Chapter 7

Logarithmic weak isomorphic isoperimetry in minimum dual mean width position

In this section we will prove the results that we stated in Section 1.6.3. We first claim that for every integer $n \geqslant 2$ and every $r > 0$ we have

$$\begin{aligned}\mathrm{iq}\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big) &= \mathrm{iq}\big([-1,1]^n \cap (rB_{\ell_2^n})\big)\\ &\gtrsim \left(\min\{\sqrt{n}, r\}\left(1-\frac{1}{\max\{1,r^2\}}\right)^{\frac{n-1}{2}}+1\right)\sqrt{n}. \end{aligned}\tag{7.1}$$

Observe that (7.1) implies (1.85). Furthermore, (7.1) implies the direction $\gtrsim$ in (1.86) because

$$\begin{aligned}&\min_{r>0} \frac{\mathrm{iq}\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big)}{\sqrt{n}}\left(\frac{\mathrm{vol}_n(B_{\ell_\infty^n})}{\mathrm{vol}_n\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big)}\right)^{\frac{1}{n}}\\ &\geqslant \min_{r>0} \frac{\mathrm{iq}\big(B_{\ell_\infty^n} \cap (rB_{\ell_2^n})\big)}{\sqrt{n}}\left(\frac{2^n}{\mathrm{vol}_n\big(rB_{\ell_2^n}\big)}\right)^{\frac{1}{n}}\\ &\gtrsim \min_{r>0}\left(\min\left\{\frac{\sqrt{n}}{r},1\right\}\left(1-\frac{1}{\max\{1,r^2\}}\right)^{\frac{n-1}{2}}+\frac{1}{r}\right)\sqrt{n}\\ &\asymp \sqrt{\log n},\end{aligned}$$

where the penultimate step uses (7.1) and the final step is elementary calculus. Since the K-convexity constant of ℓ_∞^n satisfies $K(\ell_\infty^n) \asymp \sqrt{\log n}$ (see [263, Chapter 2]), the matching upper bound in (1.86) will follow after we will prove (below) Proposition 61. This will also show that Proposition 61 is sharp, though it would be worthwhile to find out if it is sharp even for some normed space $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ for which $K(\mathbf{X}) \asymp \log n$; such a space exists by a remarkable (randomized) construction of Bourgain [44].

To prove (7.1), note first that if $0 < r \leqslant 1$, then $rB_{\ell_2^n} \subseteq [-1,1]^n$ and therefore

$$\forall 0 < r \leqslant 1, \quad \mathrm{iq}\big([-1,1]^n \cap (rB_{\ell_2^n})\big) = \mathrm{iq}(rB_{\ell_2^n}) \asymp \sqrt{n}. \tag{7.2}$$

Similarly, note that if $r \geqslant \sqrt{n}$, then $rB_{\ell_2^n} \subseteq [-1,1]^n$ and therefore

$$\forall r \geqslant \sqrt{n}, \quad \mathrm{iq}\big([-1,1]^n \cap (rB_{\ell_2^n})\big) = \mathrm{iq}\big([-1,1]^n\big) \asymp n. \tag{7.3}$$

Both (7.2) and (7.3) coincide with (7.1) in the respective ranges. The less trivial range of (7.1) is when $1 < r < \sqrt{n}$, in which case the boundary of $[-1,1]^n \cap (rB_{\ell_2^n})$ contains the disjoint union of the intersection of $rB_{\ell_2^n}$ with the $2n$ faces of $[-1,1]^n$, each of which is isometric to the following set:

$$[-1,1]^{n-1} \cap \left(\sqrt{r^2-1}B_{\ell_2^{n-1}}\right).$$

Together with the straightforward inclusion

$$[-1,1]^{n-1} \cap \left(\sqrt{r^2-1}B_{\ell_2^{n-1}}\right) \supseteq \sqrt{1-\frac{1}{r^2}}\left([-1,1]^{n-1} \cap (rB_{\ell_2^{n-1}})\right),$$

the above observation implies that if $1 < r < \sqrt{n}$, then

$$\begin{aligned}
&\mathrm{vol}_{n-1}\left(\partial\left([-1,1]^n \cap (rB_{\ell_2^n})\right)\right)\\
&\geqslant 2n\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}\left([-1,1]^{n-1} \cap (rB_{\ell_2^{n-1}})\right)\\
&= n\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}} \mathrm{vol}_n\left(\left([-1,1]^{n-1} \cap (rB_{\ell_2^{n-1}})\right) \times [-1,1]\right)\\
&\geqslant n\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}} \mathrm{vol}_n\left([-1,1]^n \cap (rB_{\ell_2^n})\right),
\end{aligned} \tag{7.4}$$

where the final step (7.4) is a consequence of the straightforward inclusion

$$\left([-1,1]^{n-1} \cap (rB_{\ell_2^{n-1}})\right) \times [-1,1] \supseteq [-1,1]^n \cap (rB_{\ell_2^n}).$$

By combining (7.4) with the definition (1.11) of the isoperimetric quotient, we see that

$$\begin{aligned}
\mathrm{iq}\left([-1,1]^n \cap (rB_{\ell_2^n})\right) &\geqslant \frac{n\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}} \mathrm{vol}_n\left([-1,1]^n \cap (rB_{\ell_2^n})\right)}{\mathrm{vol}_n\left([-1,1]^n \cap (rB_{\ell_2^n})\right)^{\frac{n-1}{n}}}\\
&= n\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}} \mathrm{vol}_n\left([-1,1]^n \cap (rB_{\ell_2^n})\right)^{\frac{1}{n}}.
\end{aligned} \tag{7.5}$$

When $r \leqslant \sqrt{n}$ we have

$$[-1,1]^n \cap (rB_{\ell_2^n}) \supseteq \left[-\frac{r}{\sqrt{n}}, \frac{r}{\sqrt{n}}\right]^n.$$

In combination with (7.5), this implies that

$$\forall 1 < r < \sqrt{n}, \quad \mathrm{iq}\left([-1,1]^n \cap (rB_{\ell_2^n})\right) \geqslant 2r\sqrt{n}\left(1-\frac{1}{r^2}\right)^{\frac{n-1}{2}}.$$

As also $\mathrm{iq}([-1,1]^n \cap (rB_{\ell_2^n})) \gtrsim \sqrt{n}$ by the isoperimetric theorem (1.12), this completes the proof of (7.1). ∎

Passing to the proof of Proposition 61, observe first that for every $r > 0$ we have

$$\frac{\mathrm{vol}_n\big(B_{\mathbf{X}} \cap (rB_{\ell_2^n})\big)}{\mathrm{vol}_n(rB_{\ell_2^n})} = \frac{\mathrm{vol}_n\big(\{x \in rB_{\ell_2^n} : \|x\|_{\mathbf{X}} \leqslant 1\}\big)}{\mathrm{vol}_n(rB_{\ell_2^n})} \geqslant 1 - \fint_{rB_{\ell_2^n}} \|x\|_{\mathbf{X}}\, \mathrm{d}x = 1 - \frac{nr}{n+1} M(\mathbf{X}), \tag{7.6}$$

where the penultimate step in (7.6) is Markov's inequality and the final step in (7.6) is integration in polar coordinates using the following standard notation for the mean of the norm on the Euclidean sphere:

$$M(\mathbf{X}) \stackrel{\mathrm{def}}{=} \fint_{S^{n-1}} \|z\|_{\mathbf{X}}\, \mathrm{d}z.$$

We will also use the common notation $M^*(\mathbf{X}) \stackrel{\mathrm{def}}{=} M(\mathbf{X}^*)$. By setting $r = 1/(2M(\mathbf{X}))$ in (7.6) we get that

$$\mathrm{vol}_n\left(B_{\mathbf{X}} \cap \left(\frac{1}{2M(\mathbf{X})} B_{\ell_2^n}\right)\right)^{\frac{1}{n}} \geqslant \left(\frac{1}{2}\mathrm{vol}_n\left(\frac{1}{2M(\mathbf{X})} B_{\ell_2^n}\right)\right)^{\frac{1}{n}} \asymp \frac{1}{M(\mathbf{X})\sqrt{n}}. \tag{7.7}$$

This simple consideration gives the following general elementary lemma.

Lemma 175. *Let $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$ be a normed space. If we denote $r = 1/(2M(\mathbf{X}))$ and $L = B_{\mathbf{X}} \cap (rB_{\ell_2^n})$, then we have*

$$\mathrm{vol}_n(L)^{\frac{1}{n}} \gtrsim \frac{1}{M(\mathbf{X})\sqrt{n}} \quad \text{and} \quad \frac{\mathrm{MaxProj}(L)}{\mathrm{vol}_n(L)^{\frac{n-1}{n}}} \lesssim 1. \tag{7.8}$$

Proof. The first inequality in (7.8) follows from (7.7). For the second inequality in (7.8), observe that $\mathsf{Proj}_{z^\perp}(L) \subseteq \mathsf{Proj}_{z^\perp}(rB_{\ell_2^n})$ for every $z \in S^{n-1}$, since $L \subseteq rB_{\ell_2^n}$. Consequently,

$$\mathrm{MaxProj}(L) \leqslant \mathrm{MaxProj}\big(rB_{\ell_2^n}\big) = r^{n-1}\,\mathrm{vol}_{n-1}\big(B_{\ell_2^{n-1}}\big) \asymp \mathrm{vol}_n\big(rB_{\ell_2^n}\big)^{\frac{n-1}{n}} \lesssim \mathrm{vol}_n(L)^{\frac{n-1}{n}}, \tag{7.9}$$

where the penultimate step of (7.9) is a standard computation using Stirling's formula and the final step of (7.9) uses the first inequality in (7.7). ■

By (1.55), the second inequality in (7.8) implies that $\mathrm{iq}(L) \lesssim \sqrt{n}$. Hence, in order to use Lemma 175 in the context of Conjecture 10 it would be beneficial to choose $S \in \mathsf{SL}_n(\mathbb{R})$ for which $M(S\mathbf{X})$ is small. So, fix $\delta > 0$ and suppose that

$$\delta M(S\mathbf{X}) \leqslant \min_{T \in \mathsf{SL}_n(\mathbb{R})} M(T\mathbf{X}). \tag{7.10}$$

By compactness, this holds for some $S \in \mathsf{SL}_n(\mathbb{R})$ with $\delta = 1$, in which case the polar of $SB_{\mathbf{X}}$ is in *minimum mean width position* and we will say that $S\mathbf{X}$ is in *minimum dual mean width position* (the terminology that is used in [108] is that $SB_{\mathbf{X}}$ *has minimal* M). By [107], the matrix in $\mathsf{SL}_n(\mathbb{R})$ at which $\min_{T\in\mathsf{SL}_n(\mathbb{R})} M(T\mathbf{X})$ is attained is unique up to orthogonal transformations. We allow the flexibility of working with some universal constant $0 < \delta < 1$ rather than considering only the minimum dual mean width position since this will encompass other commonly used positions, such as the ℓ-position (see [55, Section 1.11]). By [107], $\mathbf{X}$ is in minimum dual mean width position if and only if the measure $\mathrm{d}\nu_{\mathbf{X}}(z) = \|z\|_{\mathbf{X}}\,\mathrm{d}z$ on S^{n-1} is isotropic. Since $\nu_{\mathbf{X}}$ is evidently $\mathsf{Isom}(\mathbf{X})$-invariant, by (1.69) if $\mathbf{X}$ is canonically positioned, then it is in minimum dual mean width position.

Let γ denote the standard Gaussian measure on $\mathbb{R}$, i.e., its density at $u \in \mathbb{R}$ equals $\exp(-u^2/2)/\sqrt{2\pi}$. The (Gaussian) K-convexity constant $K(\mathbf{X})$ of $\mathbf{X}$ is defined [204] to be the infimum over those $K > 0$ that satisfy

$$\begin{aligned}\bigg(\int_{\mathbb{R}^{\aleph_0}} \bigg\| \sum_{i=1}^{\infty} \mathsf{g}_i' \int_{\mathbb{R}^{\aleph_0}} \mathsf{g}_i f(\mathsf{g})\,\mathrm{d}\gamma^{\otimes\aleph_0}(\mathsf{g}) \bigg\|_{\mathbf{X}}^2 \mathrm{d}\gamma^{\otimes\aleph_0}(\mathsf{g}')\bigg)^{\frac12} \\ \leqslant K\bigg(\int_{\mathbb{R}^{\aleph_0}} \|f(\mathsf{g})\|_{\mathbf{X}}^2\,\mathrm{d}\gamma^{\otimes\aleph_0}(\mathsf{g})\bigg)^{\frac12},\end{aligned}$$

for every measurable $f:\mathbb{R}^{\aleph_0}\to\mathbf{X}$ with $\int_{\mathbb{R}^{\aleph_0}} \|f(\mathsf{g})\|_{\mathbf{X}}^2\,\mathrm{d}\gamma^{\otimes\aleph_0}(\mathsf{g}) < \infty$. By [100] there is $T \in \mathsf{SL}_n(\mathbb{R})$ such that $M(T\mathbf{X})M^*(T\mathbf{X}) \leqslant K(\mathbf{X})$. By the above assumption (7.10) we know that $\delta M(S\mathbf{X}) \leqslant M(T\mathbf{X})$, so $\delta M(S\mathbf{X}) \leqslant K(\mathbf{X})/M^*(T\mathbf{X})$. Next, we always have

$$M(\mathbf{X}) \geqslant \bigg(\frac{\mathrm{vol}_n(B_{\ell_2^n})}{\mathrm{vol}_n(B_{\mathbf{X}})}\bigg)^{\frac1n};$$

see, e.g., [218, Section 2] and [132, Lemma 30] for two derivations of this well-known volumetric lower bound on $M(\mathbf{X})$. Applying this lower bound to the dual of $T\mathbf{X}$, we get $M^*(T\mathbf{X}) \geqslant (\mathrm{vol}_n(B_{\ell_2^n})/\,\mathrm{vol}_n(B_{\mathbf{X}^*}))^{1/n}$. The Blaschke–Santaló inequality [39, 278] states that

$$\frac{\mathrm{vol}_n(B_{\ell_2^n})}{\mathrm{vol}_n(B_{\mathbf{X}^*})} \geqslant \frac{\mathrm{vol}_n(B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\ell_2^n})},$$

so we conclude that $\delta M(S\mathbf{X})\sqrt{n} \lesssim K(\mathbf{X})/\sqrt[n]{\mathrm{vol}_n(B_{\mathbf{X}})}$. A substitution of this bound into Lemma 175 gives the following proposition:

Proposition 176. *Fix* $0 < \delta \leqslant 1$ *and a normed space* $\mathbf{X} = (\mathbb{R}^n, \|\cdot\|_{\mathbf{X}})$. *Suppose that* $S \in \mathsf{SL}_n(\mathbb{R})$ *satisfies*

$$\delta M(S\mathbf{X}) \leqslant \min_{T\in\mathsf{SL}_n(\mathbb{R})} M(T\mathbf{X}).$$

Then, denoting $r = 1/(2M(S\mathbf{X}))$ *we have*

$$\mathrm{vol}_n\big((SB_{\mathbf{X}}) \cap (rB_{\ell_2^n})\big)^{\frac{1}{n}} \gtrsim \frac{\delta}{K(\mathbf{X})}\,\mathrm{vol}_n(B_{\mathbf{X}})^{\frac{1}{n}}$$

and

$$\mathrm{MaxProj}\big((SB_{\mathbf{X}}) \cap (rB_{\ell_2^n})\big) \lesssim \mathrm{vol}_n\big((SB_{\mathbf{X}}) \cap (rB_{\ell_2^n})\big)^{\frac{n-1}{n}}.$$

Furthermore, if $\mathbf{X}$ *is canonically positioned, then this holds when* S *is the identity matrix and* $\delta = 1$.

By (1.55), Proposition 176 implies Proposition 61, with the additional information that the conclusion of Proposition 61 holds with S the identity matrix if $\mathbf{X}$ is in minimum dual mean width position, in which case we obtain an upper bound on $\mathrm{MaxProj}(L)$. Hence, by the reasoning in Section 1.6, if $\mathbf{X}$ is in minimum dual mean width position, then

$$\mathsf{SEP}(\mathbf{X}) \lesssim K(\mathbf{X})\frac{\mathrm{diam}_{\ell_2^n}(B_{\mathbf{X}})}{\mathrm{vol}_n(B_{\mathbf{X}})^{\frac{1}{n}}}.$$