Chapter 1

Preliminaries and definitions

1.1 The Wigner distribution

Let u, v be given functions in $L^2(\mathbb{R}^n)$. The function Ω, defined on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\mathbb{R}^n \times \mathbb{R}^n \ni (z, x) \mapsto u\Big(x + \frac{z}{2}\Big)\bar{v}\Big(x - \frac{z}{2}\Big) = \Omega(u, v)(x, z), \tag{1.1.1}$$

belongs to $L^2(\mathbb{R}^{2n})$ from the identity

$$\int_{\mathbb{R}^{2n}} |\Omega(u, v)(x, z)|^2 dxdz = \|u\|^2_{L^2(\mathbb{R}^n)} \|v\|^2_{L^2(\mathbb{R}^n)}. \tag{1.1.2}$$

We have also

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} |\Omega(x, z)| dz \le 2^n \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}. \tag{1.1.3}$$

We may then give the following definition (the reader will find some reminders on the Fourier transformation in Section A.1 of our appendix).

Definition 1.1.1. Let u, v be given functions in $L^2(\mathbb{R}^n)$. We define the joint Wigner distribution $\mathcal{W}(u, v)$ as the partial Fourier transform with respect to z of the function Ω defined in (1.1.1). We have for $(x, \xi) \in \mathbb{R}^n_x \times \mathbb{R}^n_\xi$, using (1.1.3),

$$\mathcal{W}(u, v)(x, \xi) = \int_{\mathbb{R}^n} e^{-2i\pi z \cdot \xi} u\Big(x + \frac{z}{2}\Big)\bar{v}\Big(x - \frac{z}{2}\Big) dz. \tag{1.1.4}$$

The Wigner distribution of u is defined as $\mathcal{W}(u, u)$.

N.B. By inverse Fourier transformation we get, in a weak sense,

$$u(x_1) \otimes \bar{v}(x_2) = \int \mathcal{W}(u, v)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1 - x_2)\cdot\xi} d\xi. \tag{1.1.5}$$

Lemma 1.1.2. *Let u, v be given functions in $L^2(\mathbb{R}^n)$. The function $\mathcal{W}(u, v)$ belongs to $L^2(\mathbb{R}^{2n})$ and we have*

$$\|\mathcal{W}(u, v)\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}. \tag{1.1.6}$$

We have also

$$\overline{\mathcal{W}(u, v)(x, \xi)} = \mathcal{W}(v, u)(x, \xi), \tag{1.1.7}$$

so that $\mathcal{W}(u, u)$ is real-valued.

Proof. Note that the function $\mathcal{W}(u, v)$ is in $L^2(\mathbb{R}^{2n})$ and satisfies (1.1.6) from (1.1.2) and the definition of $\mathcal{W}$ as the partial Fourier transform of Ω. Property (1.1.7) is immediate and entails that $\mathcal{W}(u, u)$ is real-valued. ■

Remark 1.1.3. We note also that the real-valued function $\mathcal{W}(u, u)$ can take negative values, choosing, for instance,

$$u_1(x) = xe^{-\pi x^2}$$

on the real line, we get

$$\mathcal{W}(u_1, u_1)(x, \xi) = 2^{1/2}e^{-2\pi(x^2+\xi^2)}\Big(x^2 + \xi^2 - \frac{1}{4\pi}\Big).$$

In fact, the real-valued function $\mathcal{W}(u, u)$ will take negative values unless u is a Gaussian function, thanks to a Theorem due to E. Lieb (see [37] and books [16] and [41]). As a matter of fact, this range of $\mathcal{W}(u, u)$ intersecting $\mathbb{R}_-$ for most "pulses" u in $L^2(\mathbb{R}^n)$ makes rather weird the qualification of $\mathcal{W}(u, u)$ as a "quasi-probability" (anyhow the emphasis must be on *quasi*, not on *probability*).

Remark 1.1.4. We have also by Fourier inversion formula, say for $u \in \mathscr{S}(\mathbb{R}^n)$,

$$u\Big(x + \frac{z}{2}\Big)\bar{u}\Big(x - \frac{z}{2}\Big) = \Omega(x, z) = \int \mathcal{W}(u, u)(x, \xi)e^{2i\pi z\cdot\xi}d\xi, \tag{1.1.8}$$

so that, with $z = 2x = y$, we get the reconstruction formula,

$$u(y)\bar{u}(0) = \int \mathcal{W}(u, u)\Big(\frac{y}{2}, \xi\Big)e^{2i\pi y\cdot\xi}d\xi,$$

as well as

$$|u(x)|^2 = \int \mathcal{W}(u, u)(x, \xi)d\xi, \quad |\hat{u}(\xi)|^2 = \int \mathcal{W}(u, u)(x, \xi)dx, \tag{1.1.9}$$

the former formula following from (1.1.8) and the latter from

$$\begin{aligned}\int \mathcal{W}(u, u)(x, \xi)dx &= \iint e^{-2i\pi z\xi}u\Big(x + \frac{z}{2}\Big)\bar{u}\Big(x - \frac{z}{2}\Big)dzdx \\ &= \iint e^{-2i\pi\xi(x_1-x_2)}u(x_1)\bar{u}(x_2)dx_1dx_2 = |\hat{u}(\xi)|^2.\end{aligned}$$

Lemma 1.1.5. *Let u be a function in $L^2(\mathbb{R}^n)$ which is even or odd. Then, $\mathcal{W}(u, u)$ is an even function.*

Proof. Using the notation

$$\check{u}(x) = u(-x), \tag{1.1.10}$$

we check

$$\begin{aligned}\mathcal{W}(u,v)(-x,-\xi) &= \int_{\mathbb{R}^n} e^{2i\pi z\cdot\xi} u\Big(-x+\frac{z}{2}\Big)\bar{v}\Big(-x-\frac{z}{2}\Big)dz \\ &= \int_{\mathbb{R}^n} e^{2i\pi z\cdot\xi} \check{u}\Big(x-\frac{z}{2}\Big)\bar{\check{v}}\Big(x+\frac{z}{2}\Big)dz \\ &= \int_{\mathbb{R}^n} e^{-2i\pi z\cdot\xi} \check{u}\Big(x+\frac{z}{2}\Big)\bar{\check{v}}\Big(x-\frac{z}{2}\Big)dz \\ &= \mathcal{W}(\check{u},\check{v})(x,\xi),\end{aligned}$$

so that if $\check{u} = \pm u$, we get $\mathcal{W}(u,u)(-x,-\xi) = \mathcal{W}(u,u)(x,\xi)$. ∎

N.B. This lemma is a very particular case of the symplectic covariance property displayed below in (1.2.49).

N.B. In part 1 of volume IV in the collected works [54] of Eugene P. Wigner, we find the first occurrence of what will be called later on the *Wigner distribution* along with a physicist point of view.

It turns out that most of the properties of the Wigner distribution (in particular, Lemma 1.1.5) are inherited from its links with the Weyl quantization introduced by H. Weyl in 1926 in the first edition of [53] and our next remarks are devised to stress that link.

1.2 Weyl quantization, composition formulas, positive quantizations

1.2.1 Weyl quantization

The main goal of Hermann Weyl in his seminal paper [53] was to give a simple formula, also providing symplectic covariance, ensuring that real-valued Hamiltonians $a(x,\xi)$ get quantized by formally self-adjoint operators. The standard way of dealing with differential operators does not achieve that goal since for instance the standard quantization of the Hamiltonian $x\xi$ (indeed real-valued) is the operator xD_x, which is not symmetric (D_x is defined in (A.1.4)); H. Weyl's choice in that case was

$$x\xi \text{ should be quantized by the operator } \quad \frac{1}{2}(xD_x + D_x x), \quad \text{(indeed symmetric)},$$

and more generally, say for $a \in \mathscr{S}(\mathbb{R}^{2n})$, $u \in \mathscr{S}(\mathbb{R}^n)$, the quantization of the Hamiltonian $a(x,\xi)$, denoted by $\mathrm{Op_w}(a)$, should be given by the formula

$$(\mathrm{Op_w}(a)u)(x) = \iint e^{2i\pi(x-y)\cdot\xi} a\Big(\frac{x+y}{2},\xi\Big)u(y)dyd\xi.$$

For $v \in \mathscr{S}(\mathbb{R}^n)$, we may consider

$$\begin{aligned}\langle \mathrm{Op_w}(a)u, v\rangle_{L^2(\mathbb{R}^n)} &= \iiint a(x,\xi)e^{-2i\pi z\cdot\xi}u\Big(x+\frac{z}{2}\Big)\bar{v}\Big(x-\frac{z}{2}\Big)dz dx d\xi \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} a(x,\xi)\mathcal{W}(u,v)(x,\xi)dx d\xi,\end{aligned}$$

and the latter formula allows us to give the following definition.

Definition 1.2.1. Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. We define the Weyl quantization $\mathrm{Op_w}(a)$ of the Hamiltonian a, by the formula

$$(\mathrm{Op_w}(a)u)(x) = \iint e^{2i\pi(x-y)\cdot\xi}a\Big(\frac{x+y}{2},\xi\Big)u(y)dy d\xi,$$

to be understood weakly as

$$\langle \mathrm{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}. \tag{1.2.1}$$

We note that the sesquilinear mapping

$$\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n) \ni (u,v) \mapsto \mathcal{W}(u,v) \in \mathscr{S}(\mathbb{R}^{2n}),$$

is continuous so that the above bracket of duality $\langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$ makes sense. We note as well that a temperate distribution $a \in \mathscr{S}'(\mathbb{R}^{2n})$ gets quantized by a continuous operator $\mathrm{Op_w}(a)$ from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. This very general framework is not really useful since we want to compose our operators $\mathrm{Op_w}(a)\mathrm{Op_w}(b)$. A first step in this direction is to look for sufficient conditions ensuring that the operator $\mathrm{Op_w}(a)$ is bounded on $L^2(\mathbb{R}^n)$. Moreover, for $a \in \mathscr{S}'(\mathbb{R}^{2n})$ and b a polynomial in $\mathbb{C}[x,\xi]$, we have the composition formula,

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \mathrm{Op_w}(a\sharp b), \tag{1.2.2}$$

$$(a\sharp b)(x,\xi) = \sum_{k\geq 0}\frac{1}{(4i\pi)^k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)(\partial_x^\alpha\partial_\xi^\beta b)(x,\xi), \tag{1.2.3}$$

which involves here a finite sum. This follows from [33, formula (2.1.26)] where several generalizations can be found (see in particular in that reference the integral formula (2.1.18) which can be given a meaning for quite general classes of symbols). As a consequence of (1.2.3), we get that

$$\begin{aligned}&(a\sharp b) = \sum_{k\geq 0}\omega_k(a,b), \quad \omega_0(a,b) = ab, \quad \omega_1(a,b) = \frac{1}{4i\pi}\{a,b\}, \\ &\{a,b\} = \partial_\xi a\cdot\partial_x b - \partial_x a\partial_\xi b,\end{aligned} \tag{1.2.4}$$

where $\{a,b\}$ is called the *Poisson bracket* of a and b.

Proposition 1.2.2. *Let a be a tempered distribution on $\mathbb{R}^{2n}$. Then, we have*

$$\|\mathrm{Op}_\mathrm{w}(a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le \min\big(2^n\|a\|_{L^1(\mathbb{R}^{2n})}, \|\hat{a}\|_{L^1(\mathbb{R}^{2n})}\big). \tag{1.2.5}$$

Proof. In fact, we have from (1.2.1), $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \mathrm{Op}_\mathrm{w}(a)u, v\rangle_{L^2(\mathbb{R}^n)} = \iiint a(x,\xi)u(2x-y)\bar{v}(y)e^{-4i\pi(x-y)\cdot\xi}2^n dydxd\xi,$$

and we define for $(x,\xi) \in \mathbb{R}^{2n}$ the operator $\sigma_{x,\xi}$ by

$$(\sigma_{x,\xi}u)(y) = u(2x-y)e^{-4i\pi(x-y)\cdot\xi}. \tag{1.2.6}$$

Claim 1.2.3. The operator $\sigma_{x,\xi}$ (called *phase symmetry*, also known as the *Grossman–Royer operator*) is unitary and self-adjoint.

Proof of Claim 1.2.3. Indeed, we have

$$\begin{aligned}(\sigma_{x,\xi}^2 u)(y) &= (\sigma_{x,\xi}u)(2x-y)e^{-4i\pi(x-y)\cdot\xi}\\ &= u(2x-(2x-y))e^{-4i\pi(x-(2x-y))\cdot\xi}e^{-4i\pi(x-y)\cdot\xi}\\ &= u(y), \text{ so that } \sigma_{x,\xi}^2 = \mathrm{Id}.\end{aligned}$$

We have also

$$\begin{aligned}\langle \sigma_{x,\xi}^* u, v\rangle_{L^2(\mathbb{R}^n)} &= \langle u, \sigma_{x,\xi}v\rangle_{L^2(\mathbb{R}^n)}\\ &= \overline{\mathcal{W}(v,u)(x,\xi)} = \mathcal{W}(u,v)(x,\xi)\\ &= \langle \sigma_{x,\xi}u, v\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

proving that $\sigma_{x,\xi}^* = \sigma_{x,\xi}$. ■

We have thus

$$\mathrm{Op}_\mathrm{w}(a) = 2^n \iint a(x,\xi)\sigma_{x,\xi}dxd\xi, \tag{1.2.7}$$

and the previous claim is proving the first estimate of the proposition. As a consequence of (1.2.7), we obtain that

$$(\mathrm{Op}_\mathrm{w}(a))^* = \mathrm{Op}_\mathrm{w}(\bar{a}), \text{ so that for } a \text{ real-valued, } (\mathrm{Op}_\mathrm{w}(a))^* = \mathrm{Op}_\mathrm{w}(a).$$

To prove the second estimate, we introduce the so-called ambiguity function $\mathcal{A}(u,v)$ as the inverse Fourier transform of the Wigner function $\mathcal{W}(u,v)$, so that for u, v in the Schwartz class, we have

$$(\mathcal{A}(u,v))(\eta,y) = \iint \mathcal{W}(u,v)(x,\xi)e^{2i\pi(x\cdot\eta+\xi\cdot y)}dxd\xi,$$

i.e.,

$$(\mathcal{A}(u,v))(\eta,y)=\int u\Big(x+\frac{y}{2}\Big)\bar v\Big(x-\frac{y}{2}\Big)e^{2i\pi x\cdot\eta}dx, \tag{1.2.8}$$

which reads as well as

$$(\mathcal{A}(u,v))(\eta,y)=\int u\Big(\frac{y}{2}+\frac{z}{2}\Big)\bar{\check v}\Big(\frac{y}{2}-\frac{z}{2}\Big)e^{2i\pi z\cdot\frac{\eta}{2}}dz2^{-n}=\mathcal{W}(u,\check v)\Big(\frac{y}{2},-\frac{\eta}{2}\Big)2^{-n}. \tag{1.2.9}$$

N.B. The ambiguity function is called the *Fourier–Wigner transform* in G. B. Folland's book [15].

Remark 1.2.4. With $\Omega(u,v)$ defined by (1.1.1), we have

$$\mathcal{W}(u,v)=\mathcal{F}_2(\Omega(u,v)), \tag{1.2.10}$$

where $\mathcal{F}_2$ stands for the Fourier transformation with respect to the second variable. Taking the Fourier transform with respect to the second variable in the previous formula gives, with $\mathcal{F}_j$ (resp., $\mathcal{F}$) standing for the Fourier transform with respect to the jth variable (resp., all variables),

$$\mathcal{F}_2\mathcal{W}=\mathcal{C}_2\Omega,\quad \mathcal{F}\mathcal{W}=\mathcal{F}_1\mathcal{C}_2\Omega,\quad \mathcal{A}=\mathcal{C}\mathcal{F}\mathcal{W}=\mathcal{F}_1\mathcal{C}_1\Omega,$$

where $\mathcal{C}$ (resp., $\mathcal{C}_1$ or $\mathcal{C}_2$) stands for the "check" operator $\mathcal{C}$ in $\mathbb{R}^n\times\mathbb{R}^n$ given by (1.1.10) (resp., with respect to the first or second variable), the latter formula being (1.2.8).

Applying Plancherel formula on (1.2.1), we get

$$\langle \operatorname{Op_w}(a)u,v\rangle_{L^2(\mathbb{R}^n)}=\langle \hat a,\mathcal{A}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}.$$

We note that a consequence of (1.2.3) is that for a linear form $L(x,\xi)$, we have

$$L\sharp L=L^2 \text{ and more generally } L^{\sharp N}=L^N.$$

As a result, considering for $(y,\eta)\in\mathbb{R}^{2n}$, the linear form $L_{\eta,y}$ defined by

$$L_{\eta,y}(x,\xi)=x\cdot\eta+\xi\cdot y,$$

we see that

$$\mathcal{A}(u,v)(\eta,y)=\langle \operatorname{Op_w}(e^{2i\pi(x\cdot\eta+\xi\cdot y)})u,v\rangle_{L^2(\mathbb{R}^n)},$$

and thus we get Hermann Weyl's original formula

$$\operatorname{Op_w}(a)=\iint \hat a(\eta,y)e^{i\operatorname{Op_w}(L_{\eta,y})}dyd\eta,$$

which implies the second estimate in the proposition. ∎

Proposition 1.2.5. *Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. The distribution kernel $k_a(x,y)$ of the operator* $\operatorname{Op_w}(a)$ *is*

$$k_a(x,y) = \hat{a}^{[2]}\Big(\frac{x+y}{2}, y-x\Big), \tag{1.2.11}$$

where $a^{[2]}$ stands for the Fourier transform of a with respect to the second variable. Let $k \in \mathscr{S}'(\mathbb{R}^{2n})$ be the distribution kernel of a continuous operator A from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$. Then, the Weyl symbol a of A is

$$a(x,\xi) = \int e^{-2\pi i t\cdot\xi} k\Big(x+\frac{t}{2}, x-\frac{t}{2}\Big)dt,$$

where the integral sign means that we take the Fourier transform with respect to t of the distribution $k(x+\frac{t}{2}, x-\frac{t}{2})$ on $\mathbb{R}^{2n}$ (see Section A.1.1 for the definition of the Fourier transformation on tempered distributions).

Proof. With $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have defined $\operatorname{Op_w}(a)$ via formula (1.2.1) and using Remark 1.2.4, we get

$$\begin{aligned}\langle \operatorname{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} &= \big\langle a(x,\xi), \widehat{\Omega}^{[2]}(x,\xi)\big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} \\ &= \Big\langle \hat{a}^{[2]}(t,z), u\Big(t+\frac{z}{2}\Big)\bar{v}\Big(t-\frac{z}{2}\Big)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} \\ &= \Big\langle \hat{a}^{[2]}\Big(\frac{x+y}{2}, y-x\Big), u(y)\bar{v}(x)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},\end{aligned}$$

proving (1.2.11). As a consequence, we find that

$$k_a\Big(x+\frac{t}{2}, x-\frac{t}{2}\Big) = \hat{a}^{[2]}(x,-t),$$

and by Fourier inversion, this entails

$$\begin{aligned}a(x,\xi) &= \operatorname{Fourier}_t\Big(k_a\Big(x+\frac{t}{2}, x-\frac{t}{2}\Big)\Big)(\xi) \\ &= \int e^{-2\pi i t\cdot\xi} k_a\Big(x+\frac{t}{2}, x-\frac{t}{2}\Big)dt, \end{aligned} \tag{1.2.12}$$

where the integral sign means that we perform a Fourier transformation with respect to the variable t. ∎

A particular case of Segal's formula (see, e.g., [33, Theorem 2.1.2]) is with $\mathcal{F}$ standing for the Fourier transformation on $\mathbb{R}^n$,

$$\mathcal{F}^*\operatorname{Op_w}(a)\mathcal{F} = \operatorname{Op_w}(a(\xi,-x)).$$

1.2.2 The symplectic group

We define the canonical symplectic form σ on $\mathbb{R}^n \times \mathbb{R}^n$ by

$$\langle \sigma X, Y\rangle = [X, Y] = \xi \cdot y - \eta \cdot x \quad \text{with } X = (x, \xi), Y = (y, \eta). \tag{1.2.13}$$

The symplectic group[1] $\mathrm{Sp}(n, \mathbb{R})$ is the subgroup of $S \in \mathrm{Gl}(2n, \mathbb{R})$ such that

$$\forall X, Y \in \mathbb{R}^{2n}, \quad [SX, SY] = [X, Y], \text{ i.e., } S^* \sigma S = \sigma, \tag{1.2.14}$$

where S^* is the transpose and

$$\sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.2.15}$$

It is easy to prove directly from (1.2.14) that $\mathrm{Sp}(1, \mathbb{R}) = \mathrm{Sl}(2, \mathbb{R})$.

Theorem 1.2.6. *Let n be an integer ≥ 1. The group $\mathrm{Sp}(n, \mathbb{R})$ is included in $\mathrm{Sl}(2n, \mathbb{R})$ and generated by the following mappings*

$$\begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix}, \quad \text{where } A \text{ is an } n \times n \text{ symmetric matrix,} \tag{1.2.16}$$

$$\begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix}, \quad B \in \mathrm{Gl}(n, \mathbb{R}), \tag{1.2.17}$$

$$\begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix}, \quad \text{where } C \text{ is an } n \times n \text{ symmetric matrix.} \tag{1.2.18}$$

For A, B, C as above, the mapping

$$\Xi_{A,B,C} = \begin{pmatrix} B^{-1} & -B^{-1}C \\ AB^{-1} & B^* - AB^{-1}C \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix}, \tag{1.2.19}$$

belongs to $\mathrm{Sp}(n, \mathbb{R})$. Moreover, we define on $\mathbb{R}^n \times \mathbb{R}^n$ the generating function S of the symplectic mapping $\Xi_{A,B,C}$ by the identity

$$S(x, \eta) = \frac{1}{2}(\langle Ax, x\rangle + 2\langle Bx, \eta\rangle + \langle C\eta, \eta\rangle) \text{ so that } \Xi\Big(\frac{\partial S}{\partial \eta} \oplus \eta\Big) = x \oplus \frac{\partial S}{\partial x}. \tag{1.2.20}$$

For a symplectic mapping Ξ, to be of the form (1.2.19) is equivalent to the assumption that the mapping $x \mapsto \pi_{\mathbb{R}^n \times \{0\}} \Xi(x \oplus 0)$ is invertible from $\mathbb{R}^n$ to $\mathbb{R}^n$; moreover, if this mapping is not invertible, the symplectic mapping Ξ is the product of two mappings of the type $\Xi_{A,B,C}$.

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[1] This is obviously a group since for $S_1, S_2 \in Sp(n, \mathbb{R})$, the last equation in (1.2.14) implies that $|\det S| = 1$ and $[S_1 S_2^{-1} X, S_1 S_2^{-1} Y] = [S_2^{-1} X, S_2^{-1} Y] = [X, Y]$, since $[S_2^{-1} X, S_2^{-1} Y] = [S_2 S_2^{-1} X, S_2 S_2^{-1} Y] = [X, Y]$. We shall prove below that the determinant of a symplectic mapping is actually 1.

Proof. The expression of Ξ above as well as (1.2.20) follow from a simple direct computation left to the reader. The inclusion of the symplectic group in the special linear group follows from the statement on the generators. We consider now Ξ in $\mathrm{Sp}(n,\mathbb{R})$: we have

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \quad \text{where } P, Q, R, S, \text{ are } n \times n \text{ matrices.} \tag{1.2.21}$$

The equation

$$\Xi^* \sigma \Xi = \sigma,$$

is satisfied with $\sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, which means

$$P^*R = (P^*R)^*, \quad Q^*S = (Q^*S)^*, \quad P^*S - R^*Q = I_n. \tag{1.2.22}$$

We can note also that the mapping $\Xi \mapsto \Xi^*$ is an isomorphism of $\mathrm{Sp}(n,\mathbb{R})$ since $\Xi \in \mathrm{Sp}(n,\mathbb{R})$ means

$$\Xi^*\sigma\Xi = \sigma \Longrightarrow \Xi^{-1}\sigma^{-1}(\Xi^*)^{-1} = \sigma^{-1} \Longrightarrow \Xi^{-1}(-\sigma^{-1})(\Xi^*)^{-1} = (-\sigma^{-1}),$$

and since $(-\sigma^{-1}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, we get that $\Xi^* \in \mathrm{Sp}(n,\mathbb{R})$. As a result,

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in \mathrm{Sp}(n,\mathbb{R}), \tag{1.2.23}$$

is also equivalent to

$$PQ^* = (PQ^*)^*, \quad RS^* = (RS^*)^*, \quad PS^* - QR^* = I_n. \tag{1.2.24}$$

Let us assume that the mapping P is invertible, which is the assumption in the last statement of the theorem. We define then the mappings A, B, C by

$$A = RP^{-1}, \quad B = P^{-1}, \quad C = -P^{-1}Q,$$

so that we have

$$A^* = P^{*-1}R^*PP^{-1} = P^{*-1}P^*RP^{-1} = RP^{-1} = A,$$

as well as

$$C^* = -Q^*P^{*-1} = -P^{-1}PQ^*P^{*-1} = -P^{-1}QP^*P^{*-1} = -P^{-1}Q = C,$$

and

$$P = B^{-1}, \quad R = AB^{-1}, \quad Q = -B^{-1}C,$$
$$S = P^{*-1}(I_n + R^*Q) = B^*(I_n - B^{*-1}A^*B^{-1}C) = B^* - AB^{-1}C.$$

We have thus proven that any symplectic matrix Ξ as above such that P is invertible is indeed given by the product appearing in Theorem 1.2.6.

Let us now consider the case where a symplectic mapping Ξ (given by (1.2.23)) is such that $\det P = 0$; writing $\mathbb{R}^n = \ker P \oplus N$ we have that P is an isomorphism from N onto $\operatorname{ran} P$. Let $B_1 \in \mathrm{Gl}(n, \mathbb{R})$ such that $B_1 P$ is the identity on N (see footnote[2]). We have

$$\begin{pmatrix} B_1 & 0 \\ 0 & B_1^{*-1} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} B_1 P & B_1 Q \\ B_1^{*-1} R & B_1^{*-1} S \end{pmatrix}. \tag{1.2.25}$$

If $p = \dim(\ker P)$, we have for the $n \times n$ matrix $B_1 P$ the following block decomposition

$$B_1 P = \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix},$$

where $0_{r,s}$ stands for an $r \times s$ matrix with only 0 as an entry. On the other hand, we know from (1.2.22) that the mapping

$$(B_1 P)^* B_1^{*-1} R = P^* R$$

is symmetric. Writing $B_1^{*-1} R = \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix}$, where $\tilde{R}_{r,s}$ stands for an $r \times s$ matrix, this gives the symmetry of

$$\begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} = \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ \tilde{R}_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix},$$

implying that $\tilde{R}_{n-p,p} = 0$. The symplectic matrix (1.2.25) is thus equal to

$$\begin{pmatrix} \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} & B_1 Q \\ \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} & B_1^{*-1} S \end{pmatrix}, \quad \text{where } B_1 Q \text{ and } B_1^{*-1} S \text{ are } n \times n \text{ blocks.}$$

The invertibility of (1.2.25) implies that $\tilde{R}_{p,p}$ is invertible. We consider now the $n \times n$ symmetric matrix

$$C = \begin{pmatrix} I_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & 0_{n-p,n-p} \end{pmatrix},$$

[2]This is indeed possible: choosing a supplement space M for $P(N)$, we have

$$\mathbb{R}^n = \underbrace{\ker P}_{\dim p} \oplus \underbrace{N}_{\dim n-p} = \underbrace{P(N)}_{\dim n-p} \oplus \underbrace{M}_{\dim p},$$

and we can define B_1 on $P(N)$ by $B_1(Px) = x$ (without ambiguity since for $x_1, x_2 \in N$ with $Px_1 = Px_2$ we get $x_1 - x_2 \in \ker P \cap N = \{0\}$) and $B_{1|M} : M \to \ker P$ can be chosen as an isomorphism, so that $B_1(P(N)) + B_1(M) = N + \ker P$, which implies $\operatorname{rank} B_1 = n$.

and the symplectic mapping

$$\begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_1^{*-1} \end{pmatrix} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B_1 P & B_1 Q \\ B_1^{*-1} R & B_1^{*-1} S \end{pmatrix}, \quad (1.2.26)$$

which is a symplectic mapping $\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix}$ with

$$\begin{aligned} P' &= B_1 P + C B_1^{*-1} R \\ &= \begin{pmatrix} 0_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & I_{n-p} \end{pmatrix} + \begin{pmatrix} I_{p,p} & 0_{p,n-p} \\ 0_{n-p,p} & 0_{n-p,n-p} \end{pmatrix} \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{R}_{n-p,n-p} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{R}_{p,p} & \tilde{R}_{p,n-p} \\ 0_{n-p,p} & \tilde{I}_{n-p} \end{pmatrix}, \end{aligned}$$

which is an invertible mapping. From equation (1.2.26) and the first part of our discussion, we get that

$$\begin{pmatrix} P' & Q' \\ R' & S' \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ A' & I_n \end{pmatrix} \begin{pmatrix} B'^{-1} & 0 \\ 0 & B'^{*} \end{pmatrix} \begin{pmatrix} I_n & -C' \\ 0 & I_n \end{pmatrix},$$

with A', C' symmetric and B' invertible and

$$\Xi = \begin{pmatrix} B_1{}^{-1} & 0 \\ 0 & B_1^{*} \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ A' & I_n \end{pmatrix} \begin{pmatrix} B'^{-1} & 0 \\ 0 & B'^{*} \end{pmatrix} \begin{pmatrix} I_n & -C' \\ 0 & I_n \end{pmatrix},$$

proving that the $\Xi_{A,B,C}$ generate the symplectic group and more precisely that every Ξ in the symplectic group is the product of at most two mappings of type $\Xi_{A,B,C}$. This completes the proof of Theorem 1.2.6. ■

Corollary 1.2.7. *We have* $\mathrm{Sp}(n,\mathbb{R}) \subset \mathrm{Sl}(2n,\mathbb{R})$.

Proof. Indeed, the symplectic mappings (1.2.16), (1.2.17), and (1.2.18) do have determinants equal to 1 and since Theorem 1.2.6 implies that they generate the symplectic group, this proves the sought result. ■

Remark 1.2.8. Of course for $n \ge 2$, $\mathrm{Sp}(n,\mathbb{R})$ is a proper subgroup of $\mathrm{Sl}(2n,\mathbb{R})$. Indeed, the following matrix:

$$\mathtt{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

has determinant 1 but fails to be symplectic: using notation (1.2.21), we see that the first and the third equation are satisfied, which is not the case for the second one.

N.B. Since the matrix $-I_{2n}$ belongs to $\mathrm{Sp}(n,\mathbb{R})$ ((1.2.14) holds trivially), we find that $S \in \mathrm{Sp}(n,\mathbb{R})$ is equivalent to $-S \in \mathrm{Sp}(n,\mathbb{R})$.

Claim 1.2.9. The symplectic group is also generated by the mappings

$$\begin{aligned}
&(x,\xi) \mapsto (B^{-1}x, B^*\xi), \quad B \in \mathrm{Gl}(n,\mathbb{R}),\\
&(x,\xi) \mapsto (\xi, -x),\\
&(x,\xi) \mapsto (x, \xi + Ax), \quad A \in \mathrm{Sym}(n,\mathbb{R}).
\end{aligned}$$

Another set of generators of the symplectic group is given by the mappings

$$\begin{aligned}
&(x,\xi) \mapsto (B^{-1}x, B^*\xi), \quad B \in \mathrm{Gl}(n,\mathbb{R}),\\
&(x,\xi) \mapsto (\xi, -x),\\
&(x,\xi) \mapsto (x - C\xi, \xi), \quad C \in \mathrm{Sym}(n,\mathbb{R}).
\end{aligned}$$

Proof. Indeed, we have for $C^* = C$ a real symmetric $n \times n$ matrix

$$\underbrace{\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}}_{\sigma^{-1}} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \underbrace{\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}}_{\sigma} = \begin{pmatrix} I_n & 0 \\ C & I_n \end{pmatrix},$$

proving the claim. ∎

Remark 1.2.10. The symplectic matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} = 2^{-1/2}\begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} 2^{-1/2}\begin{pmatrix} I_n & I_n \\ -I_n & I_n \end{pmatrix} = \Xi^2_{-I_n, 2^{1/2}I_n, -I_n}, \tag{1.2.27}$$

is not of the form $\Xi_{A,B,C}$ but is the square of such a matrix. It is also the case of all the mappings $(x_k, \xi_k) \mapsto (\xi_k, -x_k)$ with the other coordinates fixed. Similarly, the symplectic matrix

$$\begin{pmatrix} 0 & -I_n \\ I_n & I_n \end{pmatrix} = \begin{pmatrix} I_n & -I_n \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ I_n & I_n \end{pmatrix},$$

is not of the form $\Xi_{A,B,C}$ but is the product $\Xi_{0,I,I}\,\Xi_{I,I,0}$.

1.2.3 The metaplectic group

Proposition 1.2.11. *Let A, B, C be as in Theorem 1.2.6, and let S be the generating function of $\Xi_{A,B,C}$* (cf. (1.2.20)). *We define the operator $M_{A,B,C}$ on $\mathscr{S}(\mathbb{R}^n)$ by*

$$(M_{A,B,C}v)(x) = \int_{\mathbb{R}^n} e^{2i\pi S(x,\eta)} \hat{v}(\eta) d\eta (\det B)^{1/2}, \tag{1.2.28}$$

where $(\det B)^{1/2}$ *is a square-root of* $\det B$*. This operator is an automorphism of* $\mathscr{S}'(\mathbb{R}^n)$ *and of* $\mathscr{S}(\mathbb{R}^n)$ *which is unitary on* $L^2(\mathbb{R}^n)$*, and such that, for all* $a \in \mathscr{S}'(\mathbb{R}^{2n})$*,*

$$M^*_{A,B,C}\operatorname{Op_w}(a)M_{A,B,C} = \operatorname{Op_w}(a \circ \Xi_{A,B,C}), \tag{1.2.29}$$

where $\Xi_{A,B,C}$ *is defined in Theorem 1.2.6.*

N.B. We have for A, B, C as above,

$$(M_{A,I,0}v)(x) = e^{i\pi\langle Ax,x\rangle}v(x), \tag{1.2.30}$$

$$(M_{0,B,0}v)(x) = (\det B)^{1/2}v(Bx), \tag{1.2.31}$$

$$(M_{0,I,C}v)(x) = \big(e^{i\pi\langle CD_x,D_x\rangle}v\big)(x), \tag{1.2.32}$$

three operators which are obviously automorphisms of $\mathscr{S}(\mathbb{R}^n)$ and of $\mathscr{S}'(\mathbb{R}^n)$ as well as unitary operators in $L^2(\mathbb{R}^n)$.

Proof. Formula (1.2.29) is easily checked for each operator (1.2.30), (1.2.31), and (1.2.32). Since we have

$$\Xi_{A,B,C} = \Xi_{A,I,0}\ \Xi_{0,B,0}\ \Xi_{0,I,C}$$

and

$$M_{A,B,C} = M_{A,I,0}\ M_{0,B,0}\ M_{0,I,C}, \tag{1.2.33}$$

we get (1.2.29) and the proposition. ∎

Remark 1.2.12. We define

$$m(B) = \frac{\arg(\det B)}{\pi} = \begin{cases} \frac{k2\pi}{\pi} = 2k \in \{0,2\} \mod 4 & \text{for } \det B > 0, \\ \frac{k2\pi+\pi}{\pi} = 2k+1 \in \{1,3\} \mod 4 & \text{for } \det B < 0, \end{cases} \tag{1.2.34}$$

so that

$$\det B = |\det B|e^{i\pi m(B)}, \quad (\det B)^{1/2} \in |\det B|^{1/2}\big\{e^{i\frac{\pi}{2}m(B)}, e^{i\frac{\pi}{2}(m(B)+2)}\big\}.^{3}$$

We will consider $m(B)$ as an element of $\mathbb{Z}/4\mathbb{Z}$, so that the function $m(B)\mapsto e^{i\frac{\pi}{2}m(B)}$ is well-defined. For A, B, C as in Proposition 1.2.11, we may define

$$\big(M^{\{m(B)\}}_{A,B,C}v\big)(x) = e^{\frac{i\pi m(B)}{2}}|\det B|^{1/2}\int_{\mathbb{R}^n} e^{i\pi(Ax^2+2Bx\cdot\eta+C\eta^2)}\hat{v}(\eta)d\eta,^{4} \tag{1.2.35}$$

[3]This is a synthetic way to write

$$(\det B)^{1/2} \in \{(\pm 1)|\det B|^{1/2}\} \text{ if } \det B > 0, \quad (\det B)^{1/2} \in \{(\pm i)|\det B|^{1/2}\} \text{ if } \det B < 0.$$

[4]We can of course define $M^{\{m\}}_{A,B,C}$ for any m, but to stay in the metaplectic group (cf. Definition 1.2.13), we have to make sure that $m \in \{m(B), m(B)+2\}$ modulo 4.

but we shall omit the super-script $m(B)$ when we do not want to distinguish between the two roots of det B. We note in particular that we have

$$M^{\{0\}}_{0,I_n,0} = \mathrm{Id}_{L^2(\mathbb{R}^n)}, \quad M^{\{2\}}_{0,I_n,0} = -\mathrm{Id}_{L^2(\mathbb{R}^n)},$$

and also with the notation (1.2.6),

$$M^{\{n\}}_{0,-I_n,0} = e^{\frac{i\pi n}{2}}\sigma_0, \quad M^{\{n+2\}}_{0,-I_n,0} = -e^{\frac{i\pi n}{2}}\sigma_0.$$

More generally, we have

$$\text{for } \det B > 0,\ M^{\{0\}}_{A,B,C} = -M^{\{2\}}_{A,B,C}, \quad \text{for } \det B < 0,\ M^{\{1\}}_{A,B,C} = -M^{\{3\}}_{A,B,C}. \tag{1.2.36}$$

We note also that for $B \in \mathrm{Gl}(n,\mathbb{R})$, we have

$$m(B^*) = m(B) = m(B^{-1}),$$

since $\det B = \det B^*$ and $\det(B^{-1}) = (\det B)^{-1}$ so that

$$\arg(\det B) = \arg(\det B^{-1}).$$

Moreover, we have for $B \in \mathrm{Gl}(n,\mathbb{R})$,

$$\det(-B) = (-1)^n \det B, \quad \arg(\det(-B)) = \begin{cases} \arg(\det B) & \text{if } n \text{ is even,} \\ \arg(\det B) + \pi & \text{if } n \text{ is odd,} \end{cases}$$

so that

$$m(-B) = n + m(B). \tag{1.2.37}$$

Examples. Let us start with a one-dimensional example: in Remark 1.2.10, we have seen, in particular, that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \left\{ 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}^2, \quad 2^{-1/2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \Xi_{-1,2^{1/2},-1},$$

where we have used (1.2.19) to get the second equation. We have also with the notations of Theorem 1.2.6,

$$(M_{-1,2^{1/2},-1} v)(x) = \int_{\mathbb{R}} e^{2i\pi \frac{1}{2}(-x^2 + 2^{3/2} x\eta - \eta^2)} \hat{v}(\eta) d\eta 2^{1/4},$$

so that the kernel $k_1(x,y)$ of the operator $M_{-1,2^{1/2},-1}$ is

$$k_1(x,y) = 2^{1/4} \int e^{i\pi(-x^2 + 2^{3/2} x\eta - \eta^2)} e^{-2i\pi y\eta} d\eta \underbrace{=}_{\text{use (A.1.7)}} 2^{1/4} e^{-i\pi/4} e^{i\pi(x^2+y^2)} e^{-2^{3/2} i\pi xy},$$

so that the kernel k_2 of the operator $(M_{-1,2^{1/2},-1})^2$ is (using again (A.1.7)),

$$k_2(x,y) = \int k_1(x,z)k_1(z,y)dz$$
$$= 2^{1/2}e^{-i\pi/2}e^{i\pi(x^2+y^2)}\int e^{2i\pi z^2}e^{-2i\pi z 2^{1/2}(x+y)}dz = e^{-i\pi/4}e^{-2i\pi xy},$$

so that

$$(M_{-1,2^{1/2},-1})^2 = e^{-i\pi/4}\mathcal{F}_1,$$

with $\mathcal{F}_1$ standing for the 1d Fourier transformation. We get similarly that in n dimensions,

$$(M_{-I_n,2^{1/2}I_n,-I_n})^2 = e^{-i\pi n/4}\mathcal{F}, \tag{1.2.38}$$

with $\mathcal{F}$ standing for the Fourier transformation. Similar expressions can be obtained for $\mathcal{F}_k$, the Fourier transformation with respect to the variable x_k in n dimensions, $k \in [\![1,n]\!]$ with

$$(M_{A_k,B_k,C_k})^2 = e^{-i\pi/4}\mathcal{F}_k,$$

where B_k is the $n \times n$ diagonal matrix with diagonal entries equal to 1 except for the kth equal to $2^{1/2}$, the $n \times n$ diagonal matrices $A_k = C_k$ with diagonal entries equal to 0, except for the kth equal to -1.

Definition 1.2.13. The metaplectic group $\mathrm{Mp}(n)$ is defined as the subgroup of the group of unitary operators on $L^2(\mathbb{R}^n)$ generated by

$M_{A,I,0}$, where A is an $n \times n$ symmetric matrix, cf. (1.2.30), (1.2.39)

$M_{0,B,0}$, with $B \in \mathrm{Gl}(n,\mathbb{R})$, with $(\det B)^{\frac12} = |\det B|^{\frac12} e^{\frac{i\pi m(B)}{2}}$, cf. (1.2.34), (1.2.31), (1.2.40)

$M_{0,I,C}$, where C is an $n \times n$ symmetric matrix, cf. (1.2.32). (1.2.41)

Claim 1.2.14. If M belongs to $\mathrm{Mp}(n)$, then $-M$ belongs to $\mathrm{Mp}(n)$.

Proof. According to (1.2.36), we have

$$M^{\{2\}}_{0,I_n,0} = -M^{\{0\}}_{0,I_n,0} = -\mathrm{Id}_{L^2(\mathbb{R}^n)}$$

so that $-\mathrm{Id}_{L^2(\mathbb{R}^n)}$ belongs to $\mathrm{Mp}(n)$, proving the claim. ∎

Proposition 1.2.15. *The metaplectic group* $\mathrm{Mp}(n)$ *is generated by*

$M_{A,I,0}$, *where* A *is an* $n \times n$ *symmetric matrix, cf.* (1.2.30), (1.2.42)

$M_{0,B,0}$, *with* $B \in \mathrm{Gl}(n,\mathbb{R})$, *with* $(\det B)^{\frac12} = |\det B|^{\frac12} e^{\frac{i\pi m(B)}{2}}$, *cf.* (1.2.34), (1.2.31), (1.2.43)

$e^{-\frac{i\pi n}{4}}\mathcal{F}$, *where* $\mathcal{F}$ *is the Fourier transformation.* (1.2.44)

Proof. We check for C symmetric $n \times n$ matrix,

$$\big(M^{\{0\}}_{C,I,0}(e^{-i\pi n/4}\mathcal{F} v)\big)(\eta) = e^{-i\pi n/4} e^{i\pi C\eta^2}\hat{v}(\eta),$$

so that

$$e^{i\pi n/4}\big(\mathcal{F}^{-1}(e^{-i\pi n/4} e^{i\pi C\eta^2}\hat{v}(\eta))\big)(x) = \int e^{2i\pi x\eta} e^{i\pi C\eta^2}\hat{v}(\eta)d\eta = (M^{\{0\}}_{0,I,C} v)(x),$$

yielding

$$e^{i\pi n/4}\mathcal{F}^{-1} M^{\{0\}}_{0,I,C} e^{-i\pi n/4}\mathcal{F} = M^{\{0\}}_{0,I,C},$$

so that the group generated by (1.2.42), (1.2.43), (1.2.44) contains (1.2.39), (1.2.40), and (1.2.41) and thus contains $\mathrm{Mp}(n)$. Moreover, (1.2.38) shows that (1.2.44) is included in $\mathrm{Mp}(n)$ so that the group generated by (1.2.42), (1.2.43), and (1.2.44) is included in $\mathrm{Mp}(n)$, proving the proposition. ■

Remark 1.2.16. According to (A.1.6) in our appendix and to (1.2.36), we find

$$(e^{-i\pi n/4}\mathcal{F})^* = e^{i\pi n/4}\mathcal{F}\sigma_0 = e^{-i\pi n/4}\mathcal{F} e^{i\pi n/2}\sigma_0 = e^{-i\pi n/4}\mathcal{F} M^{\{n\}}_{0,-I_n,0}.$$

As a consequence, $e^{-i\pi n/4}\mathcal{F}, e^{-i\pi n/2}\sigma_0, e^{i\pi n/2}\sigma_0$ belong to the metaplectic group.

Lemma 1.2.17. *For $Y \in \mathbb{R}^{2n}$, we define the linear form L_Y on $\mathbb{R}^{2n}$ by*

$$L_Y(X) = \langle \sigma Y, X\rangle = [Y, X].$$

For any $M \in \mathrm{Mp}(n)$ there exists a unique $\chi \in \mathrm{Sp}(n,\mathbb{R})$ such that

$$\forall Y \in \mathbb{R}^{2n}, \quad M^*\mathrm{Op_w}(L_Y)M = \mathrm{Op_w}(L_{\chi^{-1}Y}). \tag{1.2.45}$$

Proof. Indeed, thanks to (1.2.29) and Definition 1.2.13, we can find $\chi \in \mathrm{Sp}(n,\mathbb{R})$ such that

$$M^*\mathrm{Op_w}(L_Y)M = \mathrm{Op_w}(L_Y \circ \chi) = \mathrm{Op_w}(L_{\chi^{-1}Y}),$$

since

$$(L_Y \circ \chi)(X) = \langle \sigma Y, \chi X\rangle = \langle \chi^*\sigma\chi\chi^{-1}Y, X\rangle = \langle \sigma\chi^{-1}Y, X\rangle = L_{\chi^{-1}Y}(X).$$

Moreover, if $\chi_1, \chi_2 \in \mathrm{Sp}(n,\mathbb{R})$ are such that for all $Y \in \mathbb{R}^{2n}$,

$$0 = \mathrm{Op_w}(L_{\chi_2^{-1}Y} - L_{\chi_1^{-1}Y}) = \mathrm{Op_w}(L_{(\chi_2^{-1}-\chi_1^{-1})Y}),$$

we get

$$L_{(\chi_2^{-1}-\chi_1^{-1})Y} = 0,$$

implying $\forall Y \in \mathbb{R}^{2n}, (\chi_2^{-1} - \chi_1^{-1})Y = 0$, i.e., $\chi_1 = \chi_2$. ■

We can thus define a mapping

$$\Psi : \mathrm{Mp}(n) \to \mathrm{Sp}(n, \mathbb{R}) \quad \text{with } \Psi(M) = \chi \text{ satisfying (1.2.45).} \tag{1.2.46}$$

In particular, we have from (1.2.29) in Proposition 1.2.11 and (1.2.38) that

$$\Psi(M_{A,B,C}) = \Xi_{A,B,C}, \quad \Psi\big(e^{-\frac{i\pi n}{4}}\mathcal{F}\big) = \sigma = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{1.2.47}$$

Theorem 1.2.18. *The mapping Ψ defined in (1.2.46) is a surjective homomorphism of groups with kernel $\{\pm \mathrm{Id}_{L^2(\mathbb{R}^n)}\}$.*

Proof. This mapping is a homomorphism of groups: if M_1, M_2 belong to $\mathrm{Mp}(n)$, we have with $\chi_j = \Psi(M_j)$,

$$\begin{aligned}(M_1M_2)^*\mathrm{Op}_{\mathrm{w}}(L_Y)M_1M_2 &= M_2^*\mathrm{Op}_{\mathrm{w}}(L_{\chi_1^{-1}Y})M_2 \\ &= \mathrm{Op}_{\mathrm{w}}(L_{\chi_2^{-1}\chi_1^{-1}Y}) = \mathrm{Op}_{\mathrm{w}}(L_{(\chi_1\circ\chi_2)^{-1}Y}),\end{aligned}$$

proving that $\Psi(M_1M_2) = \Psi(M_1)\Psi(M_2)$. Moreover, the homomorphism Ψ is onto, thanks to (1.2.29) and Theorem 1.2.6. The kernel of Ψ is made with $M \in \mathrm{Mp}(n)$ such that for all $Y \in \mathbb{R}^{2n}$,

$$M^*\mathrm{Op}_{\mathrm{w}}(L_Y)M = \mathrm{Op}_{\mathrm{w}}(L_Y),$$

i.e.,

$$[\mathrm{Op}_{\mathrm{w}}(L_Y), M] = 0,$$

so that, thanks to (1.2.3), (1.2.4), if $\mu(x,\xi)$ is the Weyl symbol of M (M is an endomorphism of $\mathscr{S}'(\mathbb{R}^n)$ and thus has a distribution kernel as well as a Weyl symbol via formula (1.2.12)), we get for all $(y,\eta) \in \mathbb{R}^{2n}$,

$$0 = \{\eta\cdot x - y\cdot\xi, \mu(x,\xi)\} \text{ so that } d\mu = 0,$$

and μ is a constant so that $M = c\,\mathrm{Id}_{L^2(\mathbb{R}^n)}$, necessarily with $|c| = 1$ (since M is unitary). Applying Theorem A.2.11 gives $c \in \{\pm 1\}$, concluding the proof. ∎

N.B. The proof of Theorem A.2.11 is relegated in our appendix, and requires some effort.

Corollary 1.2.19. *For $\chi \in \mathrm{Sp}(n,\mathbb{R})$, the fiber $\Psi^{-1}\{\chi\}$ contains exactly two metaplectic transformations and more precisely*

$$\Psi^{-1}\{\chi\} = \{M, -M\},$$

where M is a metaplectic transformation.

Proof. This corollary is an immediate consequence of Theorem 1.2.18. ∎

Theorem 1.2.20 (Symplectic covariance of the Weyl calculus). *Let a be in $\mathscr{S}'(\mathbb{R}^{2n})$ and let χ be in $\mathrm{Sp}(n,\mathbb{R})$. Then, for a metaplectic operator M such that $\Psi(M) = \chi$, we have*

$$M^*\mathrm{Op_w}(a)M = \mathrm{Op_w}(a \circ \chi). \tag{1.2.48}$$

For $u, v \in \mathscr{S}(\mathbb{R}^n)$, we have

$$\mathcal{W}(Mu, Mv) = \mathcal{W}(u,v) \circ \chi^{-1}, \tag{1.2.49}$$

where $\mathcal{W}$ is the Wigner distribution given in (1.1.4).

Proof. The first property follows from (1.2.29) and Definition 1.2.13 whereas (1.2.49) is a consequence of (1.2.1) and (1.2.48). ∎

We note also that for $Y = (y,\eta) \in \mathbb{R}^{2n}$, the symmetry S_Y is defined by

$$S_Y(X) = 2Y - X$$

and is quantized by the phase symmetry σ_Y as defined by (1.2.6) with the formula

$$\mathrm{Op_w}(a \circ S_Y) = \sigma_Y^*\mathrm{Op_w}(a)\sigma_Y = \sigma_Y\mathrm{Op_w}(a)\sigma_Y. \tag{1.2.50}$$

Similarly, the translation T_Y is defined on the phase space by

$$T_Y(X) = X + Y$$

and is quantized by the *phase translation* τ_Y,

$$(\tau_{(y,\eta)}u)(x) = u(x-y)e^{2i\pi(x-\frac{y}{2})\cdot\eta}, \tag{1.2.51}$$

and we have

$$\mathrm{Op_w}(a \circ T_Y) = \tau_Y^*\mathrm{Op_w}(a)\tau_Y = \tau_{-Y}\mathrm{Op_w}(a)\tau_Y.$$

Remark 1.2.21. Property (1.2.49) can be extended to the affine symplectic group and we have with the phase translations defined in (1.2.51),

$$\forall (X,Y) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \quad \mathcal{W}(\tau_Y u, \tau_Y v)(X) = \mathcal{W}(u,v)(X-Y).$$

We will define the *affine group* $\mathrm{Mp}_a(n)$ as the group of unitary transformations of $L^2(\mathbb{R}^n)$ generated by transformations (1.2.30), (1.2.31), and (1.2.32) and phase translations given by (1.2.51).

N.B. More information on the metaplectic group is given in J. Leray's book [31], the same author's articles [30,32], as well as A. Weil's paper [52] (see also V. S. Buslaev's article [5], K. Gröchenig's book [16, Chapter 9], H. Reiter's lecture notes [43]).

Theorem 1 in E. Lieb's classical article [37] gives a more precise version of (1.2.53), (1.2.54), and (1.2.55) below.

Theorem 1.2.22. *Let u, v be in $L^2(\mathbb{R}^n)$. Then, $\mathcal{W}(u,v)$ is a uniformly continuous function belonging to $L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ and using the definitions* (1.2.51), (1.2.6) *for the phase translations and phase symmetry, we have*

$$\mathcal{W}(u,v)(X) = 2^n \langle \sigma_X u, v\rangle_{L^2(\mathbb{R}^n)} = 2^n \langle \tau_X^* u, \tau_X \check{v}\rangle_{L^2(\mathbb{R}^n)}$$
$$= 2^n \langle \sigma_0 \tau_{-2X} u, v\rangle_{L^2(\mathbb{R}^n)}, \tag{1.2.52}$$
$$\|\mathcal{W}(u,v)\|_{L^2(\mathbb{R}^{2n})} = \|u\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)}, \tag{1.2.53}$$
$$\forall p \in [1, +\infty], \quad \|\mathcal{W}(u,v)\|_{L^\infty(\mathbb{R}^{2n})} \le 2^n \|u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^{p'}(\mathbb{R}^n)}. \tag{1.2.54}$$

More generally, for $q \ge 2$ and $r \in [q', q]$, we have[5]

$$\|\mathcal{W}(u,v)\|_{L^q(\mathbb{R}^{2n})} \le 2^{\frac{n(q-2)}{q}} \|u\|_{L^r(\mathbb{R}^n)} \|v\|_{L^{r'}(\mathbb{R}^n)}. \tag{1.2.55}$$

Moreover, we have

$$\lim_{\mathbb{R}^{2n} \ni X, |X| \to +\infty} \left[\mathcal{W}(u,v)(X)\right] = 0.$$

Proof. We have with $\check{v}(x) = v(-x) = (\sigma_0 v)(x)$,

$$\begin{aligned}\mathcal{W}(u,v)(x,\xi) &= 2^n \int u(x+z)\bar{v}(x-z) e^{-4i\pi z\xi} dz \\ &= 2^n \int u(z-(-x)) e^{2i\pi(z-\frac{-x}{2})(-\xi)} \bar{\check{v}}(z-x) e^{-2i\pi(z-\frac{x}{2})\xi} \\ &\quad \times e^{-4i\pi z\xi + 2i\pi(z-\frac{-x}{2})\xi + 2i\pi(z-\frac{x}{2})\xi} dz \\ &= 2^n \int (\tau_{(-x,-\xi)} u)(z) \overline{(\tau_{(x,\xi)}\check{v})(z)} dz = 2^n \langle \tau_{(x,\xi)}^* u, \tau_{(x,\xi)} \check{v}\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

or for short

$$\mathcal{W}(u,v)(X) = 2^n \langle \tau_X^* u, \tau_X \check{v}\rangle_{L^2(\mathbb{R}^n)}.$$

As a consequence, we find from (1.2.7) that

$$\langle \mathrm{Op_w}(a) u, v\rangle = \int a(X) 2^n \langle \sigma_0 \tau_{2X}^* u, v\rangle dX,$$

and since $(\sigma_{x,\xi} u)(y) = u(2x - y) e^{-4i\pi(x-y)\cdot\xi}$, we can verify directly that

$$\sigma_0 \tau_{-2X} = \sigma_X. \tag{1.2.56}$$

Indeed, composing the translation of vector $-2X$ in $\mathbb{R}^{2n}$ with the symmetry with respect to 0, we have

$$Y \mapsto Y - 2X \mapsto 2X - Y = Y', \quad \frac{1}{2}(Y + Y') = X,$$

[5] We use the standard notation: for $p \in [1, +\infty]$ we define p' by the equality $\frac{1}{p} + \frac{1}{p'} = 1$.

that is the symmetry with respect to X. Quantifying this equality, we use

$$(\tau_{(-2x,-2\xi)}u)(z) = u(z+2x)e^{2i\pi(z-\frac{-2x}{2})(-2\xi)} = u(z+2x)e^{-4i\pi(z+x)\xi},$$

so that we obtain

$$\sigma_0(\tau_{(-2x,-2\xi)}u)(z) = u(-z+2x)e^{-4i\pi(-z+x)\xi} = (\sigma_{x,\xi}u)(z),$$

which proves (1.2.56) and thus (1.2.52). Formula (1.2.53) is already proven in (1.1.6) and (1.2.54) follows from (1.2.52), Hölder's inequality and the fact that τ_X is an endomorphism of $L^p(\mathbb{R}^n)$ with norm 1 (cf. the expression (1.2.51)). To prove (1.2.55) we note that from the expression (1.2.10), the Hausdorff–Young's inequality implies

$$\|\mathcal{W}(u,v)\|_{L^q\otimes L^q} \le \|\Omega(u,v)\|_{L^q\otimes L^{q'}} \le \||u|^{q'} * |v|^{q'}\|^{1/q'}_{L^{q/q'}} 2^{n\frac{q-2}{q}}, \tag{1.2.57}$$

and since Young's inequality[6] gives

$$\||u|^{q'} * |v|^{q'}\|_{L^{q/q'}} \le \||u|^{q'}\|_{L^{a/q'}} \||v|^{q'}\|_{L^{b/q'}},$$

$a, b \ge q'$ with

$$1 - \frac{q'}{q} = 1 - \frac{q'}{a} + 1 - \frac{q'}{b},$$

i.e.,

$$q'\left(\frac{1}{a} + \frac{1}{b}\right) = 1 + \frac{q'}{q},$$

that is

$$\frac{1}{a} + \frac{1}{b} = 1,$$

so that

$$\||u|^{q'} * |v|^{q'}\|_{L^{q/q'}} \le \|u\|^{q'}_{L^a} \|v\|^{q'}_{L^b},$$

in such a way that (1.2.57) yields

$$\|\mathcal{W}(u,v)\|_{L^q\otimes L^q} \le 2^{n\frac{q-2}{q}} \|u\|_{L^a} \|v\|_{L^b}, \quad a, b \ge q', \quad \frac{1}{a} + \frac{1}{b} = 1,$$

which is (1.2.55). We are left with the proof of uniform continuity of $\mathcal{W}(u,v)$. We have for $X, Y \in \mathbb{R}^{2n}$,

$$\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X) = 2^n \langle(\sigma_Y - \sigma_X)u, v\rangle_{L^2(\mathbb{R}^n)},$$

and since $\sigma_Y^2 = \mathrm{Id}$ (see Claim 1.2.3), we find

$$\begin{aligned}\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X) &= 2^n \langle(\sigma_Y\sigma_X - \mathrm{Id})\sigma_X u, v\rangle_{L^2(\mathbb{R}^n)} \\ &= 2^n \langle\sigma_X u, (\sigma_X\sigma_Y - \mathrm{Id})v\rangle_{L^2(\mathbb{R}^n)}.\end{aligned}$$

[6]For $p, q, r \in [1, +\infty]$ with $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$, we have, $\|f * g\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}$.

According to [33, formula (2.1.16)], we have

$$\sigma_X \sigma_Y = \tau_{2X-2Y} e^{4i\pi[Y,X]},$$

and this implies

$$|\mathcal{W}(u,v)(Y) - \mathcal{W}(u,v)(X)| \le 2^n \|u\|_{L^2(\mathbb{R}^n)} \|\tau_{2(X-Y)} v\|_{L^2(\mathbb{R}^n)}. \tag{1.2.58}$$

We have from (1.2.50),

$$\begin{aligned}\tau_{z,\zeta} v(x) - v(x) &= v(x-z) e^{2i\pi(x-\frac{z}{2})\zeta} - v(x) \\ &= \big(v(x-z) - v(x)\big) e^{2i\pi(x-\frac{z}{2})\zeta} + v(x)\big(e^{2i\pi(x-\frac{z}{2})\zeta} - 1\big),\end{aligned}$$

and thus

$$\begin{aligned}&\|\tau_Z v - v\|_{L^2(\mathbb{R}^n)} \\ &\quad\le \Big(\int |v(x-z) - v(x)|^2 dx\Big)^{1/2} + \Big(\int |v(x)|^2 |e^{2i\pi(x-\frac{z}{2})\zeta} - 1|^2 dx\Big)^{1/2}.\end{aligned}$$

We have the classical result, due to the density in L^2 of continuous compactly supported functions,

$$\lim_{\mathbb{R}^n \ni z \to 0} \int |v(x-z) - v(x)|^2 dx = 0,$$

and moreover the Lebesgue dominated convergence theorem implies

$$\lim_{(z,\zeta)\to(0,0)} \int \underbrace{|v(x)|^2}_{\in L^1(\mathbb{R}^n)} \underbrace{|e^{2i\pi(x-\frac{z}{2})\zeta} - 1|^2}_{\le 4} dx = 0,$$

so that

$$\lim_{\mathbb{R}^{2n} \ni Z \to 0} \|\tau_Z v - v\|_{L^2(\mathbb{R}^n)} = 0.$$

As a consequence, (1.2.58) implies the uniform continuity of $\mathcal{W}(u,v)$. Moreover, we have, for $\phi, \psi \in \mathscr{S}(\mathbb{R}^n)$,

$$\mathcal{W}(u,v) = \mathcal{W}(u-\phi, v) + \mathcal{W}(\phi, v-\psi) + \mathcal{W}(\phi,\psi),$$

so that

$$\begin{aligned}|\mathcal{W}(u,v)(x,\xi)| \le &\int \Big|(u-\phi)\Big(x+\frac{z}{2}\Big)\Big|\Big|v\Big(x-\frac{z}{2}\Big)\Big| dz \\ &+ \iint \Big|(v-\psi)\Big(x-\frac{z}{2}\Big)\Big|\Big|\phi\Big(x+\frac{z}{2}\Big)\Big| dz + |\mathcal{W}(\phi,\psi)(x,\xi)| \\ \le\ & 2^n \|u-\phi\|_{L^2(\mathbb{R}^n)} \|v\|_{L^2(\mathbb{R}^n)} + 2^n \|v-\psi\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\mathbb{R}^n)} \\ &+ |\mathcal{W}(\phi,\psi)(x,\xi)|.\end{aligned}$$

We choose now sequences (ϕ_k), (ψ_k) of $\mathscr{S}(\mathbb{R}^n)$ converging respectively in $L^2(\mathbb{R}^n)$ towards u, v. We obtain for all $k \in \mathbb{N}$,

$$|\mathcal{W}(u,v)(x,\xi)| \le 2^n \|u-\phi_k\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + 2^n\|v-\psi_k\|_{L^2(\mathbb{R}^n)}\|\phi_k\|_{L^2(\mathbb{R}^n)} + |\mathcal{W}(\phi_k,\psi_k)(x,\xi)|,$$

so that using that $\mathcal{W}(\phi_k,\psi_k)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, we get

$$\limsup_{\mathbb{R}^{2n}\ni X, |X|\to+\infty} \big[|\mathcal{W}(u,v)(X)|\big] \le 2^n \|u-\phi_k\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + 2^n\|v-\psi_k\|_{L^2(\mathbb{R}^n)}\|\phi_k\|_{L^2(\mathbb{R}^n)},$$

and thus, taking the limit when $k \to +\infty$, we obtain

$$\lim_{\mathbb{R}^{2n}\ni X, |X|\to+\infty} \big[|\mathcal{W}(u,v)(X)|\big] = 0,$$

completing the proof of Theorem 1.2.22. ∎

Remark 1.2.23. Let u be in $L^2(\mathbb{R}^n)$ be an even function. We then have

$$\mathcal{W}(u,u)(0,0) = 2^n\|u\|^2_{L^2(\mathbb{R}^n)} = \|\mathcal{W}(u,u)\|_{L^\infty(\mathbb{R}^{2n})}.$$

On the other hand, if u is odd, we have

$$\mathcal{W}(u,u)(0,0) = -2^n\|u\|^2_{L^2(\mathbb{R}^n)} = -\|\mathcal{W}(u,u)\|_{L^\infty(\mathbb{R}^{2n})},$$

showing that for odd functions the minimum of the Wigner distribution is negative (we assume $u \ne 0$ in $L^2(\mathbb{R}^n)$) and attained at 0. Let us check for instance the (odd) function u_1 of Remark 1.1.3. We have

$$2\|u_1\|^2_{L^2(\mathbb{R})} = 2\int x^2 e^{-2\pi x^2}dx = 4\int_0^{+\infty} \frac{t}{2\pi}e^{-t}(2\pi)^{-1/2}\frac{1}{2}t^{-1/2}dt = \frac{2\Gamma(3/2)}{(2\pi)^{3/2}} = \frac{\Gamma(1/2)}{(2\pi)^{3/2}} = \frac{1}{2^{3/2}\pi} = -\mathcal{W}(u_1,u_1)(0,0).$$

1.2.4 On weak versions of the Wigner distribution

Let u, v be in the space $\mathscr{S}'(\mathbb{R}^n)$ of tempered distributions. Then, we can define as above the tempered distribution $\Omega(u,v)$ in $\mathbb{R}^{2n}$: we set

$$\langle \Omega(u,v)(x,z), \Phi(x,z)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \Big\langle u(x_1)\otimes \bar v(x_2), \Phi\Big(\frac{x_1+x_2}{2}, x_1-x_2\Big)\Big\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

and then we define the Wigner distribution $\mathcal{W}(u,v)$ as the Fourier transform with respect to z of $\Omega(u,v)$, meaning that

$$\langle \mathcal{W}(u,v), \Psi\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = \langle \Omega(u,v), \mathcal{F}_2\Psi\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where

$$(\mathcal{F}_2\Psi)(x,\xi) = \int_{\mathbb{R}^n} e^{-2i\pi z\cdot\xi}\Psi(x,z)dz.$$

Of course, $\mathcal{W}(u,v)$ is only a tempered distribution on $\mathbb{R}^{2n}$ and we have the inversion formula, using the notations of Remark 1.2.4,

$$\Omega(u,v) = \mathcal{F}_2\mathcal{C}_2\mathcal{W}(u,v).$$

The above remarks show that there is no difficulty to extend the definition of the joint Wigner distribution $\mathcal{W}(u,v)$ to the case where u,v are both tempered distributions on $\mathbb{R}^n$. Some properties are surviving from the L^2 theory, in particular the inversion formula, but one should be reasonably cautious at avoiding writing brackets of duality as integrals. Theorem 2 in [37] gives a more complete version of the following result.

Theorem 1.2.24. *Let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u,u) \in L^1(\mathbb{R}^{2n})$. Then, u belongs to $L^p(\mathbb{R}^n)$ for all $p \in [1,+\infty]$ and we have*

$$\|u\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)} \le 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})}.$$

N.B. We refer the reader to our Section 6.3 and, in particular, Theorem 6.3.3 showing that the set of u in $L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u,u)$ belongs to $L^1(\mathbb{R}^{2n})$ is meager.

Proof. Thanks to Theorem 1.2.22, we have $\mathcal{W}(u,u) \in L^p(\mathbb{R}^{2n})$ for all $p \in [1,+\infty]$ and we have in a weak sense,

$$u\Big(x+\frac{z}{2}\Big)\bar{u}\Big(x-\frac{z}{2}\Big) = \int e^{2i\pi z\cdot\xi}\mathcal{W}(u,u)(x,\xi)d\xi,$$

so that

$$u(x_1)\bar{u}(x_2) = \int e^{2i\pi(x_1-x_2)\cdot\xi}\mathcal{W}(u,u)\Big(\frac{x_1+x_2}{2},\xi\Big)d\xi,$$

and thus we get

$$\int |u(x_1)||u(x_2)|dx_1 \le \iint \Big|\mathcal{W}(u,u)\Big(\frac{x_1+x_2}{2},\xi\Big)\Big|d\xi dx_1 = 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})},$$

i.e.,

$$\|u\|_{L^1(\mathbb{R}^n)}\|u\|_{L^\infty(\mathbb{R}^n)} \le 2^n\|\mathcal{W}(u,u)\|_{L^1(\mathbb{R}^{2n})},$$

proving the lemma. ∎

1.2.5 Composition formulas

The following lemma is classical (see, e.g., [19], [46]); however we shall provide a proof for the convenience of the reader.

Lemma 1.2.25. *Let u, v, f, g be in $L^2(\mathbb{R}^n)$. Then*

$$\langle u, g\rangle_{L^2(\mathbb{R}^n)}\langle f, v\rangle_{L^2(\mathbb{R}^n)} = \iint \mathcal{W}(u, v)(x, \xi)\mathcal{W}(f, g)(x, \xi)dxd\xi. \tag{1.2.59}$$

In other words, the Weyl symbol of the rank-one operator $u \mapsto \langle u, g\rangle_{L^2(\mathbb{R}^n)} f$ is $\mathcal{W}(f, g)$. In particular, if $f = g$ is a unit vector in $L^2(\mathbb{R}^n)$ we find that $\mathcal{W}(f, f)$ is the Weyl symbol of the orthogonal projection onto $\mathbb{C} f$.

Proof. Both functions $\mathcal{W}(u, v)$, $\mathcal{W}(f, g)$ belong to $L^2(\mathbb{R}^{2n})$, so that the integral on the right-hand side of (1.2.59) actually makes sense. Also, $\mathcal{W}(u, v)$ is the partial Fourier transform with respect to the variable z of $(x, z) \mapsto u(x + z/2)\bar{v}(x - z/2)$, thus applying Plancherel formula[7], we obtain that

$$\begin{aligned}
&\iint \mathcal{W}(u, v)(x, \xi)\mathcal{W}(f, g)(x, \xi)dxd\xi \\
&\quad = \iint u(x + z/2)\bar{v}(x - z/2) f(x - z/2)\bar{g}(x + z/2)dxdz \\
&\quad = \langle u, g\rangle_{L^2(\mathbb{R}^n)}\langle f, v\rangle_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

The last property follows from (1.2.1). ∎

Using [33, Section 2.1.5], we obtain that for $a, b \in \mathscr{S}(\mathbb{R}^{2n})$

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} a(Y)b(Z)2^{2n}\sigma_Y\sigma_Z dYdZ.$$

We get

$$\mathrm{Op_w}(a)\mathrm{Op_w}(b) = \mathrm{Op_w}(a\sharp b), \tag{1.2.60}$$

[7] We refer of course to the formula

$$\langle \hat{u}, \hat{v}\rangle_{L^2(\mathbb{R}^n)} = \langle u, v\rangle_{L^2(\mathbb{R}^n)},$$

when using the *complex* Hilbert space $L^2(\mathbb{R}^n)$. Note however that formula (A.1.3) is using the *real* duality between $\mathscr{S}(\mathbb{R}^n)$ and $\mathscr{S}'(\mathbb{R}^n)$ so that to check, with $\mathscr{S}^*(\mathbb{R}^N)$ standing for the anti-dual of $\mathscr{S}(\mathbb{R}^N)$ (i.e., continuous anti-linear forms on $\mathscr{S}(\mathbb{R}^N)$), we have also

$$\begin{aligned}
\langle \widehat{T}, \widehat{\phi}\rangle_{\mathscr{S}^*(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} &= \langle \widehat{T}, \overline{\widehat{\phi}}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \widehat{\widehat{\overline{\phi}}}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} \\
&= \langle T, \overline{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \phi\rangle_{\mathscr{S}^*(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)}.
\end{aligned}$$

with

$$(a\sharp b)(X) = 2^{2n} \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[X-Y,X-Z]} a(Y)b(Z)dYdZ \tag{1.2.61}$$

$$= \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-2i\pi\langle\Xi,Z\rangle} a\Big(X + \frac{\sigma^{-1}\Xi}{2}\Big) b(Z+X)d\Xi dZ \tag{1.2.62}$$

$$= \int_{\mathbb{R}^{2n}} e^{2i\pi\langle X,\Xi\rangle} a\Big(X + \frac{\sigma^{-1}\Xi}{2}\Big)\hat{b}(\Xi)d\Xi, \tag{1.2.63}$$

where $[\cdot,\cdot]$ is the symplectic form (1.2.13) and σ is (1.2.15). Formula (1.2.62) is interesting since very close to the group J^t defined in [33, formula (4.1.14)].

Lemma 1.2.26. *Let u_0, u_1, u_2, u_3 be in $L^2(\mathbb{R}^n)$. Then, we have for all $X \in \mathbb{R}^{2n}$,*

$$|\langle u_1, u_2\rangle_{L^2}||\mathcal{W}(u_0,u_3)(X)| \le 2^n\big(|\mathcal{W}(u_0,u_2)| * |\mathcal{W}(\check{u}_1,u_3)|\big)(X).$$

Proof. According to Lemma 1.2.25, we have for $v \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\operatorname{Op_w}(\mathcal{W}(u_0,u_2))\operatorname{Op_w}(\mathcal{W}(u_1,u_3))v &= \operatorname{Op_w}(\mathcal{W}(u_0,u_2))\big(\langle v,u_3\rangle_{L^2(\mathbb{R}^n)}u_1\big)\\ &= \langle v,u_3\rangle_{L^2(\mathbb{R}^n)}\langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}u_0\\ &= \langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}\operatorname{Op_w}(\mathcal{W}(u_0,u_3))v,\end{aligned}$$

so that with the notation (1.2.60), we get

$$\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3) = \langle u_1,u_2\rangle_{L^2(\mathbb{R}^n)}\mathcal{W}(u_0,u_3), \tag{1.2.64}$$

and using (1.2.63), we get

$$\begin{aligned}&\big(\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3)\big)(x,\xi)\\ &= \iint e^{2i\pi(x\cdot\eta+\xi\cdot y)}\mathcal{W}(u_0,u_2)\Big(x-\frac{y}{2},\xi+\frac{\eta}{2}\Big)\overbrace{\mathcal{F}\big(\mathcal{W}(u_1,u_3)\big)(\eta,y)}^{\mathcal{A}(u_1,u_3)(-\eta,-y)}dyd\eta,\end{aligned}$$

where $\mathcal{F}$ stands for the Fourier transformation and $\mathcal{A}$ for the ambiguity function (cf. (1.2.8)). With formula (1.2.9), we obtain

$$\begin{aligned}&\big(\mathcal{W}(u_0,u_2)\sharp\mathcal{W}(u_1,u_3)\big)(x,\xi)\\ &\quad= \iint e^{4i\pi(-x\cdot\eta+\xi\cdot y)}\mathcal{W}(u_0,u_2)(x-y,\xi-\eta)\mathcal{W}(\check{u}_1,u_3)(y,\eta)dyd\eta 2^n,\end{aligned}$$

yielding from (1.2.64) for any $X \in \mathbb{R}^{2n}$,

$$\langle u_1,u_2\rangle_{L^2}\mathcal{W}(u_0,u_3)(X) = \int_{\mathbb{R}^{2n}} e^{4i\pi[X,Y]}\mathcal{W}(u_0,u_2)(X-Y)\mathcal{W}(\check{u}_1,u_3)(Y)dY2^n,$$

which implies the lemma. ∎

1.2.6 L^2-boundedness

Theorem 1.2.27. *Let a be a semi-classical symbol on* $\mathbb{R}^{2n}$*, i.e., a smooth function of* (x,ξ) *depending on* $h \in (0,1]$ *such that*

$$\forall l \in \mathbb{N}, \quad p_l(a) = \sup_{\substack{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1] \\ |\alpha|+|\beta|\le l}} |(\partial_x^\alpha \partial_\xi^\beta a)(x,\xi,h)| h^{-\frac{|\alpha|+|\beta|}{2}} < +\infty. \quad (1.2.65)$$

Then, the operator $\mathrm{Op_w}(a(x,\xi,h))$ *is bounded on* $L^2(\mathbb{R}^n)$ *and such that*

$$\|\mathrm{Op_w}(a(x,\xi,h))\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \le c_n p_{\ell_n}(a),$$

where c_n *and* ℓ_n *depend only on* n.

Proof. Theorem 1.2 in A. Boulkhemair's article [3] is providing that result (and more) with

$$\ell_n = [n/2] + 1.$$

Note also that [33, Theorem 1.1.4] is providing an elementary proof of the above result for the ordinary quantization of a given by

$$\begin{aligned}(\mathrm{Op}_0(a)u)(x) &= \int e^{2i\pi x\cdot\xi} a(x,\xi,h)\hat{u}(\xi)d\xi \\ &= \iint e^{2i\pi(x-y)\cdot\xi} a(x,\xi,h)u(y)dyd\xi.\end{aligned}$$

∎

N.B. Formula (1.2.63) appears as

$$(a\sharp b)(X) = \Big[\mathrm{Op}_0\Big(a\big(X - \frac{\sigma\,\Xi}{2}\big)\Big)b\Big](X),$$

where $\mathrm{Op}_0(\cdot)$ stands for the ordinary quantization in $2n$ dimensions.

The following classical result is a consequence of Theorem 1.2.27.

Theorem 1.2.28. *Let* $C_b^\infty(\mathbb{R}^{2n})$ *be the set of bounded smooth complex-valued functions on* $\mathbb{R}^{2n}$ *such that all derivatives are bounded and let a be in* $C_b^\infty(\mathbb{R}^{2n})$*. Then, the operator* $\mathrm{Op_w}(a)$ *is bounded on* $L^2(\mathbb{R}^n)$ *and the* $\mathcal{B}(L^2(\mathbb{R}^n))$ *norm of* $\mathrm{Op_w}(a)$ *is bounded above by a fixed semi-norm of a in the Fréchet space* $C_b^\infty(\mathbb{R}^{2n})$.

1.2.7 On the Heisenberg Uncertainty Relations

Let $u \in \mathscr{S}(\mathbb{R})$. We have, using the notations (A.1.4),

$$2\,\mathrm{Re}\langle D_x u, ixu\rangle_{L^2(\mathbb{R})} = \langle [D_x, ix]u, u\rangle_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|u\|^2_{L^2(\mathbb{R})}, \quad (1.2.66)$$

implying, in particular,

$$\|D_x u\|_{L^2(\mathbb{R})}\|xu\|_{L^2(\mathbb{R})} \ge \frac{1}{4\pi}\|u\|^2_{L^2(\mathbb{R})},$$

which is an equality for $u(x) = e^{-\pi x^2}$; moreover we infer also from (1.2.66) that

$$\langle \pi(D_x^2 + x^2)u, u\rangle \ge \frac{1}{2}\|u\|^2_{L^2(\mathbb{R})},$$

and for

$$q_\mu(x,\xi) = \sum_{1\le j\le n} \mu_j(x_j^2 + \xi_j^2), \quad 0 \le \mu_1 \le \dots \le \mu_n,$$

the inequality

$$\langle \operatorname{Op_w}\big(\pi q_\mu(x,\xi)\big)u, u\rangle_{L^2(\mathbb{R}^n)} \ge \|u\|^2_{L^2(\mathbb{R}^n)} \frac{1}{2} \underbrace{\sum_{1\le j\le n} \mu_j}_{\substack{\text{defined as}\\ \text{trace}_+(q_\mu)}}, \tag{1.2.67}$$

which is an equality for $u(x) = e^{-\pi|x|^2}$. Note that the above (optimal) inequality can be reformulated as

$$\iint_{\mathbb{R}^{2n}} \pi q_\mu(x,\xi)\mathcal{W}(u,u)(x,\xi)dxd\xi \ge \|u\|^2_{L^2(\mathbb{R}^n)}\frac{1}{2}\operatorname{trace}_+(q_\mu).$$

Note also that with the symplectic matrix σ defined by (1.2.15), the so-called fundamental matrix of q_μ is defined by

$$F_{q_\mu} = \sigma^{-1}Q_\mu = \begin{pmatrix} 0 & -I\\ I & 0\end{pmatrix}\begin{pmatrix} M & 0\\ 0 & M\end{pmatrix} = \begin{pmatrix} 0 & -M\\ M & 0\end{pmatrix} \quad \text{with } M = \operatorname{diag}(\mu_1,\dots,\mu_n)$$

so that

$$\text{Spectrum } F_{q_\mu} = \{\pm i\mu_j\}_{1\le j\le n}, \quad \operatorname{trace}_+(q_\mu) = \sum_{\substack{\lambda \text{ eigenvalue of } F_{q_\mu}\\ \text{with } \operatorname{Im}\lambda>0}} \lambda/i.$$

With the notations

$$\begin{cases} C_j = D_{x_j} + ix_j, & \text{creation operators,}\\ C_j^* = D_{x_j} - ix_j, & \text{annihilation operators,}\end{cases}$$

we see that

$$\pi[C_j^*, C_j] = \pi[D_{x_j} - ix_j, D_{x_j} + ix_j] = I,$$

and

$$\operatorname{Op_w}(q_\mu) = \pi \sum_{1\le j\le n} \mu_j C_j C_j^* + \frac{1}{2}\operatorname{trace}_+(q_\mu),$$

which provides another proof of (1.2.67).

Lemma 1.2.29 (Quantum Mechanics must deal with unbounded operators[8]). *Let* $\mathbb{H}$ *be a Hilbert space and let* $J, K \in \mathcal{B}(\mathbb{H})$*; then the commutator* $[J, K] \neq \mathrm{Id}$.

Proof. Let J, K be bounded operators with $[J, K]=\mathrm{Id}$. Then, for all $N \in \mathbb{N}^*$, we have

$$[J, K^N] = NK^{N-1}. \tag{1.2.68}$$

Indeed, this is true for $N = 1$ and if it holds for some $N \geq 1$, we find that

$$\begin{aligned}[J, K^{N+1}] &= JK^N K - K^{N+1} J = [J, K^N]K + K^N JK - K^{N+1} J \\ &= [J, K^N]K + K^N(JK - KJ) = [J, K^N]K + K^N = (N+1)K^N.\end{aligned}$$

Note that (1.2.68) implies that for all $N \in \mathbb{N}^*$, we have $K^N \neq 0$: of course $K \neq 0$ since $[J, K] = \mathrm{Id}$ and if we had $K^N = 0$ for some $N \geq 2$, (1.2.68) would imply $K^{N-1} = 0$ and eventually $K = 0$. As a consequence, we get from (1.2.68) that for all $N \geq 2$,

$$N\|K^{N-1}\|_{\mathcal{B}(\mathbb{H})} \leq 2\|J\|_{\mathcal{B}(\mathbb{H})}\|K^N\|_{\mathcal{B}(\mathbb{H})} \leq 2\|J\|_{\mathcal{B}(\mathbb{H})}\|K\|_{\mathcal{B}(\mathbb{H})}\|K^{N-1}\|_{\mathcal{B}(\mathbb{H})},$$

implying since $\|K^{N-1}\|_{\mathcal{B}(\mathbb{H})} > 0$, that

$$\forall N \geq 2, \quad N \leq 2\|J\|\|K\|,$$

which is impossible and proves the lemma. ∎

Lemma 1.2.30 (Hardy's inequality: the study of non-self-adjoint operators may be useful to determine lowerbounds of self-adjoint operators). *Let* $n \in \mathbb{N}, n \geq 3$*; let* u *in* $L^2(\mathbb{R}^n)$ *such that* $\nabla u \in L^2(\mathbb{R}^n)$, $|x|^{-1}u \in L^2(\mathbb{R}^n)$. *Then, we have*

$$\|\nabla u\|^2_{L^2(\mathbb{R}^n)} \geq \left(\frac{n-2}{2}\right)^2 \||x|^{-1}u\|^2_{L^2(\mathbb{R}^n)}.$$

Proof. We write first

$$\begin{aligned}&\sum_{1\leq j\leq n} \|(D_{x_j} - i\phi_j)u\|^2_{L^2(\mathbb{R}^n)} \\ &\qquad = \langle |D|^2 u, u\rangle_{L^2(\mathbb{R}^n)} + \langle |\phi|^2 u, u\rangle_{L^2(\mathbb{R}^n)} - \frac{1}{2\pi}\langle (\operatorname{div}\phi)u, u\rangle_{L^2(\mathbb{R}^n)},\end{aligned}$$

so that with $\phi(x) = \frac{\nu x}{2\pi|x|^2}$, we get the operator inequality

$$|D|^2 + \frac{\nu^2}{4\pi^2|x|^2} \geq \frac{\nu(n-2)}{4\pi^2|x|^2}, \quad \text{so that } -\Delta \geq |x|^{-2} \underbrace{\nu(n-2-\nu)}_{\text{largest at } \nu=(n-2)/2},$$

proving the lemma. ∎

[8]Thus, QM must involve infinite-dimensional Hilbert spaces and unbounded operators on them.

N.B. A modern approach to the Heisenberg uncertainty principle should certainly begin with reading C. Fefferman's article [8] as well as E. Lieb's book [38].

1.2.8 Non-negative quantizations formulas

Lemma 1.2.31. *Let χ be an even function in $\mathscr{S}(\mathbb{R}^{2n})$ with $L^2(\mathbb{R}^{2n})$ norm equal to 1. We define*

$$\Gamma_\chi = \bar{\chi}\sharp\chi. \tag{1.2.69}$$

Then, the function Γ_χ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, is real-valued even and is such that

$$\int_{\mathbb{R}^{2n}} \Gamma_\chi(X)dX = 1.$$

Let u be in $L^2(\mathbb{R}^n)$. Then, the convolution $\mathcal{W}(u,u) * \Gamma_\chi$ is non-negative. As a result, the operator with Weyl symbol $a * \Gamma_\chi$ is a non-negative operator whenever a is a non-negative function.

Proof. Following the book [33], the composition formula (1.2.61) is bilinear continuous from $\mathscr{S}(\mathbb{R}^{2n})^2$ into $\mathscr{S}(\mathbb{R}^{2n})$ and we have also

$$\overline{a\sharp b} = \bar{b}\sharp\bar{a}.$$

So that Γ_χ is indeed real-valued. Moreover, we have

$$\begin{aligned}\int_{\mathbb{R}^{2n}} \Gamma_\chi(X)dX &= 2^{2n}\iiint_{(\mathbb{R}^{2n})^3} e^{-4i\pi[X-Y,Y-Z]}\bar{\chi}(Y)\chi(Z)dYdZdX\\ &= \int |\chi(Y)|^2dY = 1,\end{aligned}$$

and

$$\begin{aligned}\Gamma_\chi(-X) &= 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[-X-Y,-X-Z]}\bar{\chi}(Y)\chi(Z)dYdZ\\ &= 2^{2n}\iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} e^{-4i\pi[-X+Y,-X+Z]}\bar{\chi}(Y)\chi(Z)dYdZ = \Gamma_\chi(X).\end{aligned}$$

We have also

$$\begin{aligned}&\big(\mathcal{W}(u,u) * \Gamma_\chi\big)(Y)\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(Y-X)\Gamma_\chi(X)dX = \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(Y+X)\Gamma_\chi(X)dX\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(u,u)(T_Y(X))\Gamma_\chi(X)dX = \int_{\mathbb{R}^{2n}} \mathcal{W}(\tau_{-Y}u,\tau_{-Y}u)(X)\Gamma_\chi(X)dX\\ &= \int_{\mathbb{R}^{2n}} \mathcal{W}(\tau_{-Y}u,\tau_{-Y}u)(X)(\bar{\chi}\sharp\chi)(X)dX\\ &= \langle \mathrm{Op_w}(\bar{\chi}\sharp\chi)\tau_{-Y}u,\tau_{-Y}u\rangle_{L^2(\mathbb{R}^n)} = \|\mathrm{Op_w}(\chi)\tau_{-Y}u\|^2_{L^2(\mathbb{R}^n)} \ge 0,\end{aligned}$$

proving the first statement of non-negativity. Let a be a non-negative function, say in $L^1(\mathbb{R}^{2n})$; we have

$$\begin{aligned}
&\operatorname{Op_w}(a * \Gamma_\chi)\\
&= 2^n \iint a(Y)\Gamma_\chi(X-Y)\sigma_X\, dY dX = \int a(Y)\int (\bar\chi \sharp \chi)(X-Y)2^n \sigma_X\, dX dY\\
&= \int a(Y)\int (\bar\chi \sharp \chi)(T_{-Y}(X))2^n \sigma_X\, dX dY = \int a(Y)\tau_Y \operatorname{Op_w}(\bar\chi \sharp \chi)\tau_{-Y}\, dY\\
&= \int a(Y)\tau_Y \operatorname{Op_w}(\bar\chi)\operatorname{Op_w}(\chi)\tau_{-Y}\, dY\\
&= \int a(Y)\underbrace{[\operatorname{Op_w}(\chi)\tau_{-Y}]^*[\operatorname{Op_w}(\chi)\tau_{-Y}]}_{\text{non-negative operator}}\, dY \geq 0,
\end{aligned}$$

if $a(Y) \geq 0$ for all $Y \in \mathbb{R}^{2n}$ and this concludes the proof. ∎

We can write as well

$$\begin{aligned}
\operatorname{Op_w}(a * \Gamma_\chi) &= \int_{\mathbb{R}^{2n}} a(Y)\big[\tau_Y \operatorname{Op_w}(\chi)\tau_{-Y}\big]^*\big[\tau_Y \operatorname{Op_w}(\chi)\tau_{-Y}\big] dY\\
&= \int_{\mathbb{R}^{2n}} a(Y)\Sigma_\chi(Y) dY, \qquad (1.2.70)
\end{aligned}$$

with

$$\Sigma_\chi(Y) = [\tau_Y \operatorname{Op_w}(\chi)\tau_{-Y}]^*[\tau_Y \operatorname{Op_w}(\chi)\tau_{-Y}] = \big(\operatorname{Op_w}(\chi(\cdot - Y))\big)^* \operatorname{Op_w}(\chi(\cdot - Y)). \qquad (1.2.71)$$

Remark 1.2.32. The Gaussian case in the previous lemma gives rise to the standard non-negativity properties of coherent states. In fact, choosing $\chi(X) = 2^n e^{-2\pi|X|^2}$, we see that χ is even, belongs to the Schwartz space and

$$\|\chi\|^2_{L^2(\mathbb{R}^{2n})} = 2^{2n}\int_{\mathbb{R}^{2n}} e^{-4\pi|X|^2} dX = 2^{2n}4^{-2n/2} = 1.$$

We have also[9]

$$\begin{aligned}
\Gamma_\chi(X) &= 2^{4n}\iint_{(\mathbb{R}^{2n})^2} e^{-4i\pi[X-Y,X-Z]}e^{-2\pi(|Y|^2+|Z|^2)} dY dZ\\
&= 2^{3n}\int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]}e^{-2\pi(|X+Y|^2+|Y|^2)} dY\\
&= 2^{3n}\int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]}e^{-2\pi(|Y+\frac{X}{2}|^2+|Y-\frac{X}{2}|^2)} dY\\
&= 2^{3n}e^{-\pi|X|^2}\int_{\mathbb{R}^{2n}} e^{4i\pi[Y,X]}e^{-4\pi|Y|^2} dY = 2^{3n}e^{-\pi|X|^2}4^{-n}e^{-\pi|X|^2} = \chi(X).
\end{aligned}$$

[9] [33, Proposition 4.1.1] is useful to compute the Fourier transform of Gaussian functions and is a notable asset of the Fourier normalization given in Section A.1.1.

In that case we find that $\operatorname{Op}_{\mathrm{w}}(\chi)$ is a rank-one orthogonal projection on the fundamental state Ψ_0 of the harmonic oscillator $\pi(|D_x|^2 + |x|^2)$. According to (A.1.16) the one-dimensional kth Hermite function is

$$\psi_k(x) = \frac{(-1)^k}{2^k\sqrt{k!}} 2^{1/4} e^{\pi x^2} \Big(\frac{d}{\sqrt{\pi}dx}\Big)^k (e^{-2\pi x^2}), \tag{1.2.72}$$

so that $\Psi_0(x) = 2^{n/4} e^{-\pi|x|^2}$. We calculate

$$\begin{aligned}\Gamma(x,\xi) = \mathcal{W}(\Psi_0,\Psi_0)(x,\xi) &= 2^{n/2}\int_{\mathbb{R}^n} e^{-\pi(|x+z/2|^2+|x-z/2|^2)} e^{-2i\pi z\xi} dz \\ &= 2^{n/2} e^{-2\pi|x|^2}\int_{\mathbb{R}^n} e^{-\pi z^2/2} e^{-2i\pi z\xi} dz = 2^n e^{-2\pi|x|^2} e^{-2\pi|\xi|^2} = \chi(x,\xi).\end{aligned}$$

The anti-Wick quantization of a symbol a is defined as (see, e.g., M. Shubin's book [47])

$$\operatorname{Op}_{\mathrm{aw}}(\mathrm{a}) = \int_{\mathbb{R}^{2n}} \mathrm{a}(Y)\Sigma_Y dY, \tag{1.2.73}$$

where Σ_Y is the rank-one orthogonal projection given by

$$\Sigma_{y,\eta} u = \langle u, \tau_{y,\eta}\Psi_0\rangle \tau_{y,\eta}\Psi_0.$$

Remark 1.2.33. It is interesting to notice that to produce non-negativity of the operator with Weyl symbol $\mathrm{a} * \Gamma_\chi$ when a is a non-negative function, we do not use the non-negativity of Γ_χ as a function, which by the way does not always hold (except in the Gaussian cases), but we use the fact that the quantization of Γ_χ is non-negative, as it is defined as $\operatorname{Op}_{\mathrm{w}}(\bar{\chi}\sharp\chi) = (\operatorname{Op}_{\mathrm{w}}(\chi))^* \operatorname{Op}_{\mathrm{w}}(\chi)$.

Remark 1.2.34. Another important remark is concerned with the Taylor expansion of $\mathrm{a} * \Gamma_\chi$, we have

$$\begin{aligned}(\mathrm{a} * \Gamma_\chi)(X) &= \int \mathrm{a}(X-Y)\Gamma_\chi(Y)dY = \int \mathrm{a}(X+Y)\Gamma_\chi(Y)dY \\ &= \int \Big(\mathrm{a}(X) + \mathrm{a}'(X)Y + \int_0^1 (1-\theta)\mathrm{a}''(X+\theta Y)Y^2\Big)\Gamma_\chi(Y)dY \\ &= \mathrm{a}(X) + \iint_0^1 (1-\theta)\mathrm{a}''(X+\theta Y)Y^2\Gamma_\chi(Y)dY.\end{aligned}$$

As a result the difference $(\mathrm{a} * \Gamma_\chi) - \mathrm{a}$ depends only on the second derivative of a. If for instance a is a semi-classical symbol, i.e., a smooth function of (x,ξ) depending on $h \in (0,1]$ such that

$$\forall(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n, \quad \sup_{(x,\xi)\in\mathbb{R}^{2n}, h\in(0,1]} |(\partial_x^\alpha\partial_\xi^\beta \mathrm{a})(x,\xi,h)| h^{-\frac{|\alpha|+|\beta|}{2}} < +\infty, \tag{1.2.74}$$

then the difference $\mathrm{Op}_{\mathrm{aw}}(\mathrm{a}) - \mathrm{Op}_{\mathrm{w}}(\mathrm{a})$ is bounded on $L^2(\mathbb{R}^n)$ with an $O(h)$ operator-norm, so that if a happens also to be non-negative, we find

$$\mathrm{Op}_{\mathrm{w}}(\mathrm{a}) = \underbrace{\mathrm{Op}_{\mathrm{w}}(\mathrm{a}) - \mathrm{Op}_{\mathrm{w}}(\mathrm{a} * \Gamma_\chi)}_{\substack{O(h)\\ \text{as an operator,}\\ \text{cf. Theorem 1.2.27}}} + \underbrace{\mathrm{Op}_{\mathrm{w}}(\mathrm{a} * \Gamma_\chi)}_{\substack{\geq 0\\ \text{as an operator}}},$$

and we obtain a version of the so-called Sharp Gårding inequality,

$$\mathrm{Op}_{\mathrm{w}}(\mathrm{a}) + Ch \geq 0 \quad \text{(as an operator).}$$

Theorem 1.2.35. *Let χ be an even function in the Schwartz space $\mathscr{S}(\mathbb{R}^{2n})$ with $L^2(\mathbb{R}^{2n})$ norm equal to 1 and let Γ_χ be given by (1.2.69). For $a \in L^\infty(\mathbb{R}^{2n})$, we define*

$$\mathrm{Op}(\chi, a) = \mathrm{Op}_{\mathrm{w}}(a * \Gamma_\chi).$$

Then, $\mathrm{Op}(\chi, a)$ *is a bounded operator in $L^2(\mathbb{R}^n)$ and we have*

$$\|\mathrm{Op}(\chi, a)\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \leq \|a\|_{L^\infty(\mathbb{R}^{2n})}. \tag{1.2.75}$$

Moreover, if a is valued in some interval J of the real line, we have the operator inequalities

$$\inf J \leq \mathrm{Op}(\chi, a) \leq \sup J. \tag{1.2.76}$$

In particular, if $a(x,\xi) \geq 0$ for all $(x,\xi) \in \mathbb{R}^{2n}$, we have the operator-inequality $\mathrm{Op}(\chi, a) \geq 0$.

N.B. The non-negativity of the anti-Wick quantization (1.2.73) and its avatars Husimi [25], Coherent States, Gabor wavelets (see, e.g., [11]), are particular cases of the above theorem. More information on this topic is available in Section 2.4 of the book [33]. Another remark is that this result can easily be extended to matrix-valued symbols as in Remark 2 page 79 of L. Hörmander's [24] and even to symbols valued in $\mathcal{B}(\mathbb{H})$, where $\mathbb{H}$ is a Hilbert space.

Proof. We start with Formulas (1.2.70), (1.2.71), entailing

$$\mathrm{Op}(\chi, a) = \int_{\mathbb{R}^{2n}} a(Y)\Sigma_\chi(Y)dY,$$

with $\Sigma_\chi(Y) = [\mathrm{Op}_{\mathrm{w}}(\chi(\cdot - Y))]^* \mathrm{Op}_{\mathrm{w}}(\chi(\cdot - Y)) = \tau_Y \mathrm{Op}_{\mathrm{w}}(\bar\chi \sharp \chi)\tau_{-Y}$. We note that

$$\mathrm{Op}(\chi, 1) = \int_{\mathbb{R}^{2n}} \tau_Y \mathrm{Op}_{\mathrm{w}}(\bar\chi \sharp \chi)\tau_{-Y} dY,$$

so has Weyl symbol $X \mapsto \int_{\mathbb{R}^{2n}} \Gamma_\chi(X - Y)dY = 1$ from Lemma 1.2.31 and thus $\mathrm{Op}(\chi, 1) = \mathrm{Id}$. We infer that for $u, v \in \mathscr{S}(\mathbb{R}^n)$,

$$\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(Y)\langle \mathrm{Op}_{\mathrm{w}}(\chi(\cdot - Y))u, \mathrm{Op}_{\mathrm{w}}(\chi(\cdot - Y))v\rangle dY,$$

so that with any $\nu > 0$,

$$
\begin{aligned}
&|\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)}| \\
&\le \|a\|_{L^\infty(\mathbb{R}^{2n})}\int_{\mathbb{R}^{2n}} \frac{1}{2}\big(\nu\|\mathrm{Op_w}(\chi(\cdot - Y))u\|^2_{L^2(\mathbb{R}^n)} + \nu^{-1}\|\mathrm{Op_w}(\chi(\cdot - Y))v\|^2_{L^2(\mathbb{R}^n)}\big)dY \\
&= \|a\|_{L^\infty(\mathbb{R}^{2n})}\frac{1}{2}\big(\nu\langle \mathrm{Op}(\chi, 1)u, u\rangle_{L^2(\mathbb{R}^n)} + \nu^{-1}\langle \mathrm{Op}(\chi, 1)v, v\rangle_{L^2(\mathbb{R}^n)}\big) \\
&= \|a\|_{L^\infty(\mathbb{R}^{2n})}\frac{1}{2}\big(\nu\|u\|^2_{L^2(\mathbb{R}^n)} + \nu^{-1}\|v\|^2_{L^2(\mathbb{R}^n)}\big),
\end{aligned}
$$

and taking the infimum of the right-hand side with respect to ν, we obtain

$$|\langle \mathrm{Op}(\chi, a)u, v\rangle_{L^2(\mathbb{R}^n)}| \le \|a\|_{L^\infty(\mathbb{R}^{2n})}\|u\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)},$$

proving (1.2.75). To prove (1.2.76), it is enough to prove the last statement in the theorem which follows immediately from (1.2.70), (1.2.71) since each operator Σ_Y is non-negative. The proof of the theorem is complete. ∎

It is nice to have examples of non-negative quantizations, but somehow more importantly, it is crucial to relate these quantizations to the mainstream quantization, that is to the Weyl quantization. This is what we do in the next theorem, dealing with semi-classical symbols.

Theorem 1.2.36 (Sharp Gårding inequality). *Let a be a function defined on $\mathbb{R}^n \times \mathbb{R}^n \times (0, 1]$ such that $a(x, \xi, h)$ is smooth for all $h \in (0, 1]$ and such that*

$$\forall(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n, \qquad \sup_{(x,\xi,h)\in\mathbb{R}^n\times\mathbb{R}^n\times(0,1]} |(\partial_x^\alpha \partial_\xi^\beta a)(x, \xi, h)|h^{-|\beta|} < +\infty. \tag{1.2.77}$$

Let us assume that the function a is valued in $\mathbb{R}_+$. Then, there exists a constant C such that

$$\mathrm{Op_w}(a) + Ch \ge 0.$$

Proof. We have given a proof of this result in Remark 1.2.34 but with a different definition for a semi-classical symbol (see (1.2.74)). Starting with our definition above in (1.2.77), we define

$$b(x, \xi, h) = a(h^{1/2}x, h^{-1/2}\xi, h),$$

and we see that b satisfies the estimates (1.2.74) and is a non-negative function so that, applying Remark 1.2.34, we can find a constant C such that

$$\mathrm{Op_w}(b) + Ch \ge 0.$$

We note now that Segal's formula (1.2.48) applied to the symplectic mapping

$$(x, \xi) \mapsto (h^{1/2}x, h^{-1/2}\xi),$$

shows that $\mathrm{Op_w}(b)$ is unitarily equivalent to $\mathrm{Op_w}(a)$, providing the sought result. ∎

N.B. Several versions of the above theorem can be found in the literature, in particular, [24, Theorem 18.1.14]. The first proof of this result was given in 1966 by L. Hörmander in [21] for scalar-valued symbols and a proof for systems was given by P. Lax and L. Nirenberg in [28] on the same year. Far-reaching refinements of that inequality were given by C. Fefferman and D. H. Phong, who proved in [9] in 1978 that, under the same assumption as in Theorem 1.2.36 for scalar-valued symbols, they obtain the much stronger

$$\mathrm{Op_w}(a) + Ch^2 \geq 0. \tag{1.2.78}$$

A thorough discussion of these questions is given in [24, Section 18.6] and in [33, Section 2.5] (see also [1]).

1.3 Examples

1.3.1 Hermite functions

We can easily calculate the Wigner distribution of Hermite functions and since the Wigner distributions respect tensor products as partial Fourier transforms, it is enough to do in one dimension. With ψ_k given in (1.2.72), the Wigner distribution $\mathcal{W}(\psi_k, \psi_k)$ appears as the Weyl symbol of $\mathbb{P}_{k;1} = \mathbb{P}_k$ as defined in (A.1.17). We find that the Weyl symbol of $\mathbb{P}_{0;n}$, following (A.3.2), is

$$2^n e^{-2\pi(|x|^2+|\xi|^2)}.$$

More generally, the paper [27] provides in one dimension

$$\mathcal{W}(\psi_k, \psi_k)(x, \xi) = (-1)^k 2 e^{-2\pi(x^2+\xi^2)} L_k(4\pi(x^2 + \xi^2)), \tag{1.3.1}$$

where L_k is the standard Laguerre polynomial with degree k (see (A.4.1)). As a result, the Weyl symbol of $\mathbb{P}_{k;n}$ is equal to $\pi_{k,n}(x, \xi)$ with

$$\pi_{k,n}(x, \xi) = (-1)^k 2^n e^{-2\pi(|x|^2+|\xi|^2)} \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \prod_{1\leq j\leq n} L_{\alpha_j}(4\pi(x_j^2 + \xi_j^2)).$$

Note that the leading term in the polynomial $(-1)^k L_k(t)$ is $t^k/k!$ and this implies that the set

$$\{(x, \xi) \in \mathbb{R}^2, \mathcal{W}(\psi_k, \psi_k)(x, \xi) < 0\},$$

where $\mathcal{W}(\psi_k, \psi_k)$ is given by (1.3.1) is a relatively compact open subset of $\mathbb{R}^2$. Indeed, we have

$$W(\psi_k, \psi_k)(X) = 2e^{-2\pi|X|^2} \left\{\frac{(4\pi|X|^2)^k}{k!}\right\} \underbrace{\left(1 + \sum_{0\leq l\leq k-1} a_l (4\pi|X|^2)^{-(k-l)}\right)}_{\geq 1/2 \text{ for } |X| \geq R_0}$$

which implies that

$$\{X \in \mathbb{R}^2, |X| \geq \max(R_0, 1)\} \subset \{X \in \mathbb{R}^2, W(\psi_k, \psi_k)(X) > 0\},$$

and thus

$$\{W(\psi_k, \psi_k)(X) \leq 0\} \subset \{|X| < \max(R_0, 1)\}.$$

1.3.2 One-sided exponentials

Let us define for $a > 0$, $f_a(t) = H(t)a^{1/2}e^{-at/2}$. We have

$$\begin{aligned}\mathcal{W}(f_a, f_a)(x, \xi) &= aH(x)\int_{|z|\leq 2x} e^{-2i\pi z\xi}e^{-\frac{a}{2}(x+z/2)}e^{-\frac{a}{2}(x-z/2)}dz\\ &= aH(x)e^{-xa}\int_{|z|\leq 2x} e^{-2i\pi z\xi}dz\\ &= 2aH(x)e^{-xa}\int_0^{2x}\cos(z2\pi\xi)dz\\ &= aH(x)e^{-xa}\frac{\sin(4\pi x\xi)}{\pi\xi}.\end{aligned}$$

We can check

$$\iint \mathcal{W}(f_a, f_a)(x, \xi)dxd\xi = \frac{a}{\pi}\int_{x=0}^{+\infty} e^{-ax}\int\frac{\sin(4\pi x\xi)}{\xi}d\xi dx = 1 = \|f_a\|^2_{L^2(\mathbb{R})},$$

and since

$$\int_{\mathbb{R}}\frac{\sin^2 t}{t^2}dt = \pi,$$

we verify (see Lemma 1.2.25 and (1.1.4)),

$$\iint \mathcal{W}(f_a, f_a)(x, \xi)^2dxd\xi = \frac{a^2}{\pi^2}\int_{x=0}^{+\infty} e^{-2ax}\int\frac{\sin^2(4\pi x\xi)}{\xi^2}d\xi dx = 1 = \|f_a\|^4_{L^2(\mathbb{R})}.$$

On the other hand, the ambiguity function $\mathcal{A}(f_a, f_a)$ is the inverse Fourier transform of $\mathcal{W}$ and we have

$$\begin{aligned}\mathcal{A}(f_a, f_a)(\eta, y) &= \frac{a}{\pi}\iint H(x)e^{-x(a-2i\pi\eta)}\frac{\sin\xi}{\xi}e^{2i\pi\frac{y}{4\pi x}\xi}dxd\xi\\ &= a\int_{|y|/2}^{+\infty} e^{-x(a-2i\pi\eta)}dx = \frac{ae^{-\frac{1}{2}|y|(a-2i\pi\eta)}}{a-2i\pi\eta},\end{aligned}$$

which corresponds to [17, formula (9)] noting that with our notations, we have

$$\mathcal{A}(f, f)(\eta, y) = \widetilde{\mathcal{A}}(f, f)(y, -\eta),$$

where $\widetilde{\mathcal{A}}(f, f)$ is the normalization chosen in [17]. Going back to the Wigner distribution, that simple example is interesting since we have

$$\begin{aligned}&\big\{(x,\xi), \mathcal{W}(f_a,f_a)(x,\xi)<0\big\}\\&=\bigcup_{k\in\mathbb{N}}\Big\{(x,\xi)\in(0,+\infty)\times\mathbb{R}^*, \frac{k}{2}+\frac{1}{4}<x|\xi|<\frac{k}{2}+\frac{1}{2}\Big\},\end{aligned}$$

and we see that the Lebesgue measure of

$$E_k=\Big\{(x,\xi)\in(0,+\infty)\times\mathbb{R}^*, \frac{k}{2}+\frac{1}{4}<x|\xi|<\frac{k}{2}+\frac{1}{2}\Big\},$$

is infinite since

$$|E_k|=2\int_0^{+\infty}\frac{dx}{4x}=+\infty.$$

Moreover, the function $\mathcal{W}(f_a,f_a)(x,\xi)$ does not belong to $L^1(\mathbb{R}^2)$ since

$$\iint H(x)e^{-xa}\left|\frac{\sin(4\pi x\xi)}{\pi\xi}\right|dx d\xi\geq\iint_{(0,+\infty)^2}e^{-xa}\left|\frac{\sin\eta}{\pi\eta}\right|dx d\eta=+\infty.$$

As a consequence, we have, using the notation for $\alpha\in\mathbb{R}$,

$$\alpha_\pm=\max(\pm\alpha,0),$$

$$\iint\big(\mathcal{W}(f_a,f_a)(x,\xi)\big)_+dx d\xi=\iint\big(\mathcal{W}(f_a,f_a)(x,\xi)\big)_-dx d\xi=+\infty,$$

since the real-valued function $\mathcal{W}(f_a,f_a)$ does not belong to $L^1(\mathbb{R}^2)$ and is such that

$$\iint\mathcal{W}(f_a,f_a)(x,\xi)dx d\xi=\|f_a\|^2_{L^2(\mathbb{R})}=1.$$

We will see in Section 6.4 several important consequences of that phenomenon for the quantization of the indicatrix of some subsets of $\mathbb{R}^2$, such as

$$E_\pm=\big\{(x,\xi),\pm\mathcal{W}(f_a,f_a)(x,\xi)>0\big\}.$$

1.3.3 Box functions

We start with $\beta_0(t)=\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)$, for which a straightforward calculation gives

$$\mathcal{W}(\beta_0,\beta_0)(x,\xi)=\mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(x)\frac{\sin(2\pi(1-2|x|)\xi)}{\pi\xi}.$$

More generally, for real parameters $a\leq b$, defining

$$\beta(t)=(b-a)^{-1/2}\mathbf{1}_{[a,b]}(t)e^{2i\pi\omega t},$$

we find

$$\mathcal{W}(\beta,\beta)(x,\xi) = [(b-a)\pi(\xi-\omega)]^{-1} \times \Big(\mathbf{1}_{[a,\frac{a+b}{2}]}(x)\sin[4\pi(\xi-\omega)(x-a)] + \mathbf{1}_{[\frac{a+b}{2},b]}(x)\sin[4\pi(\xi-\omega)(b-x)]\Big).$$

Checking now $\beta_1(t) = \mathbf{1}_{[-\frac{1}{2},\frac{1}{2}]}(t)\operatorname{sign} t$, we find after a simple (but this time a bit tedious) calculation,

$$\mathcal{W}(\beta_1,\beta_1)(x,\xi) = \mathbf{1}\Big(|x| \le \frac{1}{4}\Big)\frac{2\sin(4\pi|x|\xi) - \sin\big(2\pi(1-2|x|)\xi\big)}{\pi\xi} + \mathbf{1}\Big(\frac{1}{4} \le |x| \le \frac{1}{2}\Big)\frac{\sin(2\pi(1-2|x|)\xi)}{\pi\xi}.$$

1.4 Integrals of the Wigner distribution on subsets of the phase space

Lemma 1.4.1. *Let E be a measurable subset with finite Lebesgue measure of the phase space $\mathbb{R}^n \times \mathbb{R}^n$ and let $\mathbf{1}_E$ be the indicator function of the set E. Then, the operator with Weyl symbol $\mathbf{1}_E$ is bounded self-adjoint on $L^2(\mathbb{R}^n)$ and for any $u \in L^2(\mathbb{R}^n)$, we have*

$$\langle \operatorname{Op_w}(\mathbf{1}_E)u,u\rangle_{L^2(\mathbb{R}^n)} = \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi. \tag{1.4.1}$$

Proof. It follows immediately from (1.2.1) and (1.2.5). ■

Remark 1.4.2. A consequence of the above formula is that a spectral analysis of the operator $\operatorname{Op_w}(\mathbf{1}_E)$ would display interesting extremalization properties for the right-hand side of (1.4.1); for instance, if

$$\lambda_- = \inf\big(\text{spectrum}(\operatorname{Op_w}(\mathbf{1}_E))\big), \quad \lambda_+ = \sup\big(\text{spectrum}(\operatorname{Op_w}(\mathbf{1}_E))\big),$$

we obtain that for u normalized in $L^2(\mathbb{R}^n)$, we have

$$\lambda_- \le \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \lambda_+.$$

In particular, if λ_- is an eigenvalue related to a normalized eigenfunction u_-, (resp., if λ_+ is an eigenvalue related to a normalized eigenfunction u_+), we get for all u normalized in $L^2(\mathbb{R}^n)$,

$$\iint_E \mathcal{W}(u_-,u_-)(x,\xi)\,dx\,d\xi \le \iint_E \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \iint_E \mathcal{W}(u_+,u_+)(x,\xi)\,dx\,d\xi.$$

We shall see below several examples where the operator $\mathrm{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ with an E having infinite Lebesgue measure. We may note in particular that

$$\mathrm{Op_w}(\mathbf{1}_{\mathbb{R}^{2n}}) = \mathrm{Id},$$

and for a given non-zero linear form $L(x,\xi)$ on $\mathbb{R}^{2n}$ and

$$E = \{(x,\xi) \in \mathbb{R}^{2n}, L(x,\xi) \in J\}, \quad \text{where } J \text{ is a subset of } \mathbb{R}, \tag{1.4.2}$$

we may find affine symplectic coordinates (y,η) on $\mathbb{R}^{2n}$ such that $L(x,\xi) = y_1$, implying with (1.2.48) that $\mathrm{Op_w}(\mathbf{1}_E)$ is unitarily equivalent to the orthogonal projection $u \mapsto u(y)\mathbf{1}_J(y_1)$. Although in that case, the quantization of the indicatrix of E given by (1.4.2) is trivial, we shall see below that in many cases, including some rather explicit ones, the Weyl quantization of the rough Hamiltonian $\mathbf{1}_E(x,\xi)$ could be far from a projection and may have a rather complicated spectrum with a supremum which could be strictly larger than 1 and an infimum which could be negative.

In some sense, although we have the trivial identity $\mathbf{1}_E(x,\xi)^2 = \mathbf{1}_E(x,\xi)$, we shall see that the quantization process by the Weyl formula is destroying that property; to understand integrals of the Wigner distribution on subsets of the phase space, formula (1.4.1) forces us to consider the Weyl quantization of the function $\mathbf{1}_E(x,\xi)$ and the Heisenberg Uncertainty Principle shows that non-commutation properties are governing operators and these properties are of course distorting the classical identities satisfied by classical Hamiltonians.

We must point out as well that we do not have here at our disposal a semi-classical version of our quantization which could ensure some bridge between classical properties and operator-theoretic results as it is the case for the quantization of nice smooth semi-classical symbols depending on a small parameter h such as a C^∞ function $a(x,\xi,h)$ satisfying (1.2.77). In particular, for a symbol a satisfying (1.2.77), we have the following result: if for all $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ we have $a(x,\xi,h) \le 1$, then there exists a semi-norm C of the symbol a such that

$$\mathrm{Id} - \mathrm{Op_w}(a) + Ch^2 \ge 0,$$

i.e.,

$$\mathrm{Op_w}(a) \le \mathrm{Id} + Ch^2,$$

an inequality following from the Fefferman–Phong inequality (cf. (1.2.78)) which implies as well the following lemma.

Lemma 1.4.3. *Let a be a semi-classical symbol of order 0, i.e., a smooth function satisfying (1.2.77) such that for all $(x,\xi,h) \in \mathbb{R}^n \times \mathbb{R}^n \times (0,1]$ we have $0 \le a(x,\xi,h) \le 1$. Then, there exists a semi-norm C of the symbol a such that*

$$-Ch^2 \le \mathrm{Op_w}(a) \le \mathrm{Id} + Ch^2.$$