Chapter 2

Quantization of radial functions and Mehler's formula

This section and the following are essentially based upon the author's paper [36].

2.1 Basic formulas in one dimension

In this section, we work in one dimension and consider a function F in the Schwartz class of $\mathbb{R}$. We want to calculate somewhat explicitly the Weyl quantization of $F(x^2+\xi^2)$ and also extend that computation to the case where F is merely $L^\infty(\mathbb{R})$. We have, say for F in the Wiener algebra $\mathscr{W}(\mathbb{R}) = \text{Fourier}(L^1(\mathbb{R}))$,

$$\operatorname{Op_w}(F(x^2+\xi^2)) = \int_{\mathbb{R}} \widehat{F}(\tau) \operatorname{Op_w}(e^{2i\pi\tau(x^2+\xi^2)}) d\tau,$$

as an absolutely converging integral of a function defined on $\mathbb{R}$ (equipped with the Lebesgue measure) valued in $\mathscr{B}(L^2(\mathbb{R}))$ (bounded endomorphisms of $L^2(\mathbb{R})$). In fact, applying Mehler's formula (A.3.1), we find

$$\underbrace{\operatorname{Op_w}(e^{2i\pi\tau(x^2+\xi^2)})}_{\substack{\text{operator with Weyl symbol} \\ e^{2i\pi\tau(x^2+\xi^2)}}} = \cos(\arctan\tau)\, \underbrace{e^{2i\pi(\arctan\tau)\operatorname{Op_w}(x^2+\xi^2)}}_{\substack{\text{exponential } e^{iM}, \\ \text{with } M \text{ self-adjoint operator} \\ =2\pi(\arctan\tau)\operatorname{Op_w}(x^2+\xi^2)}},$$

so that, using the spectral decomposition (A.1.17) of the harmonic oscillator

$$\operatorname{Op_w}(\pi(x^2+\xi^2)),$$

we get,

$$\begin{aligned}\operatorname{Op_w}(F(x^2+\xi^2)) &= \int_{\mathbb{R}} \widehat{F}(\tau) \sum_{k\ge 0} e^{2i(\arctan\tau)(k+\frac12)} \mathbb{P}_k \frac{d\tau}{\sqrt{1+\tau^2}} \\ &= \sum_{k\ge 0} \int_{\mathbb{R}} \widehat{F}(\tau) e^{2i(k+\frac12)\arctan\tau} \frac{d\tau}{\sqrt{1+\tau^2}} \mathbb{P}_k,\end{aligned}$$

where the use of Fubini theorem is justified by

$$\int_{\mathbb{R}} |\widehat{F}(\tau)| \frac{d\tau}{\sqrt{1+\tau^2}} < +\infty, \quad \mathbb{P}_k \ge 0, \quad \sum_{k\ge 0} \mathbb{P}_k = \operatorname{Id}.$$

We have

$$\begin{aligned}&\int_{\mathbb{R}} \widehat{F}(\tau) e^{2i(k+\frac12)\arctan\tau} \frac{d\tau}{\sqrt{1+\tau^2}} \\ &\qquad = \int_{\mathbb{R}} \widehat{F}(\tau)(\cos(\arctan\tau) + i\sin(\arctan\tau))^{2k+1} \frac{d\tau}{\sqrt{1+\tau^2}},\end{aligned}$$

and, using Section A.8.1, we get

$$\int_{\mathbb{R}} \widehat{F}(\tau) e^{2i(k+\frac{1}{2})\arctan\tau} \frac{d\tau}{\sqrt{1+\tau^2}} = \int_{\mathbb{R}} \widehat{F}(\tau)(1+i\tau)^{2k+1} \frac{d\tau}{(1+\tau^2)^{k+1}}.$$

We have proven the following lemma.

Lemma 2.1.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\widehat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}} |\widehat{F}(\tau)| \frac{d\tau}{\sqrt{1+\tau^2}} < +\infty. \tag{2.1.1}$$

Then, the operator $\mathrm{Op_w}\left(F(x^2+\xi^2)\right)$ *has the spectral decomposition*

$$\begin{aligned}\mathrm{Op_w}\left(F(x^2+\xi^2)\right) &= \sum_{k\geq 0} \int_{\mathbb{R}} \frac{\widehat{F}(\tau)(1+i\tau)^{2k+1}}{(1+\tau^2)^{k+1}} d\tau \mathbb{P}_k \\ &= \sum_{k\geq 0} \int_{\mathbb{R}} \frac{\widehat{F}(\tau)(1+i\tau)^{k}}{(1-i\tau)^{k+1}} d\tau \mathbb{P}_k,\end{aligned}$$

where the orthogonal projections $\mathbb{P}_k$ are defined in (A.1.17).

2.2 Higher-dimensional questions

We work now in n dimensions and consider a function F in the Schwartz class of $\mathbb{R}$. We want to calculate somewhat explicitly the Weyl quantization of $F(\sum_{1\leq j\leq n} \mu_j (x_j^2+\xi_j^2))$, where the μ_j are positive parameters, denoted by

$$\mathrm{Op_w}\left(F\Big(\sum_{1\leq j\leq n} \mu_j(x_j^2+\xi_j^2)\Big)\right), \quad q_\mu(x,\xi) = \sum_{1\leq j\leq n} \mu_j(x_j^2+\xi_j^2),$$

and also extend that computation to the case where F is merely $L^\infty(\mathbb{R})$. We have, say for F in the Wiener algebra $\mathscr{W}(\mathbb{R}) = \mathrm{Fourier}(L^1(\mathbb{R}))$,

$$\mathrm{Op_w}\left(F(q_\mu(x,\xi))\right) = \int_{\mathbb{R}} \widehat{F}(\tau) \mathrm{Op_w}\left(e^{2i\pi\tau\sum_{1\leq j\leq n}\mu_j(x_j^2+\xi_j^2)}\right) d\tau,$$

as an absolutely converging integral of a function defined on $\mathbb{R}$ (equipped with the Lebesgue measure) valued in $\mathcal{B}(L^2(\mathbb{R}^n))$ (bounded endomorphisms of $L^2(\mathbb{R}^n)$). In fact, applying Mehler's formula (A.3.1), we find by tensorisation,

$$\underbrace{\mathrm{Op_w}\left(e^{2i\pi\tau\sum_{1\leq j\leq n}\mu_j(x_j^2+\xi_j^2)}\right)}_{\substack{\text{operator with Weyl symbol} \\ e^{2i\pi\tau q_\mu(x,\xi)}}} = \prod_{1\leq j\leq n} \cos(\arctan(\tau\mu_j)) \underbrace{e^{2i\pi(\arctan(\tau\mu_j))\mathrm{Op_w}(x_j^2+\xi_j^2)}}_{\substack{\text{exponential } e^{iM_j}, \\ \text{with } M_j \text{ self-adjoint operator} \\ = 2\pi(\arctan(\tau\mu_j))\mathrm{Op_w}(x_j^2+\xi_j^2)}}, \tag{2.2.1}$$

so that, using the spectral decomposition (A.1.19) of the harmonic oscillator, we get

$$\begin{aligned}
&\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big)\\
&\quad= \int_{\mathbb{R}} \widehat{F}(\tau) \sum_{\alpha\in\mathbb{N}^n} \prod_{1\le j\le n} e^{2i(\arctan(\tau\mu_j))(\alpha_j+\frac12)}\mathbb{P}_{\alpha_j} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau\\
&\quad= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau\,\mathbb{P}_\alpha,
\end{aligned}$$

where the use of Fubini theorem is justified by

$$\int_{\mathbb{R}} |\widehat{F}(\tau)| \frac{d\tau}{\sqrt{1+\tau^2}} < +\infty, \quad \mathbb{P}_\alpha \ge 0, \quad \sum_\alpha \mathbb{P}_\alpha = \mathrm{Id}\,.$$

We have

$$\begin{aligned}
&\int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau\\
&= \int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} \big(\cos(\arctan(\mu_j\tau)) + i\sin(\arctan(\mu_j\tau))\big)^{2\alpha_j+1} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau,
\end{aligned}$$

and, using Section A.8.1, we get

$$\begin{aligned}
&\int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} e^{2i(\alpha_j+\frac12)\arctan(\tau\mu_j)} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau\\
&\qquad= \int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+(\tau\mu_j)^2)^{\alpha_j+\frac12}} \frac{1}{\sqrt{1+(\tau\mu_j)^2}} d\tau.
\end{aligned}$$

We have proven the following lemma.

Lemma 2.2.1. *Let F be a tempered distribution on $\mathbb{R}$ such that $\widehat{F}$ is locally integrable and such that*

$$\int_{\mathbb{R}} |\widehat{F}(\tau)| \frac{d\tau}{\sqrt{1+\tau^2}} < +\infty.$$

Then, the operator $\operatorname{Op_w}\big(F(\sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2))\big)$ has the spectral decomposition

$$\begin{aligned}
\operatorname{Op_w}\Big(F\Big(\sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2)\Big)\Big) &= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+\tau^2\mu_j^2)^{\alpha_j+1}} d\tau\,\mathbb{P}_\alpha\\
&= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}} \widehat{F}(\tau) \prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} d\tau\,\mathbb{P}_\alpha,
\end{aligned}$$

where $\mathbb{P}_\alpha$ is the rank-one orthogonal projection onto Ψ_α given by (A.1.18).

Lemma 2.2.2. *Let F be as in Lemma 2.2.2 and let us assume that all the μ_j are equal to μ (positive). Then*

$$\operatorname{Op_w}\Big(F\Big(\mu \sum_{1\le j\le n}(x_j^2+\xi_j^2)\Big)\Big) = \sum_{k\ge 0}\int_{\mathbb{R}} \widehat{F}(\tau)\frac{(1+i\tau\mu)^k}{(1-i\tau\mu)^{k+n}} d\tau \mathbb{P}_{k;n},$$

with

$$\mathbb{P}_{k;n} = \sum_{\substack{\alpha\in\mathbb{N}^n\\ |\alpha|=k}} \mathbb{P}_\alpha,$$

where $\mathbb{P}_\alpha$ is the rank-one orthogonal projection onto Ψ_α given by (A.1.18).

Proof. With all the μ_j equal to $\mu > 0$, we find

$$\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} = \prod_{1\le j\le n}\frac{(1+i\tau\mu)^{\alpha_j}}{(1-i\tau\mu)^{\alpha_j+1}} = \frac{(1+i\tau\mu)^{|\alpha|}}{(1-i\tau\mu)^{|\alpha|+n}},$$

which depends only on $|\alpha|$, so that applying the previous lemma gives

$$\Big(F\Big(\mu \sum_{1\le j\le n}(x_j^2+\xi_j^2)\Big)\Big)^w = \sum_{k\ge 0}\int_{\mathbb{R}} \widehat{F}(\tau)\frac{(1+i\tau\mu)^k}{(1-i\tau\mu)^{k+n}} d\tau \mathbb{P}_{k;n},$$

giving the sought result. ∎