Foreword

As indicated by the title of this memoir, this work is a survey of properties of integrals of the Wigner distribution on subsets of the phase space. Since it is quite lengthy, we wish in this foreword to describe the content of this article, browsing through the table of contents, expecting that the reader will find some organization with the way this memoir is written. In particular, we shall point here what is original in our survey (to the best of our knowledge) and what was well-known beforehand. There is no doubt that the fifty-five articles quoted in the references list are a small part of the literature on the topic and could be probably extended tenfold: we expect nevertheless that our choice of references will be enough to cover the most important contributions.

Chapter 1 is *Preliminaries and definitions* and is very classical. We have used J. Leray's book [31] and other lecture notes of this author at the *Collège de France* such as [30], L. Hörmander's four-volume treatise, *The analysis of linear partial differential operators* and, in particular, volume III, as well as K. Gröchenig's [16], *Foundations of time-frequency analysis*, along with G. B. Folland's [15], A. Unterberger's [50] and N. Lerner's [33]. Some details are given, in particular, on positive quantizations, but that chapter is far from being self-contained, which is probably unavoidable: the link of properties of the Wigner distribution and of the Weyl quantization of classical Hamiltonians is easy to obtain but turns out to be an important piece of information for our purpose.

Chapter 2 is stressing the link *Quantization of radial functions – Mehler's formula* and is also very classical: here also the link aforementioned is easy to get but gives some simplifications in the formulas providing the quantization of radial Hamiltonians: in one dimension for the configuration space (phase space $\mathbb{R}^2$), we are reduced to check simple integrals related to the Laguerre polynomials, following P. Flandrin's method in his 1988 article [13].

Chapter 3 is dealing with *Conics with eccentricity* < 1. The result for the disc in $\mathbb{R}^2$ is due to P. Flandrin and the result for the Euclidean ball in $\mathbb{R}^{2n}$ to E. Lieb and Y. Ostrover in [39]. Using Mehler's formula simplifies a little bit the presentation, but leaves open the case of anisotropic ellipsoids for which we formulate a conjecture.

Chapter 4 is dealing with *Epigraphs of Parabolas*. The results obtained in that chapter follow easily from Chapter 3 but nevertheless the precise diagonalisation proven there seems to be new. We formulate also a conjecture on anisotropic paraboloids which is closely related to the conjecture in Chapter 3.

Chapter 5 is concerned with *Conics with eccentricity* > 1. Many of the results in that chapter are contained in the paper [55] by J. G. Wood and A. J. Bracken; however since the latter article contains some formal calculations, using for instance test functions which do not belong to $L^2(\mathbb{R})$, we have made a mathematically sound presentation. As certainly the most important contribution of this work, we provide a "theoretical" disproof of Flandrin's conjecture on integrals of the Wigner distribution on convex subsets of the phase space: we find, in particular, some $a > 0$ and some function $u \in L^2(\mathbb{R})$ with norm 1 such that

$$\iint_{[0,a]^2} \mathcal{W}(u,u)(x,\xi)dxd\xi > 1,$$

where $\mathcal{W}(u,u)$ is the Wigner distribution of u. This fact was already proven in our joint paper [6] with B. Delourme and T. Duyckaerts, using a rigorous numerical argument.

Chapter 6 is entitled *Unboundedness is Baire generic* and most of its content is included in Chapter 12 of K. Gröchenig's book [16]. Using the Feichtinger algebra, we show that, generically in the Baire sense, the Wigner distribution of a pulse in $L^2(\mathbb{R}^n)$ does not belong to $L^1(\mathbb{R}^{2n})$, providing as a byproduct a large class of examples of subsets of the phase space $\mathbb{R}^{2n}$ on which the integral of the Wigner distribution is infinite. We raise a couple of questions, in particular, whether we can find a pulse $u \in L^2(\mathbb{R}^n)$ such that

$$E_+(u) = \{(x,\xi) \in \mathbb{R}^{2n}, \mathcal{W}(u,u)(x,\xi) > 0\} \text{ is connected.}$$

Chapter 7 is *Convex polygons in the plane*: we study there the sets defined by the intersection of N half-spaces in the plane $\mathbb{R}^2$ and the integrals of the Wigner distribution on these sets. We start with convex cones ($N = 2$) for which a complete result is known and we go on with triangles ($N = 3$) for which we find an upper bound: the integral of $\mathcal{W}(u,u)$ on a triangle of $\mathbb{R}^2$ for a normalized pulse in $L^2(\mathbb{R})$ is bounded above by a universal constant. We show also that the integral of $\mathcal{W}(u,u)$ on a convex polygon with N sides of $\mathbb{R}^2$ for a normalized pulse in $L^2(\mathbb{R})$ is bounded above by a universal constant $\times\sqrt{N}$. We raise a couple of questions: in particular it seems possible that the behaviour of convex subsets of the plane is such that there exists a constant $\alpha > 1$ such that

$$\text{for all } C \text{ convex subset of the plane } \mathbb{R}^2, \text{ for all } u \in L^2(\mathbb{R}) \text{ with } \|u\|_{L^2(\mathbb{R})} = 1,$$
$$\text{we have } \iint_C \mathcal{W}(u,u)(x,\xi)dxd\xi \le \alpha.$$

That would be a weak version of Flandrin's conjecture: the original Flandrin's conjecture was the above statement with $\alpha = 1$, which is untrue, but that does not rule out the existence of a number $\alpha > 1$ such that the above estimate holds true.

Chapter 8 is entitled *Open questions and Conjectures*: we review in that chapter the various conjectures that we meet along the text of the memoir, estimating the importance and difficulty of the various questions. Chapter A is an appendix containing only classical material, hopefully helping the reader by improving the self-containedness of this memoir.