &= \left(\sum_{k=1}^{\infty}\sum_{i\in S_k} x_i^2\right)^{\frac{1}{2}} \\ &\leqslant \left(\sum_{k=1}^{\infty}\frac{|S_k|}{2^{2(k-1)}}\right)^{\frac{1}{2}} \\ &= 2\left(\sum_{k=1}^{\infty}\frac{|S_k|^{1-\theta}}{2^{2k(1-\theta)}}|S_k|^{\theta}2^{-2k\theta}\right)^{\frac{1}{2}} \\ &\leqslant 2\left(\sum_{k=1}^{\infty}\frac{|S_k|^{\frac{q}{2}}}{2^{2kq}}\right)^{\frac{1-\theta}{q}}\left(\sum_{k=1}^{\infty}|S_k|\right)^{\frac{\theta}{2}}\left(\sum_{k=1}^{\infty}2^{-\frac{2kq\theta}{(q-2)(1-\theta)}}\right)^{(\frac{1}{2}-\frac{1}{q})(1-\theta)} \\ &\lesssim (\Lambda R_{\mathbf{E}})^{1-\theta} n^{\frac{\theta}{2}}\theta^{-(\frac{1}{2}-\frac{1}{q})}, \end{aligned} \tag{6.124}$$

where the second step of (6.124) uses (6.122), the penultimate step of (6.124) uses trilinear Hölder with exponents $1/\theta, q/(2(1-\theta))$ and $1/((1-2/q)(1-\theta))$, and the final step of (6.124) uses (6.123), the fact that

$$\sum_{k=1}^{\infty}|S_k| = n,$$

and elementary calculus. By choosing $\theta = 1/\log n$ in (6.124), we get (6.121).

By the Lozanovskiĭ factorization theorem [186] there exist $w_1, \dots, w_n > 0$ such that

$$\Big\| \sum_{i=1}^{n} w_i e_i \Big\|_{\mathbf{E}} = \Big\| \sum_{i=1}^{n} \frac{1}{w_i} e_i \Big\|_{\mathbf{E}^*} = \sqrt{n}. \tag{6.125}$$

We will call any $w_1, \dots, w_n > 0$ that satisfy (6.125) Lozanovskiĭ weights for $\mathbf{E}$. They can be found by maximizing the concave function $w \mapsto \sum_{i=1}^{n} \log w_i$ over $w \in B_{\mathbf{E}}$ (see also, e.g., [263, Chapter 3]), which can be done efficiently if $\mathbf{E}$ is given by an efficient oracle; their existence can also be established non-constructively using the Brouwer fixed point theorem [135]. By [285, Lemma 1.2] (note that we are using a different normalization of the weights than in [285]),

$$\mathrm{vol}_n(B_{\mathbf{E}})^{\frac{1}{n}} \asymp \frac{(w_1 \cdots w_n)^{\frac{1}{n}}}{\sqrt{n}}. \tag{6.126}$$

By combining (6.121) and (6.126), we get the following lemma.

Lemma 168. *Fix an integer $n \geqslant 2$ and let $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ be an unconditional normed space. Suppose that $\mathbf{E}$ satisfies a lower q-estimate with constant Λ for some $q \geqslant 2$ and $\Lambda \geqslant 1$. Then,*

$$\mathrm{evr}(\mathbf{E}) \lesssim \frac{\max_{\varnothing \neq S \subseteq \{1,\dots,n\}} \Big(\frac{\sqrt{|S|}}{\|\sum_{i \in S} e_i\|_{\mathbf{E}}} \Big)}{\sqrt[n]{w_1 \cdots w_n}} \Lambda (\log n)^{\frac{1}{2} - \frac{1}{q}},$$

for any Lozanovskiĭ weights $w_1, \dots, w_n > 0$ for $\mathbf{E}$. If the Löwner ellipsoid of $\mathbf{E}$ is a multiple of $B_{\ell_2^n}$, then

$$\frac{\max_{\varnothing \neq S \subseteq \{1,\dots,n\}} \Big(\frac{\sqrt{|S|}}{\|\sum_{i \in S} e_i\|_{\mathbf{E}}} \Big)}{\sqrt[n]{w_1 \cdots w_n}} \lesssim \mathrm{evr}(\mathbf{E})$$
$$\lesssim \frac{\max_{\varnothing \neq S \subseteq \{1,\dots,n\}} \Big(\frac{\sqrt{|S|}}{\|\sum_{i \in S} e_i\|_{\mathbf{E}}} \Big)}{\sqrt[n]{w_1 \cdots w_n}} \Lambda (\log n)^{\frac{1}{2} - \frac{1}{q}}.$$

The following corollary is a consequence of Lemma 168 because if

$$\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$$

is a normed space that satisfies the assumptions of Lemma 53 (in particular, $\mathbf{E}$ is unconditional), then by Lemma 152

$$w_1 = w_2 = \cdots = w_n = \frac{\sqrt{n}}{\|e_1 + \cdots + e_n\|_{\mathbf{E}}}$$

are Lozanovskiĭ weights for $\mathbf{E}$.

Corollary 169. *If* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *a normed space that satisfies the assumptions of Lemma* 53*, then*

$$\frac{\|e_1+\cdots+e_n\|_{\mathbf{E}}}{\sqrt{n}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right) \lesssim \mathrm{evr}(\mathbf{E}) \lesssim \frac{\|e_1+\cdots+e_n\|_{\mathbf{E}}}{\sqrt{n}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)\sqrt{\log n}.$$

Hence, by Corollary 79 *we have*

$$\|e_1+\cdots+e_n\|_{\mathbf{E}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right) \lesssim \mathsf{SEP}(\mathbf{E}) \lesssim \|e_1+\cdots+e_n\|_{\mathbf{E}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)\sqrt{\log n},$$

More succinctly, this can be written in the following form, which we already stated in Corollary 4*:*

$$\mathsf{SEP}(\mathbf{E}) = \|e_1+\cdots+e_n\|_{\mathbf{E}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)n^{o(1)}.$$

By [285, Proposition 2.2], the unitary ideal of any symmetric normed space $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ satisfies

$$\mathrm{vr}(\mathsf{S}_{\mathbf{E}}) \asymp \mathrm{vr}(\mathbf{E}). \tag{6.127}$$

This implies that

$$\mathrm{evr}(\mathsf{S}_{\mathbf{E}}) \asymp \mathrm{evr}(\mathbf{E}), \tag{6.128}$$

by (1.71) combined with $\mathsf{S}_{\mathbf{E}}^* = \mathsf{S}_{\mathbf{E}^*}$, though a straightforward adjustment of the proof of (6.127) in [285] yields (6.128) directly, without using the much deeper result (1.71). We therefore have the following corollary.

Corollary 170. *If* $\mathbf{E} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{E}})$ *is a symmetric normed space, then*

$$\frac{\|e_1+\cdots+e_n\|_{\mathbf{E}}}{\sqrt{n}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right) \lesssim \mathrm{evr}(\mathsf{S}_{\mathbf{E}}) \lesssim \frac{\|e_1+\cdots+e_n\|_{\mathbf{E}}}{\sqrt{n}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)\sqrt{\log n}.$$

Hence, by Corollary 79 *we have*

$$\|e_1+\cdots+e_n\|_{\mathbf{E}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)\sqrt{n} \lesssim \mathsf{SEP}(\mathsf{S}_{\mathbf{E}}) \lesssim \|e_1+\cdots+e_n\|_{\mathbf{E}}\left(\max_{k\in\{1,\ldots,n\}}\frac{\sqrt{k}}{\|e_1+\cdots+e_k\|_{\mathbf{E}}}\right)\sqrt{n}\log n,$$

More succinctly, this can be written in the following form, which we already stated in Corollary 4:

$$\mathsf{SEP}(\mathsf{S}_{\mathbf{E}}) = \|e_1 + \cdots + e_n\|_{\mathbf{E}} \left(\max_{k \in \{1,\ldots,n\}} \frac{\sqrt{k}}{\|e_1 + \cdots + e_k\|_{\mathbf{E}}} \right) n^{\frac{1}{2}+o(1)}.$$

Remark 171. In the above discussion, as well as in the ensuing treatment of tensor products, we prefer to consider square matrices rather than rectangular matrices because the setting of square matrices exhibits all of the key issues while being notationally simpler. Nevertheless, there are two places in which we do need to work with rectangular matrices, namely the above proof of Proposition 164 and the proof of the first inequality in (1.117). For the latter, fix $p \geqslant 1$ and $n, m \in \mathbb{N}$. As in the proof of Theorem 77, denote the Schatten–von Neumann trace class on the n-by-m real matrices $\mathsf{M}_{n\times m}(\mathbb{R})$ by $\mathsf{S}_p^{n\times m}$; recall (1.118). The following asymptotic identity implies (1.119) (recall that in the setting of (1.119) we have $r \in \{1, \ldots, n\}$)

$$\mathrm{evr}\big(\mathsf{S}_p^{n\times m}\big) \asymp \big(\min\{n, m\}\big)^{\max\{\frac{1}{p}-\frac{1}{2},0\}}. \tag{6.129}$$

Volumes of unit balls of Schatten–von Neumann trace classes have been satisfactorily estimated in the literature, starting with [293] and the comprehensive work [285], through the more precise asymptotics in [146, 277]. Unfortunately, all of these works dealt only with square matrices. Nevertheless, these references could be mechanically adjusted to treat rectangular matrices as well. Since (6.129) does not seem to have been stated in the literature, we will next sketch its derivation by mimicking the reasoning of [285], though the more precise statements of [146, 277] could be derived as well via similarly straightforward modifications of the known proofs for square matrices. We claim that

$$\mathrm{vol}_{nm}\big(B_{\mathsf{S}_p^{n\times m}}\big)^{\frac{1}{nm}} \asymp \frac{1}{\big(\min\{n, m\}\big)^{\frac{1}{p}} \sqrt{\max\{n, m\}}}. \tag{6.130}$$

(6.130) gives (6.129) since $\mathsf{S}_p^{n\times m}$ is canonically positioned, so by Hölder's inequality its Löwner ellipsoid is

$$\mathcal{L}_{\mathsf{S}_p^{n\times m}} = \big(\min\{n, m\}\big)^{\max\left\{\frac{1}{2}-\frac{1}{p},0\right\}} B_{\mathsf{S}_2^{n\times m}}.$$

To prove (6.130), note first that it follows from its special case $p = \infty$. Indeed, as $\mathsf{S}_1^{n\times m} = (\mathsf{S}_\infty^{n\times m})^*$, by the Blaschke–Santaló inequality [39, 278] and the Bourgain–Milman inequality [50] the case $p = 1$ of (6.130) follows from its case $p = \infty$. Now, (6.130) follows in full generality since by Hölder's inequality:

$$\frac{1}{\big(\min\{n, m\}\big)^{\frac{1}{p}}} B_{\mathsf{S}_\infty^{n\times m}} \subseteq B_{\mathsf{S}_p^{n\times m}} \subseteq \big(\min\{n, m\}\big)^{1-\frac{1}{p}} B_{\mathsf{S}_1^{n\times m}}.$$

The upper bound $\mathrm{vol}_{nm}(B_{\mathsf{S}_\infty^{n\times m}})^{1/(nm)} \lesssim 1/\sqrt{\max\{n,m\}}$ follows from the inclusion $B_{\mathsf{S}_\infty^{n\times m}} \subseteq \sqrt{\min\{n,m\}}B_{\mathsf{S}_2^{n\times m}}$. To justify the matching lower bound, if $\{\varepsilon_{ij}\}_{i,j\in\mathbb{N}}$ are i.i.d. Bernoulli random variables, then by [35, Theorem 1],

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n}\sum_{j=1}^{m}\varepsilon_{ij}e_i\otimes e_j\right\|_{\mathsf{S}_\infty^{n\times m}}\right] \lesssim \sqrt{\max\{n,m\}},$$

This implies the lower bound $\mathrm{vol}_{nm}(B_{\mathsf{S}_\infty^{n\times m}})^{1/(nm)} \gtrsim 1/\sqrt{\max\{n,m\}}$ by an application of [285, Lemma 1.5].

Proof of Lemma 54. By [285, equation (2.2)] we have

$$\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{E}}}\big)^{\frac{1}{n^2}} \asymp \frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}\sqrt{n}}. \tag{6.131}$$

In particular,

$$\forall q\geqslant 1,\qquad \mathrm{vol}_{n^2}\big(B_{\mathsf{S}_q^n}\big)^{\frac{1}{n^2}} \asymp \frac{1}{n^{\frac12+\frac1q}}. \tag{6.132}$$

Because S_q^n is canonically positioned (it belongs to the class of spaces in Example 40), and hence it is in its minimum surface area position, by combining [104, Proposition 3.1] and (1.55) we see that

$$\frac{\mathrm{vol}_{n^2-1}\big(\partial B_{\mathsf{S}_q^n}\big)}{\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_q^n}\big)} \asymp \frac{n\mathrm{MaxProj}\big(B_{\mathsf{S}_q^n}\big)}{\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_q^n}\big)} \overset{(1.43)}{\asymp} n^{\frac32+\frac1q}\sqrt{\min\{q,n\}}. \tag{6.133}$$

Consequently,

$$\begin{aligned}\mathrm{iq}\big(B_{\mathsf{S}_q^n}\big) &= n\frac{\mathrm{vol}_{n^2-1}\big(\partial B_{\mathsf{S}_q^n}\big)}{\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_q^n}\big)}\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_q^n}\big)^{\frac{1}{n^2}}\\ &\overset{(6.132)\wedge(6.133)}{\asymp} \frac{n^{\frac32+\frac1q}\sqrt{\min\{q,n\}}}{n^{\frac12+\frac1q}} = n\sqrt{\min\{q,n\}}.\end{aligned} \tag{6.134}$$

Because by (6.14) we have

$$\forall x\in\mathbb{R}^n,\qquad \|x\|_{\mathbf{E}} \leqslant \|e_1+\cdots+e_n\|_{\mathbf{E}}\|x\|_{\ell_\infty^n},$$

every matrix $A\in\mathsf{M}_n(\mathbb{R})$ satisfies

$$\|A\|_{\mathsf{S}_{\mathbf{E}}} \leqslant \|e_1+\cdots+e_n\|_{\mathbf{E}}\|A\|_{\mathsf{S}_\infty^n} \leqslant \|e_1+\cdots+e_n\|_{\mathbf{E}}\|A\|_{\mathsf{S}_q^n}.$$

Consequently,

$$\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}B_{\mathsf{S}_q^n} \subseteq B_{\mathsf{S}_{\mathbf{E}}}. \tag{6.135}$$

Moreover,

$$\mathrm{iq}\left(\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}B_{\mathsf{S}_q^n}\right)=\mathrm{iq}\big(B_{\mathsf{S}_q^n}\big)\overset{(6.134)}{\asymp} n\sqrt{\min\{q,n\}}$$

and

$$\mathrm{vol}_{n^2}\left(\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}B_{\mathsf{S}_q^n}\right)^{\frac{1}{n^2}}\overset{(6.132)}{\asymp}\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}n^{\frac{1}{2}+\frac{1}{q}}}\overset{(6.131)}{\asymp}\frac{\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{E}}}\big)^{\frac{1}{n^2}}}{n^{\frac{1}{q}}}.$$

By choosing $q=\log n$ we get (1.80) for the normed space $\mathbf{Y}$ whose unit ball is the left-hand side of (6.135). ■

Remark 172. An inspection of the proof of Lemma 54 reveals that if Conjecture 49 holds for S_∞^n, then also Conjecture 49 holds for $\mathsf{S}_{\mathbf{E}}$ for any symmetric normed space $\mathbf{E}=(\mathbb{R}^n,\|\cdot\|_{\mathbf{E}})$. Indeed, we would then take $\mathbf{Y}'=(\mathsf{M}_n(\mathbb{R}),\|\cdot\|_{\mathbf{Y}'})$ to be the normed space whose unit ball is

$$B_{\mathbf{Y}'}=\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}\,\mathrm{Ch}\,S_\infty^n=\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}\mathsf{S}_{\chi\ell_\infty^n},$$

where we recall Corollary 43. If Conjecture 49 holds for S_∞^n, then we would have $n\asymp\mathrm{iq}(\mathrm{Ch}\,S_\infty^n)=\mathrm{iq}(B_{\mathbf{Y}'})$, and also

$$\mathrm{vol}_{n^2}\big(\mathrm{Ch}\,S_\infty^n\big)^{\frac{1}{n^2}}\asymp\mathrm{vol}_{n^2}\big(S_\infty^n\big)^{\frac{1}{n^2}}\overset{(6.132)}{\asymp}\frac{1}{\sqrt{n}},$$

from which we see that

$$\mathrm{vol}_{n^2}\big(B_{\mathbf{Y}'}\big)^{\frac{1}{n^2}}=\frac{\mathrm{vol}_{n^2}\big(\mathrm{Ch}\,S_\infty^n\big)^{\frac{1}{n^2}}}{\|e_1+\cdots+e_n\|_{\mathbf{E}}}\asymp\frac{1}{\|e_1+\cdots+e_n\|_{\mathbf{E}}\sqrt{n}}\overset{(6.131)}{\asymp}\mathrm{vol}_{n^2}\big(B_{\mathsf{S}_{\mathbf{E}}}\big)^{\frac{1}{n^2}}.$$

This proves Conjecture 49 for $\mathsf{S}_{\mathbf{E}}$. Note in passing that this also implies that

$$\frac{1}{\sqrt{n}}\asymp\mathrm{vol}_{n^2}\big(S_{\chi\ell_\infty^n}\big)^{\frac{1}{n^2}}\overset{(6.131)}{\asymp}\frac{1}{\|e_1+\cdots+e_n\|_{\chi\ell_\infty^n}\sqrt{n}}.$$

Hence, if Conjecture 49 holds for S_∞^n, then we would have $\|e_1+\cdots+e_n\|_{\chi\ell_\infty^n}\asymp 1$. More generally, by mimicking the above reasoning we deduce that if Conjecture 49 holds for $\mathsf{S}_{\mathbf{E}}$, then $\|e_1+\cdots+e_n\|_{\chi\mathbf{E}}\asymp\|e_1+\cdots+e_n\|_{\mathbf{E}}$, which would be a modest step towards Problem 44.

Fix $n\in\mathbb{N}$ and $p,q\geqslant 1$. We claim that the volume ratio of the projective tensor product $\ell_p^n\widehat{\otimes}\ell_q^n$ satisfies

$$\mathrm{vr}\big(\ell_p^n\widehat{\otimes}\ell_q^n\big)\asymp\Phi_{p,q}(n), \tag{6.136}$$

where

$$\Phi_{p,q}(n) \stackrel{\mathrm{def}}{=} \begin{cases} 1 & \text{if } 1 \leqslant p,q \leqslant 2,\\ n^{\frac12-\frac1p} & \text{if } q \leqslant 2 \leqslant p \leqslant \frac{q}{q-1},\\ n^{\frac1q-\frac12} & \text{if } q \leqslant 2 \leqslant \frac{q}{q-1} \leqslant p,\\ n^{\frac12-\frac1q} & \text{if } p \leqslant 2 \leqslant q \leqslant \frac{p}{p-1},\\ n^{\frac1p-\frac12} & \text{if } p \leqslant 2 \leqslant \frac{p}{p-1} \leqslant q,\\ 1 & \text{if } p,q \geqslant 2 \text{ and } \frac1p+\frac1q \geqslant \frac12,\\ n^{\frac12-\frac1p-\frac1q} & \text{if } \frac1p+\frac1q \leqslant \frac12. \end{cases} \tag{6.137}$$

Assuming (6.137) for the moment, by substituting it into Theorem 3 we get that

$$\mathsf{SEP}\big(\ell_p^n \check{\otimes} \ell_q^n\big) \gtrsim n\,\mathrm{vr}\Big(\big(\ell_p^n \check{\otimes} \ell_q^n\big)^*\Big) = n\,\mathrm{vr}\big(\ell_{p^*}^n \hat{\otimes} \ell_{q^*}^n\big)$$

$$\asymp n\Phi_{p^*,q^*}(n) = \begin{cases} n & \text{if } p,q \geqslant 2,\\ n^{\frac1p+\frac12} & \text{if } \frac{q}{q-1} \leqslant p \leqslant 2 \leqslant q,\\ n^{\frac32-\frac1q} & \text{if } p \leqslant \frac{q}{q-1} \leqslant 2 \leqslant q,\\ n^{\frac1q+\frac12} & \text{if } \frac{p}{p-1} \leqslant q \leqslant 2 \leqslant p,\\ n^{\frac32-\frac1p} & \text{if } q \leqslant \frac{p}{p-1} \leqslant 2 \leqslant p,\\ n & \text{if } p,q \leqslant 2 \text{ and } \frac1p+\frac1q \leqslant \frac32,\\ n^{\frac1p+\frac1q-\frac12} & \text{if } \frac1p+\frac1q \geqslant \frac32. \end{cases}$$

Since for any two normed spaces $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ and $\mathbf{Y} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{Y}})$ the space of operators from $\mathbf{X}^*$ to $\mathbf{Y}$ is isometric to the injective tensor product $\mathbf{X}^* \check{\otimes} \mathbf{Y}$ (see, e.g., [87]), we get from this that

$$\mathsf{SEP}\big(\mathsf{M}_n(\mathbb{R}), \|\cdot\|_{\ell_p^n \to \ell_q^n}\big) = \mathsf{SEP}\big(\ell_{p^*}^n \check{\otimes} \ell_q^n\big)$$

$$\gtrsim \begin{cases} n & \text{if } p \leqslant 2 \leqslant q,\\ n^{\frac32-\frac1p} & \text{if } 2 \leqslant p \leqslant q,\\ n^{\frac32-\frac1q} & \text{if } 2 \leqslant q \leqslant p,\\ n^{\frac1q+\frac12} & \text{if } p \leqslant q \leqslant 2,\\ n^{\frac1p+\frac12} & \text{if } q \leqslant p \leqslant 2,\\ n & \text{if } \frac{2p}{p+2} \leqslant q \leqslant 2 \leqslant p,\\ n^{\frac1q-\frac1p+\frac12} & \text{if } q \leqslant \frac{2p}{p+2}. \end{cases} \tag{6.138}$$

Note that the rightmost quantity in (6.138) coincides with the right-hand side of (1.14). Since $\ell_p^n \check{\otimes} \ell_q^n$ belongs to the class of spaces in Example 40, a positive answer to Conjecture 11 for $\ell_p^n \check{\otimes} \ell_q^n$ would imply the following asymptotic evaluation of $\mathsf{SEP}(\ell_p^n \check{\otimes} \ell_q^n)$, which is equivalent to (1.14):

$$\mathsf{SEP}\big(\ell_p^n \check{\otimes} \ell_q^n\big) \asymp \begin{cases} n & \text{if } p,q \geqslant 2, \\ n^{\frac12+\frac1p} & \text{if } \frac{q}{q-1} \leqslant p \leqslant 2 \leqslant q, \\ n^{\frac32-\frac1q} & \text{if } p \leqslant \frac{q}{q-1} \leqslant 2 \leqslant q, \\ n^{\frac12+\frac1q} & \text{if } \frac{p}{p-1} \leqslant q \leqslant 2 \leqslant p, \\ n^{\frac32-\frac1p} & \text{if } q \leqslant \frac{p}{p-1} \leqslant 2 \leqslant p, \\ n & \text{if } p,q \leqslant 2 \text{ and } \frac1p+\frac1q \leqslant \frac32, \\ n^{\frac1p+\frac1q-\frac12} & \text{if } \frac1p+\frac1q \geqslant \frac32. \end{cases}$$

Furthermore, by Theorem 80 the leftmost quantity in (6.138) is bounded from above by $O(\log n)$ times the rightmost quantity in (6.138), thus implying the fourth bullet point of Corollary 4.

The asymptotic evaluation (6.136) of $\mathrm{vr}(\ell_p^n \hat{\otimes} \ell_q^n)$ was proved in [285] up to constant factors that depend on p, q, namely [285, Theorem 3.1] states that

$$\forall\, p,q > 1, \quad \mathrm{vr}\big(\ell_p^n \hat{\otimes} \ell_q^n\big) \asymp_{p,q} \Phi_{p,q}(n). \tag{6.139}$$

If $2 \in \{p,q\}$ and also $\min\{p,q\} \leqslant 2$, then (6.139) is due to Szarek and Tomczak-Jaegermann [293]. More recently, Defant and Michels [84] generalized (6.139) to projective tensor products of symmetric normed spaces that are either 2-convex or 2-concave. The proof of (6.139) in [285] yields constants that degenerate as $\min\{p,q\}$ tends to 1. We will therefore next improve the reasoning in [285] to get (6.136).

Lemma 173. *Fix $n \in \mathbb{N}$ and $p,q \geqslant 1$. Let $\{\varepsilon_{ij}\}_{i,j\in\{1,\ldots,n\}}$ be i.i.d. Bernoulli random variables (namely, they are independent and each of them is uniformly distributed over $\{-1,1\}$). Then,*

$$\mathbb{E}\left[\left\| \sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij} e_i \otimes e_j \right\|_{\ell_p^n \check{\otimes} \ell_q^n}\right] \asymp n^{\beta(p,q)} \stackrel{\mathrm{def}}{=} \begin{cases} n^{\frac1p+\frac1q-\frac12} & \textit{if } \max\{p,q\} \leqslant 2, \\ n^{\frac{1}{\min\{p,q\}}} & \textit{if } \max\{p,q\} \geqslant 2. \end{cases} \tag{6.140}$$

Citing the work [79] of Chevet, a version of Lemma 173 appears as [285, Lemma 2.3], except that in [285, Lemma 2.3] the implicit constants in (6.140) depend on p, q. An inspection of the proof of (6.139) in [285] reveals that this is the only source of the dependence of the constants on p, q (in fact, for this purpose [285] only needs half of (6.140), namely to bound from above its left-hand side by its right-hand side). Specifically, all of the steps within [285] incur only a loss of a universal constant factor, and the proof of (6.139) in [285] also appeals to inequalities in the earlier work [284] of Schütt, as well a classical inequality of Hardy and Littlewood [127]; all of the constants in these cited inequalities are universal. Therefore, (6.136) will be established after we prove Lemma 173.

Proof of Lemma 173. We will denote the random matrix whose (i,j) entry is ε_{ij} by $\mathcal{E} \in \mathsf{M}_n(\mathbb{R})$. Then, the goal is

$$\mathbb{E}\big[\|\mathcal{E}\|_{\ell_{p*}^n \to \ell_q^n}\big] \asymp n^{\beta(p,q)}. \tag{6.141}$$

In fact, the lower bound on the expected norm in (6.141) holds always, i.e., for a universal constant $c > 0$,

$$\forall A \in \mathsf{M}_n(\{-1,1\}), \quad \|A\|_{\ell_{p*}^n \to \ell_q^n} \geqslant c n^{\beta(p,q)}. \tag{6.142}$$

A justification of (6.142) appears in the *proof of* Proposition 3.2 of Bennett's work [34] (specifically, see the reasoning immediately after [34, inequality (15)]), where it is explained that we can take $c = 1$ if $\min\{p^*, q\} \geqslant 2$ or $\max\{p^*, q\} \leqslant 2$, and that we can take $c = 1/\sqrt{2}$ otherwise.

Next, let $\{\mathsf{g}_{ij}\}_{i,j \in \{1,\ldots,n\}}$ be i.i.d. standard Gaussian random variables. By [79, Lemme 3.1],

$$\mathbb{E}\left[\left\|\sum_{i=1}^n \sum_{j=1}^n \mathsf{g}_{ij} e_i \otimes e_j\right\|_{\ell_p^n \check{\otimes} \ell_q^n}\right] \asymp n^{\max\{\frac{1}{p}+\frac{1}{q}-\frac{1}{2},\frac{1}{p}\}}\sqrt{p} + n^{\max\{\frac{1}{p}+\frac{1}{q}-\frac{1}{2},\frac{1}{q}\}}\sqrt{q}. \tag{6.143}$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[\left\|\sum_{i=1}^n \sum_{j=1}^n \varepsilon_{ij} e_i \otimes e_j\right\|_{\ell_p^n \check{\otimes} \ell_q^n}\right] &\leqslant \sqrt{\frac{\pi}{2}}\,\mathbb{E}\left[\left\|\sum_{i=1}^n \sum_{j=1}^n \mathsf{g}_{ij} e_i \otimes e_j\right\|_{\ell_p^n \check{\otimes} \ell_q^n}\right] \\ &\lesssim n^{\beta(p,q)}\sqrt{\max\{p,q\}},\end{aligned} \tag{6.144}$$

where the first step of (6.144) is a standard comparison between Rademacher and Gaussian averages (a quick consequence of Jensen's inequality; e.g., [204]) and the final step of (6.144) uses (6.143). This proves the desired bound (6.140) when

$$\max\{p,q\} \leqslant 2,$$

so suppose from now on that $\max\{p,q\} \geqslant 2$.

It suffices to treat the case $p \geqslant 2$. Indeed, if $p \leqslant 2$, then $q \geqslant 2$ since $\max\{p,q\} \geqslant 2$, so by the duality

$$\|\mathcal{E}\|_{\ell_{p*}^n \to \ell_q^n} = \|\mathcal{E}^*\|_{\ell_{q*}^n \to \ell_p^n},$$

and the fact that the transpose $\mathcal{E}^*$ has the same distribution as $\mathcal{E}$, the case $p \leqslant 2$ follows from the case $p \geqslant 2$. It also suffices to treat the case $q \leqslant p$ because if $q \geqslant p$, then $\|\cdot\|_{\ell_q^n} \leqslant \|\cdot\|_{\ell_p^n}$ point-wise, and therefore

$$\|\mathcal{E}\|_{\ell_{p^*}^n \to \ell_q^n} \leqslant \|\mathcal{E}\|_{\ell_{p^*}^n \to \ell_p^n}.$$

Consequently, since $\beta(p,q) = \beta(p,p)$ when $q \geqslant p$, the case $q \geqslant p$ follows from the case $q = p$.

So, suppose from now that $p \geqslant 2$ and $q \leqslant p$. If we denote

$$r \stackrel{\mathrm{def}}{=} \frac{q(p-2)}{p-q},$$

with the convention $r = \infty$ if $q = p$, then $r \geqslant 1$ and

$$\frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{2}, \quad \text{where } \theta \stackrel{\mathrm{def}}{=} \frac{2}{p} \in [0,1]. \tag{6.145}$$

Hence, by the Riesz–Thorin interpolation theorem [272, 301] we have

$$\begin{aligned} \|\mathcal{E}\|_{\ell_{p^*}^n \to \ell_q^n} &\leqslant \|\mathcal{E}\|_{\ell_1^n \to \ell_r^n}^{1-\theta} \|\mathcal{E}\|_{\ell_2^n \to \ell_2^n}^{\theta} \\ &= \Big(\max_{i \in \{1,\ldots,n\}} \|\mathcal{E}e_i\|_{\ell_r^n} \Big)^{1-\theta} \|\mathcal{E}\|_{\ell_2^n \to \ell_2^n}^{\theta} = n^{\frac{1-\theta}{r}} \|\mathcal{E}\|_{\ell_2^n \to \ell_2^n}^{\theta}. \end{aligned}$$

By taking expectations of this inequality, we get that

$$\begin{aligned} \mathbb{E}\big[\|\mathcal{E}\|_{\ell_{p^*}^n \to \ell_q^n}\big] &\leqslant n^{\frac{1-\theta}{r}} \mathbb{E}\big[\|\mathcal{E}\|_{\ell_2^n \to \ell_2^n}^{\theta}\big] \leqslant n^{\frac{1-\theta}{r}} \big(\mathbb{E}\big[\|\mathcal{E}\|_{\ell_2^n \to \ell_2^n}\big]\big)^{\theta} \\ &\lesssim n^{\frac{1-\theta}{r} + \frac{\theta}{2}} = n^{\frac{1}{q}} = n^{\beta(p,q)}, \end{aligned} \tag{6.146}$$

where the second step of (6.146) uses Jensen's inequality, the third step of (6.146) uses the classical fact that the expectation of the operator norm from ℓ_2^n to ℓ_2^n of an $n \times n$ matrix whose entries are i.i.d. symmetric Bernoulli random variables is $O(\sqrt{n})$ (this follows from (6.144), though it is older; see, e.g., [35]), the penultimate step of (6.146) uses (6.145), and the last step of (6.146) uses the definition of $\beta(p,q)$ in (6.140) while recalling that we are now treating the case $p \geqslant 2$ and $q \leqslant p$. ■

A substitution of Lemma 173 into the proof of [285, Lemma 3.2] yields the following asymptotic evaluations of the n^2-roots of volumes of the unit balls of injective and projective tensor products; the statement of [285, Lemma 3.2] is identical, except that the constant factors depend on p, q, but that is due only to the dependence of the constants on p, q in [285, Lemma 2.3], which Lemma 173 removes

$$\mathrm{vol}_{n^2}\big(B_{\ell_p^n \check{\otimes} \ell_q^n}\big)^{\frac{1}{n^2}} \asymp n^{-\beta(p,q)} \quad \text{and} \quad \mathrm{vol}_{n^2}\big(B_{\ell_p^n \hat{\otimes} \ell_q^n}\big)^{\frac{1}{n^2}} \asymp n^{\beta(p^*,q^*)-2}. \tag{6.147}$$

Since $\ell_p^n \widehat{\otimes} \ell_q^n$ belongs to the class of spaces in Example 40, its Löwner ellipsoid is the minimal multiple of the standard Euclidean ball $B_{\mathsf{S}_2^n}$ that superscribes the unit ball of $\ell_p^n \widehat{\otimes} \ell_q^n$, namely

$$\mathcal{L}_{\ell_p^n \widehat{\otimes} \ell_q^n} = R(n,p,q) B_{\mathsf{S}_2^n},$$

where, since $B_{\ell_p^n \widehat{\otimes} \ell_q^n}$ is the convex hull of $B_{\ell_p^n} \otimes B_{\ell_q^n}$,

$$\begin{aligned} R(n,p,q) &= \max_{\substack{x \in B_{\ell_p^n} \\ y \in B_{\ell_q^n}}} \|x \otimes y\|_{\mathsf{S}_2^n} \\ &= \Big(\max_{x \in B_{\ell_p^n}} \|x\|_{\ell_2^n}\Big)\Big(\max_{y \in B_{\ell_q^n}} \|y\|_{\ell_2^n}\Big) = n^{\max\{\frac12 - \frac1p, 0\} + \max\{\frac12 - \frac1q, 0\}}. \end{aligned} \tag{6.148}$$

By combining (6.147) and (6.148) we get that

$$\begin{aligned} \mathrm{vr}\big(\ell_{p^*}^n \check{\otimes} \ell_{q^*}^n\big) &\overset{(1.71)}{\asymp} \mathrm{evr}\big(\ell_p^n \widehat{\otimes} \ell_q^n\big) \\ &= R(n,p,q) \left(\frac{\mathrm{vol}_{n^2}(B_{\mathsf{S}_2^n})}{\mathrm{vol}_{n^2}(B_{\ell_p^n \widehat{\otimes} \ell_q^n})} \right)^{\frac{1}{n^2}} \\ &\asymp n^{\max\{\frac12 - \frac1p, 0\} + \max\{\frac12 - \frac1q, 0\} - \beta(p^*, q^*) + 1} \\ &\overset{(6.140)}{=} \begin{cases} \sqrt{n} & \text{if } \max\{p,q\} \geqslant 2, \\ n^{\frac{1}{\max\{p,q\}}} & \text{if } \max\{p,q\} \leqslant 2. \end{cases} \end{aligned} \tag{6.149}$$

A substitution of (6.149) into Theorem 3 gives

$$\mathsf{SEP}\big(\ell_p^n \widehat{\otimes} \ell_q^n\big) \gtrsim \begin{cases} n^{\frac32} & \text{if } \max\{p,q\} \geqslant 2, \\ n^{1 + \frac{1}{\max\{p,q\}}} & \text{if } \max\{p,q\} \leqslant 2. \end{cases} \tag{6.150}$$

Furthermore, if Conjecture 11 holds for $\ell_p^n \widehat{\otimes} \ell_q^n$, then (6.150) is sharp, namely (1.15) holds. Also, by Theorem 80 the left-hand side of (6.150) is bounded from above by $O(\log n)$ times the right-hand side of (6.150), thus implying the fifth bullet point of Corollary 4.

Remark 174. The above results imply clustering statements (and impossibility thereof) for norms that have significance to algorithms and complexity theory. For example, the *cut norm* [101] on $\mathsf{M}_n(\mathbb{R})$ is $O(1)$-equivalent [6] to the operator norm from ℓ_∞^n to ℓ_1^n. So, by (1.13) the separation modulus of the cut norm on $\mathsf{M}_n(\mathbb{R})$ is predicted to be bounded above and below by universal constant multiples of $n^{3/2}$, and by Theorem 80 we know that it is at least a universal constant multiple of $n^{3/2}$ and at most a universal constant multiple of $n^{3/2} \log n$. As another notable example, we proved that

$$\mathsf{SEP}(\ell_\infty^n \widehat{\otimes} \ell_\infty^n) \gtrsim n^{\frac32}.$$

Moreover, if Conjecture 11 holds for $\ell_\infty^n \hat{\otimes} \ell_\infty^n$, then $\mathsf{SEP}(\ell_\infty^n \hat{\otimes} \ell_\infty^n) \asymp n^{3/2}$ and by Theorem 80 we have

$$\mathsf{SEP}(\ell_\infty^n \hat{\otimes} \ell_\infty^n) \lesssim n^{\frac{3}{2}} \log n.$$

Grothendieck's inequality [121] implies that

$$\forall A \in \mathsf{M}_n(\mathbb{R}), \quad \|A\|_{\ell_\infty^n \hat{\otimes} \ell_\infty^n} \asymp \gamma_2^{1\to\infty}(A), \tag{6.151}$$

where $\gamma_2^{1\to\infty}(A)$ is the factorization-through-ℓ_2 norm (see [261]) of A as an operator from ℓ_1^n to ℓ_∞^n, i.e.,

$$\begin{aligned}\gamma_2^{1\to\infty}(A) &\stackrel{\mathrm{def}}{=} \min_{\substack{X,Y \in \mathsf{M}_n(\mathbb{R})\\ A=XY}} \|X\|_{\ell_2^n \to \ell_\infty^n} \|Y\|_{\ell_1^n \to \ell_2^n} \\ &= \min_{\substack{X,Y \in \mathsf{M}_n(\mathbb{R})\\ A=XY}} \max_{i,j \in \{1,\ldots,n\}} \|\mathrm{row}_i(X)\|_{\ell_2^n} \|\mathrm{column}_j(Y)\|_{\ell_2^n}.\end{aligned}$$

Above, for $i, j \in \{1, \ldots, n\}$ and $M \in \mathsf{M}_n(\mathbb{R})$ we denote by $\mathrm{row}_i(M)$ and $\mathrm{column}_j(M)$ the ith row and jth column of M, respectively. See [183] for the justification of (6.151), as well as the importance of the factorization norm $\gamma_2^{1\to\infty}$ to complexity theory (see [38, 202] for further algorithmic significance of factorization norms). Thanks to the above discussion, we know that

$$n^{\frac{3}{2}} \lesssim \mathsf{SEP}\big(\mathsf{M}_n(\mathbb{R}), \gamma_2^{1\to\infty}\big) \lesssim n^{\frac{3}{2}} \log n,$$

and that $\mathsf{SEP}(\mathsf{M}_n(\mathbb{R}), \gamma_2^{1\to\infty}) \asymp n^{3/2}$ assuming Conjecture 11. To check that this does not follow from the previously known bounds (1.2), we need to know the asymptotic growth rate of the Banach–Mazur distance between $\ell_\infty^n \hat{\otimes} \ell_\infty^n$ and each of the spaces $\ell_1^{n^2}, \ell_2^{n^2}$. However, these Banach–Mazur distances do not appear in the literature. In response to our inquiry, Carsten Schütt answered this question, by showing that

$$d_{\mathrm{BM}}\big(\ell_2^{n^2}, \ell_\infty^n \hat{\otimes} \ell_\infty^n\big) \asymp d_{\mathrm{BM}}\big(\ell_1^{n^2}, \ell_\infty^n \hat{\otimes} \ell_\infty^n\big) \asymp n. \tag{6.152}$$

More generally, Schütt succeeded to evaluate the asymptotic growth rate of the Banach–Mazur distance between $\ell_p^n \hat{\otimes} \ell_q^n$ and $\ell_p^n \check{\otimes} \ell_q^n$ to each of $\ell_1^{n^2}, \ell_2^{n^2}$ for every $p, q \in [1, \infty]$ (this is a substantial matter that Schütt communicated to us privately and he will publish it elsewhere). Due to (6.152), an application of (1.2) only gives the bounds $n \lesssim \mathsf{SEP}(\ell_\infty^n \hat{\otimes} \ell_\infty^n) \lesssim n^2$, which hold for *every* n^2-dimensional normed space. More generally, Schütt's result shows that (1.13) and (1.15) do not follow from (1.2).

The volume computations of this section are only an indication of the available information. The literature contains many more volume estimates that could be substituted into Theorem 3 and Conjecture 6 to yield new results (and conjectures) on separation moduli of various spaces; examples of further pertinent results appear in [20, 85, 88, 104, 110, 115–117, 145, 146, 285].