Chapter 3

Conics with eccentricity smaller than 1

3.1 Indicatrix of a disc

Let us assume now that with some $a \geq 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi}, \frac{a}{2\pi}]},$$

so that

$$F(x^2 + \xi^2) = \mathbf{1}_{\{2\pi(x^2+\xi^2) \leq a\}}.$$

According to Section A.8.1, we have

$$\widehat{F}(\tau) = \frac{\sin a\tau}{\pi\tau},$$

so that (2.1.1) holds true. We find in this case,

$$\mathrm{Op_w}(F(x^2 + \xi^2)) = \sum_{k\geq 0} F_k(a)\mathbb{P}_k, \quad F_k(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} d\tau, \quad (3.1.1)$$

so that (note that $F_k(a)$ is real-valued since F is real-valued and thus the operator $\mathrm{Op_w}(F(x^2 + \xi^2))$ is self-adjoint), and for $a > 0$, using the result (A.8.2) in Section A.8.2, we obtain

$$\begin{aligned}
F_k'(a) &= \frac{1}{\pi} \int_{\mathbb{R}} \cos a\tau \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} d\tau \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ \frac{(1+i\tau)^k}{(1-i\tau)^{k+1}} + \frac{(1-i\tau)^k}{(1+i\tau)^{k+1}} \right\} d\tau \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ \frac{i^k(\tau-i)^k}{(-i)^{k+1}(\tau+i)^{k+1}} + \frac{(-i)^k(\tau+i)^k}{i^{k+1}(\tau-i)^{k+1}} \right\} d\tau \\
&= \frac{(-1)^k}{2i\pi} \int_{\mathbb{R}} e^{ia\tau} \left\{ -\frac{(\tau-i)^k}{(\tau+i)^{k+1}} + \frac{(\tau+i)^k}{(\tau-i)^{k+1}} \right\} d\tau.
\end{aligned}$$

We shall now calculate explicitly both integrals above: let $1 < R$ be given and let us consider the closed path (see Figure 3.1)

$$\gamma_R = [-R, R] \cup \underbrace{\{Re^{i\theta}\}_{0\leq\theta\leq\pi}}_{\gamma_{2;R}}.$$

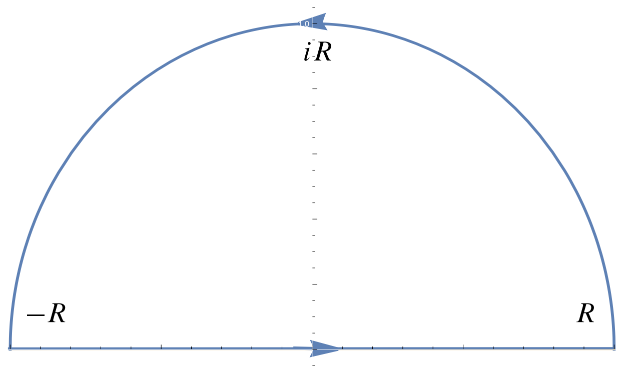

Figure 3.1. $\gamma_R = [-R, R] \cup \{Re^{i\theta}\}_{0\leq\theta\leq\pi}$.

We have

$$\frac{1}{2i\pi}\int_{\gamma_R} e^{ia\tau}\Big\{-\frac{(\tau-i)^k}{(\tau+i)^{k+1}}+\frac{(\tau+i)^k}{(\tau-i)^{k+1}}\Big\}d\tau = \mathtt{Res}\Big(e^{ia\tau}\frac{(\tau+i)^k}{(\tau-i)^{k+1}}; i\Big)$$

$$= \frac{1}{k!}(\frac{d}{d\tau})^k\{e^{ia\tau}(\tau+i)^k\}_{|\tau=i},$$

and we note that, for $a > 0$,

$$\lim_{R\to+\infty}\int_{\gamma_{2;R}} e^{ia\tau}\Big\{-\frac{(\tau-i)^k}{(\tau+i)^{k+1}}+\frac{(\tau+i)^k}{(\tau-i)^{k+1}}\Big\}d\tau = 0,$$

since for $R \geq 2$,

$$\int_0^\pi |e^{iaRe^{i\theta}}|\Big|-\frac{(Re^{i\theta}-i)^k}{(Re^{i\theta}+i)^{k+1}}+\frac{(Re^{i\theta}+i)^k}{(Re^{i\theta}-i)^{k+1}}\Big||iRe^{i\theta}|d\theta$$

$$\leq \int_0^\pi e^{-aR\sin\theta}\Big|-\frac{(e^{i\theta}-iR^{-1})^k}{(e^{i\theta}+iR^{-1})^{k+1}}+\frac{(e^{i\theta}+iR^{-1})^k}{(e^{i\theta}-iR^{-1})^{k+1}}\Big|d\theta$$

$$\leq \int_0^\pi e^{-aR\sin\theta}d\theta \sup_{0\leq\rho\leq1/2}\Big\{\frac{(1+\rho)^k}{(1-\rho)^{k+1}}+\frac{(1+\rho)^k}{(1-\rho)^{k+1}}\Big\}.$$

For $a > 0$, we obtain

$$\lim_{R\to+\infty}\int_0^\pi e^{-aR\sin\theta}d\theta = 0$$

by dominated convergence. As a result, we get

$$F_k'(a) = (-1)^k \frac{1}{k!}\left(\frac{d}{d\tau}\right)^k \{e^{ia\tau}(\tau+i)^k\}_{|\tau=i}$$

$$= (-1)^k \frac{1}{k!}\left(\frac{d}{\frac{i}{a}d\varepsilon}\right)^k \{e^{-a-\varepsilon}(i+i\frac{\varepsilon}{a}+i)^k\}_{|\varepsilon=0},$$

that is

$$F_k'(a) = \frac{(-1)^k}{k!} e^{-a}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-\varepsilon}(2a+\varepsilon)^k\}_{|\varepsilon=0}.$$

We note that F_k' belongs to $L^1(\mathbb{R}_+)$ as the product of e^{-a} by a polynomial. We have also that

$$\lim_{a\to+\infty} F_k(a) = 1 \quad \text{(see Section A.8.3)},$$

and this yields

$$F_k(a) = 1 + \int_{+\infty}^{a} F_k'(b)db = 1 - \int_a^{+\infty} \frac{(-1)^k}{k!} e^{-b}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-\varepsilon}(2b+\varepsilon)^k\}_{|\varepsilon=0} db,$$

so that

$$F_k(a) = 1 - e^{-a}P_k(a), \tag{3.1.2}$$

with

$$P_k(a) = \frac{(-1)^k}{k!}\int_0^{+\infty} e^{-t}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-2\varepsilon}(a+t+\varepsilon)^k\}_{|\varepsilon=0} dt$$

$$= \frac{(-1)^k}{k!}\int_0^{+\infty} e^{t}\left(\frac{d}{d\varepsilon}\right)^k \{e^{-2\varepsilon-2t}(a+t+\varepsilon)^k\}_{|\varepsilon=0} dt$$

$$= \frac{(-1)^k}{k!}\int_0^{+\infty} e^{t}\left(\frac{d}{dt}\right)^k \{e^{-2t}(a+t)^k\} dt. \tag{3.1.3}$$

We see that P_k is a polynomial with leading monomial $\frac{2^k a^k}{k!}$ (by a direct computation) and $P_k(0) = 1$ (since $0 = F_k(0) = 1 - P_k(0)$) and moreover, using Laguerre polynomials (see, e.g., (A.4.1) in our Section A.4), we obtain

$$P_k(a) = \frac{(-1)^k}{k!}\int_0^{+\infty} e^{-t} e^{2t+2a}\left(\frac{d}{2dt}\right)^k \{e^{-2t-2a}(2a+2t)^k\} dt$$

$$= (-1)^k \int_0^{+\infty} e^{-t} L_k(2t+2a)dt, \tag{3.1.4}$$

and this gives in particular

$$\begin{aligned}P_k'(a) &= (-1)^k \int_0^{+\infty} e^{-t} 2L_k'(2t+2a)dt \\ &= (-1)^k \left\{ [e^{-t} L_k(2t+2a)]_{t=0}^{t=+\infty} + \int_0^{+\infty} e^{-t} L_k(2t+2a)dt \right\} \\ &= (-1)^{k+1} L_k(2a) + P_k(a).\end{aligned}$$

Moreover, we have from (3.1.3), for $k \geq 1$,

$$\begin{aligned}P_k'(a) &= \frac{(-1)^k}{k!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^k \{e^{-2t} k(a+t)^{k-1}\} dt \\ &= \frac{(-1)^k}{k!} \int_0^{+\infty} e^t \frac{d}{dt} \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} dt \\ &= \frac{(-1)^k}{k!} \left\{ \left[e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} \right]_{t=0}^{t=+\infty} \right. \\ &\qquad \left. - \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} k(a+t)^{k-1}\} dt \right\} \\ &= \frac{(-1)^{k-1}}{(k-1)!} \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} (a+t)^{k-1}\}_{|t=0} \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} (a+t)^{k-1}\} dt \\ &= \frac{(-1)^{k-1}}{(k-1)!} e^{2t+2a} \left(\frac{d}{2dt}\right)^{k-1} \{e^{-2t-2a} (2a+2t)^{k-1}\}_{|t=0} \\ &\quad + \frac{(-1)^{k-1}}{(k-1)!} \int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^{k-1} \{e^{-2t} (a+t)^{k-1}\} dt \\ &= (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a),\end{aligned}$$

so that

$$\forall k \geq 1, \quad P_k'(a) = (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a) = (-1)^{k+1} L_k(2a) + P_k(a). \tag{3.1.5}$$

This implies for $N \geq 1$,

$$\sum_{1 \leq k \leq N} P_k(a) - \sum_{1 \leq k \leq N} (-1)^k L_k(2a) = \sum_{0 \leq k \leq N-1} P_k(a) + \sum_{0 \leq k \leq N-1} (-1)^k L_k(2a),$$

yielding

$$P_N(a) - \underbrace{P_0(a)}_{=1=L_0(a)} = \sum_{1\le k\le N} (-1)^k L_k(2a) + \sum_{0\le k\le N-1} (-1)^k L_k(2a),$$

and

$$P_N(a) = \sum_{0\le k\le N} (-1)^k L_k(2a) + \sum_{0\le k\le N-1} (-1)^k L_k(2a). \tag{3.1.6}$$

Note that the previous formula holds as well for $N = 0$, since $P_0 = 1 = L_0$.

Although the function $\mathbb{R}_+ \ni a \mapsto F_k(a)$ has no monotonicity properties, we prove below that $\mathbb{R}_+ \ni a \mapsto P_k(a)$ is indeed increasing. For that purpose, let us use (3.1.5), which implies

$$\begin{aligned} P_k'(a) &= (-1)^{k-1} L_{k-1}(2a) + P_{k-1}(a), \quad k \ge 1, \\ P_{k-1}(a) &= P_{k-2}(a) + (-1)^{k-2} L_{k-2}(2a) + (-1)^{k-1} L_{k-1}(2a), \quad k \ge 2, \\ P_k'(a) &= 2(-1)^{k-1} L_{k-1}(2a) + (-1)^{k-2} L_{k-2}(2a) + P_{k-2}(a), \quad k \ge 2. \end{aligned}$$

We claim that for $k \ge 1$,

$$P_k'(a) = 2 \sum_{0\le l\le k-1} (-1)^l L_l(2a). \tag{3.1.7}$$

That property holds for $k = 1$ since $P_1(a) = 1 + 2a$: we check $P_1'(a) = 2$. Moreover, we have

$$\begin{aligned} P_{k+1}'(a) &= (-1)^k L_k(2a) + P_k(a) \quad \text{(from the first equation in (3.1.5))} \\ \text{(using (3.1.6))} \quad &= (-1)^k L_k(2a) + \sum_{0\le l\le k} (-1)^l L_l(2a) + \sum_{0\le l\le k-1} (-1)^l L_l(2a) \\ &= 2 \sum_{0\le l\le k} (-1)^l L_l(2a), \end{aligned}$$

which is the sought formula. As a byproduct we find from (A.4.2)

$$\forall a \ge 0, \quad P_k'(a) \ge 0,$$

which implies that for $a \ge 0$, $P_k(a) \ge P_k(0) = 1$. We have proven the following lemma.

Lemma 3.1.1. *The polynomial*

$$P_k(a) = e^a(1 - F_k(a))$$

is increasing on $\mathbb{R}_+$*,*

$$P_k(0) = 1.$$

Let us take a look at the first P_k: we have

$$P_0(a) = 1,$$
$$P_1(a) = 1 + 2a,$$
$$P_2(a) = 1 + 2a^2,$$
$$P_3(a) = 1 + 2a - 2a^2 + \frac{4a^3}{3},$$
$$P_4(a) = 1 + 4a^2 - \frac{8a^3}{3} + \frac{2a^4}{3},$$
$$P_5(a) = 1 + 2a - 4a^2 + \frac{16a^3}{3} - 2a^4 + \frac{4a^5}{15},$$
$$P_6(a) = 1 + 6a^2 - 8a^3 + \frac{14a^4}{3} - \frac{16a^5}{15} + \frac{4a^6}{45},$$
$$P_7(a) = 1 + 2a - 6a^2 + 12a^3 - \frac{26a^4}{3} + \frac{44a^5}{15} - \frac{4a^6}{9} + \frac{8a^7}{315},$$
$$P_8(a) = 1 + 8a^2 - 16a^3 + \frac{44a^4}{3} - \frac{32a^5}{5} + \frac{64a^6}{45} - \frac{16a^7}{105} + \frac{2a^8}{315},$$
$$P_9(a) = 1 + 2a - 8a^2 + \frac{64a^3}{3} - \frac{68a^4}{3} + \frac{184a^5}{15} - \frac{32a^6}{9} + \frac{176a^7}{315} - \frac{2a^8}{45} + \frac{4a^9}{2835},$$
$$P_{10}(a) = 1 + 10a^2 - \frac{80a^3}{3} + \frac{100a^4}{3} - \frac{64a^5}{3} + \frac{344a^6}{45} - \frac{496a^7}{315} + \frac{58a^8}{315} - \frac{32a^9}{2835} + \frac{4a^{10}}{14175},$$
$$P_{11}(a) = 1 + 2a - 10a^2 + \frac{100a^3}{3} - \frac{140a^4}{3} + \frac{104a^5}{3} - \frac{664a^6}{45} + \frac{1184a^7}{315} - \frac{26a^8}{45} + \frac{148a^9}{2835} - \frac{4a^{10}}{1575} + \frac{8a^{11}}{155925},$$
$$P_{12}(a) = 1 + 12a^2 - 40a^3 + \frac{190a^4}{3} - \frac{160a^5}{3} + \frac{1184a^6}{45} - \frac{2512a^7}{315} + \frac{478a^8}{315} - \frac{512a^9}{2835} + \frac{184a^{10}}{14175} - \frac{16a^{11}}{31185} + \frac{4a^{12}}{467775}.$$

We note as well that

$$P_k(x) = \sum_{0 \le m \le k} \frac{x^m}{m!} \sum_{m \le l \le k} 2^l (-1)^{k-l} \binom{k}{l},$$

since from (3.1.3),

$$
\begin{aligned}
P_k(a) &= \frac{(-1)^k}{k!}\int_0^{+\infty} e^t \left(\frac{d}{dt}\right)^k \{e^{-2t}(a+t)^k\}dt \\
&= (-1)^k \sum_{0\le m\le k} \int_0^{+\infty} e^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}(a+t)^{k-m}dt \\
&= (-1)^k \sum_{0\le m\le k} \int_0^{+\infty} e^{-t}\frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}\sum_{0\le l\le k-m} a^l t^{k-l-m}\binom{k-m}{l}dt \\
&= (-1)^k \sum_{\substack{0\le m\le k\\ 0\le l\le k-m}} \frac{(-2)^{k-m}}{(k-m)!}\frac{k!}{(k-m)!m!}a^l(k-l-m)!\binom{k-m}{l} \\
&= \sum_{0\le l+m\le k}\frac{(-1)^m 2^{k-m}}{(k-m)!}\frac{k!}{m!}a^l\frac{1}{l!} = \sum_{0\le l\le k}\frac{a^l}{l!}\sum_{l\le m'\le k}(-1)^{k-m'}2^{m'}\binom{k}{m'},
\end{aligned}
$$

which is the sought formula.

Lemma 3.1.2. *With the polynomial P_k defined by (3.1.4), we have*

$$
\begin{cases}
P_k(a) = 2\sum_{0\le l\le k-1}(-1)^l L_l(2a) + (-1)^k L_k(2a), \\
P_k'(a) = 2\sum_{0\le l\le k-1}(-1)^l L_l(2a).
\end{cases}
$$

Proof. We may use the already proven (3.1.6), (3.1.7), but we may also prove this directly by induction on k. ■

Proposition 3.1.3. *Let F_k be given by* (3.1.2) *with P_k defined by* (3.1.3)*. We have*

$$
\begin{aligned}
&F_k(a) = 1 - e^{-a}P_k(a) \le 1 - e^{-a} = F_0(a) \quad \text{for } a \ge 0, \\
&F_k'(a) = e^{-a}\big(P_k(a) - P_k'(a)\big) = e^{-a}(-1)^k L_k(2a), \\
&F_k'(0) = (-1)^k, \quad \lim_{a\to+\infty} F_k'(a) = 0_+, \quad F_k(0) = 0, \lim_{a\to+\infty} F_k(a) = 1_-. \qquad (3.1.8)
\end{aligned}
$$

Proof. We use (3.1.2), (3.1.7), and (3.1.6) for the three first equalities, Lemma 3.1.1 for the first inequality. The fourth equality follows from $L_k(0) = 1$, while the fifth is due to the fact that the leading monomial of $(-1)^k L_k(2a)$ is $2^k a^k/k!$. The two last equalities are a consequence of the first line. ■

Remark 3.1.4. The zeroes of F_k' on the positive half-line are the positive zeroes of the Laguerre polynomial L_k divided by 2. When k is even (resp., odd) the function F_k is positive increasing (resp., negative decreasing) near 0, then oscillates with changes of monotonicity at each a such that $L_k(2a) = 0$ and when $2a$ is larger than the largest zero of L_k, the function F_k is increasing, smaller than 1, with limit 1 at infinity. Typically, we have $F_{2l}(0) = 0$, $F'_{2l}(0) = +1$,

$$0 < a_{1,2l} < a_2 < \cdots < a_{2l-1,2l} < a_{2l,2l}, \quad \text{the zeroes of } L_{2l}(2a), \tag{3.1.9}$$

F_{2l} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \le j \le 2l-1$, also at $b_{2l} > a_{2l}$: $2l+1$ zeroes with a positive (resp., negative) derivative at $b_0, b_2, \dots, b_{2l}$ (resp., at $b_1, b_3, \dots, b_{2l-1}$). Moreover, we have $F_{2l+1}(0) = 0$, $F'_{2l+1}(0) = -1$,

$$0 < a_{1,2l+1} < a_{2,2l+1} < \cdots < a_{2l,2l+1} < a_{2l+1,2l+1}, \quad \text{the zeroes of } L_{2l+1}(2a), \tag{3.1.10}$$

F_{2l+1} vanishes simply at $b_0 = 0$ and at $b_j \in (a_j, a_{j+1})$ for $1 \le j \le 2l$, also at $b_{2l+1} > a_{2l+1}$: $2l+2$ zeroes with a positive (resp., negative) derivative at $b_1, b_3, \dots, b_{2l+1}$ (resp., at $b_0, b_2, \dots, b_{2l}$).

We note as well that a consequence of the previous remark is that

$$\min_{a \ge 0} F_{2l}(a) = \min_{1 \le j \le l} \{F_{2l}(a_{2j,2l})\},$$
$$\min_{a \ge 0} F_{2l+1}(a) = \min_{0 \le j \le l} \{F_{2l+1}(a_{2j+1,2l+1})\},$$

where $(a_{p,k})_{1 \le p \le k}$ are defined in (3.1.9), (3.1.10).

Theorem 3.1.5. *Let $a \ge 0$ be given and let*

$$D_a = \left\{(x, \xi) \in \mathbb{R}^2, x^2 + \xi^2 \le \frac{a}{2\pi}\right\}. \tag{3.1.11}$$

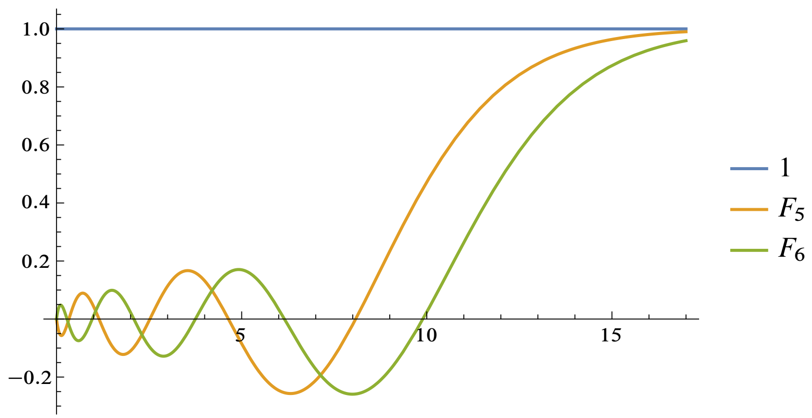

Figure 3.2. Functions F_5, F_6.

Then, we have

$$\mathrm{Op_w}(\mathbf{1}_{D_a}) = \sum_{k\ge 0} F_k(a)\mathbb{P}_k \le 1 - e^{-a}.$$

Proof. An immediate consequence of (3.1.1), (3.1.8). Note that the inequality in the above theorem is due to P. Flandrin in [13] (see also the related references [20], [14]). ∎

Curves. Let us display some curves of $\mathbb{R}_+ \ni a \mapsto F_k(a) = 1 - e^{-a} P_k(a)$.

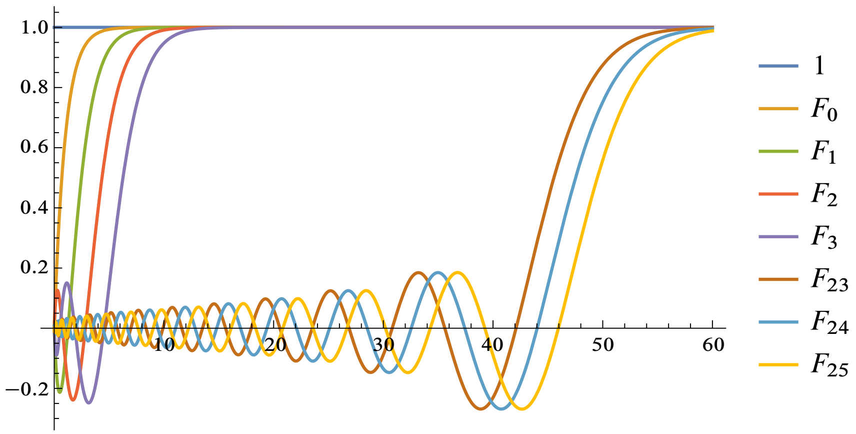

Figure 3.3. Functions F_k.

3.2 Indicatrix of a Euclidean ball

The following result displays an explicit spectral decomposition on the Hermite basis for the Weyl quantization of the characteristic function of Euclidean balls.

Theorem 3.2.1. *Let $a \ge 0$ be given and let*

$$\mathcal{Q}_{a,n} = \mathrm{Op_w}(\mathbf{1}\{2\pi(|x|^2 + |\xi|^2) \le a\}),$$

be the Weyl quantization of the characteristic function of the Euclidean ball of $\mathbb{R}^{2n}$ with center 0 and radius $\sqrt{a/(2\pi)}$. Then, we have

$$\mathcal{Q}_{a,n} = \sum_{k\ge 0} F_{k;n}(a)\mathbb{P}_{k;n},$$

with $\mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.1.18)), with $|\alpha| = \sum_{1\le j\le n} \alpha_j = k$ and

$$F_{k;n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau.$$

The spectral decomposition of the previous theorem allows a simple recovery of the result of the article [39] by E. Lieb and Y. Ostrover.

Theorem 3.2.2. *Let $a \ge 0, \mathcal{Q}_{a,n}, F_{k;n}$ be defined above. Then, we have*

$$F_{k;n}(a) \le 1 - \frac{1}{\Gamma(n)} \int_a^{+\infty} e^{-t} t^{n-1} dt = 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{3.2.1}$$

and thus we have

$$\mathcal{Q}_{a,n} \le 1 - \frac{\Gamma(n,a)}{\Gamma(n)}, \tag{3.2.2}$$

where the incomplete Gamma function $\Gamma(\cdot,\cdot)$ is defined in (A.8.3).

Proof of Theorems 3.2.1 and 3.2.2. We use the results of (the previous) Section 3.1: Let us assume now that with some $a \ge 0$,

$$F = \mathbf{1}_{[-\frac{a}{2\pi}, \frac{a}{2\pi}]},$$

so that

$$F(|x|^2 + |\xi|^2) = \mathbf{1}\{2\pi(|x|^2 + |\xi|^2) \le a\}.$$

According to Section A.8.1, we have $\widehat{F}(\tau) = \frac{\sin a\tau}{\pi\tau}$, so that (2.1.1) holds true. We find in this case, following the results of Lemma 2.2.2,

$$\operatorname{Op_w}\left(F(|x|^2 + |\xi|^2)\right) = \sum_{k\ge 0} F_{k;n}(a)\mathbb{P}_{k;n}, \quad \mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha,$$

$$F_{k;n}(a) = \int_{\mathbb{R}} \frac{\sin a\tau}{\pi\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau, \tag{3.2.3}$$

where $\mathbb{P}_\alpha$ is the orthogonal projection onto Ψ_α (defined in (A.1.18)), with

$$|\alpha| = \sum_{1\le j\le n} \alpha_j = k.$$

This completes the proof of Theorem 3.2.1.

We postpone the proof of Theorem 3.2.2 until after settling a couple of lemmas.

Lemma 3.2.3. *Let $(k,n) \in \mathbb{N} \times \mathbb{N}^*$. With $F_{k;n}(a)$ given by (3.2.3), we have*

$F_{k;n}(a) = 1 - e^{-a} P_{k,n}(a)$, *where $P_{k;n}$ is the polynomial*

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)!} \int_0^{+\infty} e^{-t}(t+a)^{n-1} \Big\{ e^s \Big(\frac{d}{ds}\Big)^{n+k-1} \big[s^k e^{-s}\big] \Big\}_{|s=2t+2a} dt, \tag{3.2.4}$$

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}} \int_0^{+\infty} (t+a)^{n-1} e^t \Big(\frac{d}{dt}\Big)^{n+k-1} \{(t+a)^k e^{-2t}\} dt.$$

Proof of Lemma 3.2.3. The lemma holds true for $n = 1$ from Proposition 3.1.3. We have for $a > 0$, $n \geq 2$,

$$F'_{k;n}(a) = \frac{1}{\pi}\int_{\mathbb{R}} \cos a\tau \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau$$
$$= \frac{1}{2\pi}\int_{\mathbb{R}} e^{ia\tau} \frac{(1+i\tau)^k}{(1-i\tau)^{k+n}} d\tau + \frac{1}{2\pi}\int_{\mathbb{R}} e^{ia\tau} \frac{(1-i\tau)^k}{(1+i\tau)^{k+n}} d\tau$$
$$= \frac{i}{2i\pi}\int_{\mathbb{R}} e^{ia\tau} \frac{i^k(\tau-i)^k}{(-i)^{k+n}(\tau+i)^{k+n}} d\tau + \frac{i}{2i\pi}\int_{\mathbb{R}} e^{ia\tau} \frac{(-i)^k(\tau+i)^k}{i^{k+n}(\tau-i)^{k+n}} d\tau,$$

so that

$$F'_{k;n}(a) = i^{1-n}(-1)^k \mathtt{Res}\left(e^{ia\tau}\frac{(\tau+i)^k}{(\tau-i)^{k+n}}; i\right)$$
$$= \frac{i^{1-n}(-1)^k}{(k+n-1)!}\left(\frac{d}{d\tau}\right)^{k+n-1}\left\{e^{ia\tau}(\tau+i)^k\right\}_{|\tau=i},$$

and thus

$$F'_{k;n}(a) = \frac{i^{1-n}(-1)^k}{(k+n-1)!}\left(\frac{d}{\frac{i}{a}d\varepsilon}\right)^{k+n-1}\left\{e^{-a-\varepsilon}\left(i+i\frac{\varepsilon}{a}+i\right)^k\right\}_{|\varepsilon=0}$$
$$= \frac{i^{1-n}(-1)^k a^{n-1}}{i^{n-1}(k+n-1)!}\left(\frac{d}{d\varepsilon}\right)^{k+n-1}\left\{e^{-a-\varepsilon}(2a+\varepsilon)^k\right\}_{|\varepsilon=0}$$
$$= e^a\frac{(-1)^{k+n-1}a^{n-1}}{(k+n-1)!}\left(\frac{d}{2d\varepsilon}\right)^{k+n-1}\left\{e^{-2a-2\varepsilon}(2a+2\varepsilon)^k\right\}_{|\varepsilon=0},$$

that is

$$F'_{k;n}(t) = \frac{(-1)^{k+n-1}}{(k+n-1)!}e^t t^{n-1}\left(\frac{d}{ds}\right)^{k+n-1}\left\{e^{-s}s^k\right\}_{|s=2t}$$
$$= \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}e^t t^{n-1}\left(\frac{d}{dt}\right)^{k+n-1}\left\{e^{-2t}t^k\right\}.$$

We have also that $\lim_{a\to+\infty} F_{k;n}(a) = 1$ (following the arguments of Section 3.1) and this yields

$$F_{k;n}(a) = 1 - \frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}\int_a^{+\infty} e^t t^{n-1}\left(\frac{d}{dt}\right)^{k+n-1}\left\{e^{-2t}t^k\right\}dt$$
$$= 1 - e^{-a}\frac{(-1)^{k+n-1}}{(k+n-1)!2^{n-1}}$$
$$\times \int_0^{+\infty}(t+a)^{n-1}e^t\left(\frac{d}{dt}\right)^{k+n-1}\left\{e^{-2t}(t+a)^k\right\}dt,$$

concluding the proof of the lemma. ∎

Let us go back to formula (3.2.4), written as

$$\frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}\left\{\frac{(2t+2a)^{n-1}}{(k+n-1)!}\Big(\frac{d}{d\varepsilon}-1\Big)^{n+k-1}\big[(\varepsilon+2t+2a)^k\big]\right\}_{|\varepsilon=0} dt$$

$$= P_{k;n}(a) = \frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t} L_{k+n-1}^{1-n}(2t+2a)dt, \tag{3.2.5}$$

where the generalized Laguerre polynomial L_{k+n-1}^{1-n} is defined by (A.4.5) (note that $1-n+k+n-1=k$ which is not negative).

Lemma 3.2.4. *Let $n \in \mathbb{N}^*$, $k \in \mathbb{N}$ and let $P_{k;n}$ be the polynomial defined in Lemma 3.2.3 (and thus in (3.2.5)). Then, we have*

$$P_{k;n}(X) - P'_{k;n}(X) = \frac{(-1)^{k+n-1}}{2^{n-1}} L_{k+n-1}^{1-n}(2X), \quad P_{k;n}(0) = 1, \tag{3.2.6}$$

$$\textit{for } n \geq 2, \quad P'_{k;n} = P_{k;n-1}. \tag{3.2.7}$$

Proof. From (3.2.5), we find

$$P'_{k;n}(a) = \frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}2(L_{k+n-1}^{1-n})'(2t+2a)dt$$

$$= \frac{(-1)^{k+n-1}}{2^{n-1}}\left\{\big[e^{-t}(L_{k+n-1}^{1-n})(2t+2a)\big]_{t=0}^{t=+\infty} + \int_0^{+\infty} e^{-t}L_{k+n-1}^{1-n}(2t+2a)dt\right\}$$

$$= \frac{(-1)^{k+n}}{2^{n-1}} L_{k+n-1}^{1-n}(2a) + P_{k;n}(a),$$

and since $0 = F_{k;n}(0) = 1 - P_{k;n}(0)$, this proves (3.2.6). Using now (3.2.5) and (A.4.7), we find that

$$\begin{aligned}
P_{k;n}(a) &= \frac{(-1)^{k+n}}{2^{n-1}}\int_0^{+\infty}\frac{d}{dt}\{e^{-t}\}L_{k+n-1}^{1-n}(2t+2a)dt \\
&= \frac{(-1)^{k+n}}{2^{n-1}}\Big\{\big[e^{-t}L_{k+n-1}^{1-n}(2t+2a)\big]_{t=0}^{t=+\infty} \\
&\qquad - \int_0^{+\infty} e^{-t}2(L_{k+n-1}^{1-n})'(2t+2a)dt\Big\} \\
&= \frac{(-1)^{k+n}}{2^{n-1}}\left\{-L_{k+n-1}^{1-n}(2a) + \int_0^{+\infty} e^{-t}2(L_{k+n-2}^{2-n})(2t+2a)dt\right\} \\
&= \underbrace{\frac{(-1)^{k+n-1}}{2^{n-1}}L_{k+n-1}^{1-n}(2a)}_{\substack{P_{k;n}(a)-P'_{k;n}(a) \\ \text{from (3.2.6)}}} + \underbrace{\frac{(-1)^{k+n-2}}{2^{n-2}}\int_0^{+\infty} e^{-t}L_{k+n-2}^{2-n}(2t+2a)dt}_{\substack{P_{k;n-1}(a) \\ \text{from (3.2.5)}}},
\end{aligned}$$

so that for $n \geq 2$, $k \in \mathbb{N}$, we obtain (3.2.7), completing the proof of the lemma. ■

Lemma 3.2.5. *Let $k, n, P_{k;n}$ be as in Lemma 3.2.4. Then, we have*

$$\forall j \in [\![0, n-1]\!], \qquad \left(\frac{d}{dX}\right)^j P_{k;n} = P_{k;n-j}. \tag{3.2.8}$$

Moreover, for all $a \ge 0$ and all $k \in \mathbb{N}$,

$$P_{k;n}(a) \ge P_{0;n}(a) = \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}(t+a)^{n-1}dt = e^a \frac{\Gamma(n,a)}{\Gamma(n)}. \tag{3.2.9}$$

Proof. Formula (3.2.8) follows immediately by induction from (3.2.7) since the latter is proving (3.2.8) for $j = 1, n \ge 2, k \in \mathbb{N}$. Assuming that (3.2.8) holds true for some $1 \le j < n$, all $k \in \mathbb{N}$, we have $P_{k;n}^{(j)} = P_{k,n-j}$ and if $j + 1 < n$, we obtain from (3.2.7) that

$$P_{k,n-j-1} = P'_{k,n-j} = P_{k;n}^{(j+1)},$$

proving (3.2.8). The property (3.2.9) holds true for $n=1$. From (3.2.7), $P_{k;n+1}(0)=1$, we find that $P_{k;n+1}(a) = 1 + \int_0^a P_{k;n}(s)ds$ and assuming that (3.2.9) holds true for n, we obtain for $a \ge 0$,

$$\begin{aligned} P_{k;n+1}(a) &\ge 1 + \int_0^a \frac{1}{(n-1)!}\int_0^{+\infty} e^{-t}(t+s)^{n-1}dtds \\ &= 1 + \int_0^{+\infty} e^{-t}\left[\frac{(t+s)^n}{n!}\right]_{s=0}^{s=a} dt \\ &= 1 + \frac{1}{n!}\int_0^{+\infty} e^{-t}((t+a)^n - t^n)dt = \frac{1}{n!}\int_0^{+\infty} e^{-t}(t+a)^n dt, \end{aligned}$$

completing the proof of the lemma. ∎

We can now prove Theorem 3.2.2, since

$$F_{k;n}(a) = 1 - e^{-a}P_{k;n}(a)$$

the estimate (3.2.8) implies indeed

$$F_{k;n}(a) \le \frac{\Gamma(n,a)}{\Gamma(n)},$$

concluding the proof. ∎

Remark 3.2.6. Our methods of proof in one and more dimensions are quite similar.

- Using Mehler's formula, we diagonalise in the Hermite basis the quantization of the indicatrix of the Euclidean ball

$$D_{a;n} = \left\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi\left(|x|^2 + |\xi|^2\right) \le a\right\}.$$

- Once we get the diagonalisation

$$\operatorname{Op_w}(\mathbf{1}_{D_{a;n}}) = \sum_{k\in\mathbb{N}} F_{k;n}(a)\mathbb{P}_{k;n},$$

  we study explicitly the functions $F_{k;n}$ and prove that

$$F_{k;n}(a) = 1 - e^{-a}P_{k;n}(a),$$

  where $P_{k;n}$ is a polynomial given in terms of the generalized Laguerre polynomials

$$P_{k;n}(a) = \frac{(-1)^{k+n-1}}{2^{n-1}}\int_0^{+\infty} e^{-t}L_{k+n-1}^{1-n}(2t+2a)dt.$$

- Following the Flandrin paper [13], we use Feldheim inequality in [12] to tackle the case $n = 1$, and next we use an induction on n, made possible by the relationship between the standard and the generalized Laguerre polynomials. It is interesting to note that the functions $F_{k;n}$ have no monotonicity properties: with value 0 at 0, they have an oscillatory behavior for $a \le a_{k,n}$ and for a large enough, increase monotonically to 1 (see for instance Figures 3.2 and 3.3 in the 1D case); the inequality

$$F_{k;n}(a) \le 1 - e^{-a}$$

  holds true for all $a \ge 0$ in all dimensions. On the other hand, the polynomials $P_{k;n}$ are increasing and larger than 1 on the positive half-line.

The key ingredients are thus Mehler's formula and Feldheim inequality, but it should be pointed out that the arguments proving Feldheim inequality (formula (6.8) and Theorem 12) in the R. Askey and G. Gasper's article [2] are also based upon a version of Mehler's formula which appears thus as the basic result for our investigation. The paper [39] by E. Lieb and Y. Ostrover has a slightly different line of arguments and takes advantage of symmetry properties of the sphere. We shall go back to this in a situation where the symmetry is absent, such as for some general ellipsoids.

3.3 Ellipsoids in the phase space

3.3.1 Preliminaries

We provide below a couple of remarks on ellipsoids in higher dimensions. Let us first recall a particular case of in [24, Theorem 21.5.3].

Theorem 3.3.1 (Symplectic reduction of quadratic forms). *Let q be a positive-definite quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the canonical symplectic form* (1.2.13).

Then, there exists S in the symplectic group $\mathrm{Sp}(n, \mathbb{R})$ *of* $\mathbb{R}^{2n}$ *and* $\mu_1, \dots, \mu_n$ *positive such that for all* $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$q(SX) = \sum_{1 \le j \le n} \mu_j (x_j^2 + \xi_j^2). \tag{3.3.1}$$

Note that an interesting consequence of this theorem is that, considering a general ellipsoid in $\mathbb{R}^{2n}$ (with center of gravity at 0),

$$\mathbb{E} = \{X \in \mathbb{R}^{2n}, q(X) \le 1\},$$

where q is a positive definite quadratic form, we are able to find symplectic coordinates such that q is given by (3.3.1). Note however that no further simplification is possible and that the μ_j are symplectic invariants of $\mathbb{E}$. Note that the volume of $\mathbb{E}$ is given by

$$|\mathbb{E}|_{2n} = \frac{\pi^n}{n!\mu_1 \cdots \mu_n}.$$

3.3.2 Spectral decomposition for the quantization of the characteristic function of the ellipsoid

Let $a_1, \dots, a_n$ be positive numbers. We consider the ellipsoid $E(a_1, \dots, a_n)$ given by

$$E(a) = E(a_1, \dots, a_n) = \left\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n, 2\pi \sum_{1 \le j \le n} \frac{x_j^2 + \xi_j^2}{a_j} \le 1\right\}. \tag{3.3.2}$$

We define on $\mathbb{R}^n$ the function

$$F(X_1, \dots, X_n) = \mathbf{1}_{[-1,1]}\left(\frac{2\pi}{a_1} X_1 + \cdots + \frac{2\pi}{a_n} X_n\right).$$

Theorem 3.3.2. *Let* $a = (a_j)_{1 \le j \le n}$ *be positive numbers and let* $E(a)$ *be defined by* (3.3.2). *Then, we have*

$$\mathrm{Op_w}(\mathbf{1}_{E(a)}) = \sum_{\alpha \in \mathbb{N}^n} F_\alpha(a) \mathbb{P}_\alpha,$$

where $\mathbb{P}_\alpha$ *is defined in* (A.1.19) *and* $F_\alpha(a) = 1 - K_\alpha(a)$, *with*

$$K_\alpha(a) = \int_{\substack{\sum t_j / a_j \ge 1 \\ t_j \ge 0}} e^{-(t_1 + \cdots + t_n)} \prod_{1 \le j \le n} (-1)^{\alpha_j} L_{\alpha_j}(2t_j) dt, \tag{3.3.3}$$

Remark 3.3.3. For all $\alpha \in \mathbb{N}^n$, the functions F_α, K_α are holomorphic on

$$\mathcal{U} = \{a \in \mathbb{C}^n, \forall j \in [\![1, n]\!], \operatorname{Re} a_j > 0\}. \tag{3.3.4}$$

Indeed, let K be a compact subset of $\mathcal{U}$; there exists $\rho > 0$ such that

$$\forall (a_1, \ldots, a_n) \in K, \qquad \min_{1\le j\le n} \operatorname{Re} a_j \ge \rho,$$

and as a result for $a \in K$, we have for $s \in \mathbb{R}^n_+$

$$\left| e^{-(a_1 s_1 + \cdots + a_n s_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) \right| \le e^{-\rho(s_1+\cdots+s_n)} C_{K,\alpha}(1+|s|)^{|\alpha|},$$

so that

$$\begin{aligned} &\int_{\substack{\sum s_j \ge 1\\ s_j\ge 0}} \sup_{a\in K} \left| e^{-(a_1 s_1 + \cdots + a_n s_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) \right| ds \\ &\quad\le \int_{\substack{\sum s_j \ge 1\\ s_j\ge 0}} e^{-\rho(s_1+\cdots+s_n)} C_{K,\alpha}(1+|s|)^{|\alpha|} ds \\ &\quad\le C_{K,\alpha} \int_{\mathbb{R}^n} e^{-\rho\sigma_n |s|}(1+|s|)^{|\alpha|} ds < +\infty. \end{aligned}$$

Since we have

$$K_\alpha(a) = \int_{\substack{\sum s_j \ge 1\\ s_j\ge 0}} e^{-(a_1 s_1 + \cdots + a_n s_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j) ds\, a_1 \cdots a_n,$$

this proves the sought holomorphy.

Proof of Theorem 3.3.2. We have

$$\begin{aligned} \operatorname{Op_w}(\mathbf{1}_{E(a)}) &= \operatorname{Op_w}\left(F(x_1^2+\xi_1^2, \ldots, x_n^2+\xi_n^2)\right) \\ &= \int_{\mathbb{R}^n} \widehat{F}(\tau) \operatorname{Op_w}\left(e^{2i\pi \sum_j \tau_j (x_j^2+\xi_j^2)}\right) d\tau \\ &= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n} \widehat{F}(\tau) \prod_{1\le j\le n} \frac{(1+i\tau_j)^{2\alpha_j+1}}{(1+\tau_j^2)^{\alpha_j+1}} d\tau \mathbb{P}_\alpha \\ &= \sum_{\alpha\in\mathbb{N}^n} \int_{\mathbb{R}^n} \widehat{F}(\tau) \prod_{1\le j\le n} \frac{(1+i\tau_j)^{\alpha_j}}{(1-i\tau_j)^{\alpha_j+1}} d\tau \mathbb{P}_\alpha, \end{aligned}$$

where $\mathbb{P}_\alpha$ is defined in (A.1.19). On the other hand, we have

$$\begin{aligned} \widehat{F}(\tau) &= \int e^{-2i\pi\tau\cdot x} \mathbf{1}_{[-1,1]}\left(\frac{2\pi}{a_1}x_1 + \cdots + \frac{2\pi}{a_n}x_n\right) dx_1 \cdots dx_n \\ &= a_1 \cdots a_n (2\pi)^{-n} \int e^{-i\sum_j \tau_j a_j y_j} \mathbf{1}_{[-1,1]}\left(\sum y_j\right) dy, \end{aligned}$$

so that, with M_k defined in (A.4.3), using (A.4.4), we get

$$\begin{aligned}
&\mathrm{Op_w}(\mathbf{1}_{E(a)})\\
&= a_1\cdots a_n \sum_{\alpha\in\mathbb{N}^n} \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{-i2\pi\sum_j \tau_j a_j y_j}\mathbf{1}_{[-1,1]}\Big(\sum y_j\Big)dy\\
&\quad\times \prod_{1\le j\le n}\frac{(1+i2\pi\tau_j)^{\alpha_j}}{(1-i2\pi\tau_j)^{\alpha_j+1}}d\tau\mathbb{P}_\alpha\\
&= a_1\cdots a_n \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i2\pi\sum_j \tau_j a_j y_j}\mathbf{1}_{[-1,1]}\Big(\sum y_j\Big)dy\prod_{1\le j\le n}\overline{\widehat{G}_{\alpha_j}(\tau_j)}d\tau\mathbb{P}_\alpha\\
&= a_1\cdots a_n \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^n}\mathbf{1}_{[-1,1]}\Big(\sum y_j\Big)\prod_{1\le j\le n}G_{\alpha_j}(a_jy_j)dy\mathbb{P}_\alpha\\
&= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}^n}\mathbf{1}_{[-1,1]}\Big(\sum t_j/a_j\Big)\prod_{1\le j\le n}(-1)^{\alpha_j}H(t_j)e^{-t_j}L_{\alpha_j}(2t_j)dt\mathbb{P}_\alpha,
\end{aligned}$$

with

$$\begin{aligned}
F_\alpha(a) &= \int_{\mathbb{R}^n}\Big(1-\mathbf{1}_{[1,+\infty]}\Big(\sum t_j/a_j\Big)\Big)\prod_{1\le j\le n}(-1)^{\alpha_j}H(t_j)e^{-t_j}L_{\alpha_j}(2t_j)dt\\
&= 1-\int_{\mathbb{R}^n}\mathbf{1}_{[1,+\infty]}\Big(\sum t_j/a_j\Big)\prod_{1\le j\le n}(-1)^{\alpha_j}H(t_j)e^{-t_j}L_{\alpha_j}(2t_j)dt, \quad (3.3.5)
\end{aligned}$$

where we have used that

$$P_{k;1}(0) = 1 \quad \text{(cf. Lemma 3.1.1)},$$

so that setting

$$K_\alpha(a) = \int_{\substack{\sum t_j/a_j\ge 1\\ t_j\ge 0}} e^{-(t_1+\cdots+t_n)}\prod_{1\le j\le n}(-1)^{\alpha_j}L_{\alpha_j}(2t_j)dt,$$

we have $F_\alpha(a) = 1-K_\alpha(a)$, concluding the proof of the theorem. ∎

Remark 3.3.4. We have from (3.3.5)

$$F_\alpha(a_1,\dots,a_n) = \int_{\mathbb{R}^n}\mathbf{1}_{[0,1]}\Bigg(\sum_{1\le j\le n} s_j\Bigg)\prod_{1\le j\le n}(-1)^{\alpha_j}H(s_j)e^{-a_js_j}L_{\alpha_j}(2a_js_j)a_jds,$$

and since the set

$$\Big\{s\in\mathbb{R}_+^n, \sum_{1\le j\le n} s_j\le 1\Big\}$$

is compact, we obtain that F_α is an entire function, as well as K_α which is indeed given by (3.3.3) on the open subset $\mathcal{U}$ defined in (3.3.4).

Lemma 3.3.5. *With the notations of Theorem 3.3.2, we have with* $\mu_j = 1/a_j$,

$$F_\alpha(a) = \Bigl(\prod_{1\le j\le n} a_j\Bigr)\int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Bigl(\prod_{1\le j\le n}\frac{(a_j+i\tau)^{\alpha_j}}{(a_j-i\tau)^{\alpha_j+1}}\Bigr)d\tau$$
$$= \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Bigl(\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Bigr)d\tau. \tag{3.3.6}$$

Proof. Mehler's formula implies in one dimension that

$$\operatorname{Op_w}(e^{2\pi i\tau(x^2+\xi^2)}) = (1+\tau^2)^{-1/2}\exp\bigl[2\pi i(\arctan\tau)(x^2+D_x^2)\bigr],$$

and a simple tensorisation gives

$$\operatorname{Op_w}(e^{2\pi i\tau\sum_j \mu_j(x_j^2+\xi_j^2)}) = \prod_j (1+(\tau\mu_j)^2)^{-1/2}\exp\Bigl[2\pi i\sum_j(\arctan(\tau\mu_j))(x_j^2+D_{x_j}^2)\Bigr],$$

so that we have

$$\operatorname{Op_w}\Bigl(F\bigl(\sum_j \mu_j(x_j^2+\xi_j^2)\bigr)\Bigr)$$
$$= \int_{\mathbb{R}} \widehat{F}(\tau)\operatorname{Op_w}\Bigl(e^{2\pi i\tau\sum_j \mu_j(x_j^2+\xi_j^2)}\Bigr)d\tau$$
$$= \int_{\mathbb{R}} \widehat{F}(\tau)\prod_j(1+(\tau\mu_j)^2)^{-1/2}\exp\Bigl[2\pi i\sum_j(\arctan(\tau\mu_j))(x_j^2+D_{x_j}^2)\Bigr]d\tau$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Bigl(\prod_j(1+(\tau\mu_j)^2)^{-1/2}\exp\Bigl[2i(\arctan(\tau\mu_j))\Bigl(\alpha_j+\frac12\Bigr)\Bigr]\Bigr)d\tau\mathbb{P}_\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Bigl(\prod_j(1+(\tau\mu_j)^2)^{-1/2}\frac{(1+i\tau\mu_j)^{2\alpha_j+1}}{(1+(\tau\mu_j)^2)^{\alpha_j+\frac12}}\Bigr)d\tau\mathbb{P}_\alpha$$
$$= \sum_{\alpha\in\mathbb{N}^n}\int_{\mathbb{R}} \widehat{F}(\tau)\Bigl(\prod_{1\le j\le n}\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Bigr)d\tau\mathbb{P}_\alpha,$$

and for $F(t) = \mathbf{1}_{[-1,1]}(2\pi t)$, we find $\widehat{F}(\tau) = \frac{\sin\tau}{\pi\tau}$ and the sought result. ∎

Remark 3.3.6. It is also possible to provide a direct checking for the above lemma, since with the notations (A.4.3), (A.4.4), we have

$$\frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} = \widecheck{\widehat{G_{\alpha_j}}}(\tau\mu_j/(2\pi)),$$

and thus

$$\begin{aligned}F_\alpha(a) &= \int_{\mathbb{R}} \widehat{F}(\tau) \prod_j \widecheck{G_{\alpha_j}}(\tau\mu_j/(2\pi))d\tau \\ &= \int_{\mathbb{R}} \widehat{F}(\tau) \int_{\mathbb{R}^n} \prod_j (-1)^{\alpha_j} L_{\alpha_j}(2t_j)H(t_j)e^{-t_j} e^{2\pi i\tau\mu_j t_j/(2\pi)} dt d\tau \\ &= \int_{\mathbb{R}^n} \prod_j (-1)^{\alpha_j} L_{\alpha_j}(2t_j)H(t_j)e^{-t_j} F\Big(\sum_j \mu_j t_j/2\pi\Big)dt.\end{aligned}$$

Now, since we have

$$F\Big(\sum_j \mu_j t_j/2\pi\Big) = \mathbf{1}_{[-1,1]}\Big(\sum_j \mu_j t_j\Big),$$

this fits with the expression of F_α in Theorem 3.3.2.

Remark 3.3.7. Another interesting remark is that the expression (3.3.6) depends obviously only on $|\alpha|$ and $a = a_1 = \cdots = a_n$ in the case where all the a_j are equal: indeed, in that case, we have with $\mu = 1/a$,

$$\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} = \frac{(1+i\tau\mu)^{|\alpha|}}{(1-i\tau\mu)^{|\alpha|+n}},$$

and this gives another (*a posteriori*) justification of our calculations in the isotropic case of Section 3.2. On the other hand, we get also the identity

$$F_{0_{\mathbb{N}^n}}(a_1,\dots,a_n) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau} \operatorname{Re}\Big(\prod_{1\le j\le n}(1-i\tau\mu_j)^{-1}\Big)d\tau,$$

where the explicit expression (3.3.7) is given for the left-hand side.

Lemma 3.3.8. *With the notations of Theorem 3.3.2, the function $K_{\alpha_1,\dots,\alpha_n}(a_1,\dots, a_n)$ is symmetric in the variables $(\alpha_1,a_1,\dots,\alpha_n,a_n)$, i.e., for a permutation π of $\{1,\dots,n\}$, we have*

$$K_{\alpha_{\pi(1)},\dots,\alpha_{\pi(n)}}(a_{\pi(1)},\dots,a_{\pi(n)}) = K_{\alpha_1,\dots,\alpha_n}(a_1,\dots,a_n).$$

Proof. Formula (3.3.3) yields

$$K_\alpha(a) = \int_{\substack{\sum s_j\ge 1\\ s_j\ge 0}} \prod_{1\le j\le n} \big(e^{-a_j s_j} a_j (-1)^{\alpha_j} L_{\alpha_j}(2a_j s_j)\big)ds,$$

and the domain of integration is invariant by permutation of the variables, entailing the sought result. ∎

Lemma 3.3.9. *With the notations of Theorem 3.3.2, we have*

$$
\begin{aligned}
&K_{\alpha_1,\dots,\alpha_n}(a_1,\dots,a_n) = e^{-a_n} P_{\alpha_n}(a_n) \\
&\quad + \int_0^{a_n} (-1)^{\alpha_n} L_{\alpha_n}(2t_n) e^{-t_n} K_{\alpha_1,\dots,\alpha_{n-1}}\big(a_1(1-t_n/a_n),\dots,a_{n-1}(1-t_n/a_n)\big) dt_n \\
&= e^{-a_n} P_{\alpha_n}(a_n) \\
&\quad + \int_0^1 (-1)^{\alpha_n} L_{\alpha_n}(2a_n\theta) e^{-\theta a_n} K_{\alpha_1,\dots,\alpha_{n-1}}\big(a_1(1-\theta),\dots,a_{n-1}(1-\theta)\big) d\theta a_n.
\end{aligned}
$$

Proof. The domain of integration is the disjoint union

$$
\left\{\frac{t_1}{a_1}+\dots+\frac{t_{n-1}}{a_{n-1}} \geq 1-\frac{t_n}{a_n}, t_j \geq 0, 0 \leq \frac{t_n}{a_n} \leq 1\right\} \sqcup \left\{\frac{t_n}{a_n} > 1, t_j \geq 0, 1 \leq j \leq n-1\right\},
$$

so that

$$
\begin{aligned}
&K_{\alpha_1,\dots,\alpha_n}(a_1,\dots,a_n) = e^{-a_n} P_{\alpha_n}(a_n) \\
&\quad + \int_0^{a_n} (-1)^{\alpha_n} L_{\alpha_n}(2t_n) e^{-t_n} K_{\alpha_1,\dots,\alpha_{n-1}}\big(a_1(1-t_n/a_n),\dots,a_{n-1}(1-t_n/a_n)\big) dt_n \\
&= e^{-a_n} P_{\alpha_n}(a_n) \\
&\quad + \int_0^1 (-1)^{\alpha_n} L_{\alpha_n}(2a_n\theta) e^{-\theta a_n} K_{\alpha_1,\dots,\alpha_{n-1}}\big(a_1(1-\theta),\dots,a_{n-1}(1-\theta)\big) d\theta a_n,
\end{aligned}
$$

which is the sought result. ∎

Lemma 3.3.10. *With the notations of Theorem 3.3.2, we have, assuming that the $(a_j)_{1\leq j\leq n}$ are positive distinct numbers,*

$$
K_{0,\dots,0}(a_1,\dots,a_n) = \sum_{1\leq j\leq n} e^{-a_j} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)}. \tag{3.3.7}
$$

Proof. The latter formula is true for $n=1$ since we have

$$
K_0(a_1) = e^{-a_1}.
$$

We have also

$$
\begin{aligned}
K_{0\in\mathbb{N}^n}(a_1,\dots,a_n) &= e^{-a_n} + a_n \int_0^1 e^{-\theta a_n} K_{0\in\mathbb{N}^{n-1}}\big(a_1(1-\theta),\dots,a_{n-1}(1-\theta)\big) d\theta \\
&= e^{-a_n} + a_n \int_0^1 e^{-\theta a_n} \sum_{1\leq j\leq n-1} e^{-a_j(1-\theta)} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)} d\theta \\
&= e^{-a_n} + a_n \sum_{1\leq j\leq n-1} \frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)} \int_0^1 e^{-\theta a_n} e^{-a_j(1-\theta)} d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} e^{-a_j} \int_0^1 e^{\theta(a_j - a_n)} d\theta \\
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} e^{-a_j} \frac{e^{a_j - a_n} - 1}{a_j - a_n} \\
&= e^{-a_n} + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{e^{-a_n} - e^{-a_j}}{(a_j - a_n)} \\
&= e^{-a_n} \left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{1}{(a_j - a_n)} \right) \\
&\quad + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k - a_j)} \frac{e^{-a_j}}{(a_n - a_j)}.
\end{aligned}
$$

We need to prove that

$$
\left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{1}{(a_j - a_n)} \right) = \frac{\prod_{1\le l\le n-1} a_l}{\prod_{1\le l\le n-1}(a_l - a_n)}.
$$

That is

$$
\prod_{1\le l\le n-1} a_l = \prod_{1\le l\le n-1} (a_l - a_n) \left(1 + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{1}{(a_j - a_n)} \right),
$$

which is

$$
\begin{aligned}
\prod_{1\le l\le n-1} a_l &= \prod_{1\le l\le n-1} (a_l - a_n) \\
&\quad + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} \frac{\prod_{1\le l\le n-1}(a_l - a_n)}{(a_j - a_n)},
\end{aligned}
$$

i.e.,

$$
\prod_{1\le l\le n-1} a_l = \prod_{1\le l\le n-1} (a_l - a_n) + \sum_{1\le j\le n-1} \frac{a_n \prod_{k\neq j, 1\le k\le n-1} a_k (a_k - a_n)}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)}. \tag{3.3.8}
$$

Let us reformulate (3.3.8) as an equality between polynomials (to be proven) with

$$
\prod_{1\le l\le n-1} (a_l - X) + \sum_{1\le j\le n-1} \frac{X \prod_{k\neq j, 1\le k\le n-1} a_k (a_k - X)}{\prod_{k\neq j, 1\le k\le n-1}(a_k - a_j)} - \prod_{1\le l\le n-1} a_l = 0, \tag{3.3.9}
$$

and let us assume that the $(a_j)_{1\le j\le n-1}$ are distinct and different from 0. The polynomial $\mathcal{Q}$ on the left-hand side has degree less than $n-1$ and we have

$$\mathcal{Q}(0) = 0 \quad \forall j \in [\![1, n-1]\!],$$
$$\mathcal{Q}(a_j) = \frac{a_j \prod_{k\ne j,1\le k\le n-1} a_k(a_k - a_j)}{\prod_{k\ne j,1\le k\le n-1}(a_k - a_j)} - \prod_{1\le l\le n-1} a_l = 0,$$

so that $\mathcal{Q}$ has degree less than $n-1$ with n distinct roots and this proves the identity (3.3.9) when the $(a_j)_{1\le j\le n-1}$ are distinct and all different from 0, proving (3.3.7) in that case; of course we may assume that all a_j are positive and noting from (3.3.3) that K_α is continuous on $(\mathbb{R}_+^*)^n$, we get formula (3.3.7) in all cases where all the a_j are positive, concluding the proof of the lemma. ■

Lemma 3.3.11. *With the notations of Theorem 3.3.2, we have, assuming $0 < a_1 \le \cdots \le a_n$, the inequality*

$$K_{0\in\mathbb{N}^n}(a_1,\dots,a_n) \ge \sum_{1\le j\le n} e^{-a_j} \frac{\prod_{1\le l<j} a_l}{(j-1)!} \ge e^{-\min_{1\le j\le n} a_j} = \max_{1\le j\le n} e^{-a_j}.$$

Remark 3.3.12. The above estimate is sharp in the sense that when all the a_j are equal to the same $a > 0$, we have proven in (3.2.1) that

$$\begin{aligned}
K_0(a) &= \frac{e^{-a}}{(n-1)!}\int_0^{+\infty} e^{-s}(s+a)^{n-1} ds \\
&= e^{-a} \sum_{0\le l\le n-1} \frac{a^l}{(n-1-l)!\,l!}\Gamma(n-l) \\
&= e^{-a} \sum_{0\le l\le n-1} \frac{a^l}{l!} = e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!} \\
&= \sum_{1\le j\le n} e^{-a_j} \frac{\prod_{1\le l<j} a_l}{(j-1)!}\Big|_{a_1=\cdots=a_n=a}.
\end{aligned}$$

Proof. The property is true for $n = 1$ since $K_0(a_1) = e^{-a_1}$. We check the case $n = 2$ with $a_1 < a_2$, and we find

$$\begin{aligned}
K_{(0,0)}(a_1, a_2) &= e^{-a_1} + \int_0^{a_1} e^{-t_1} e^{-a_2(1-t_1/a_1)} dt_1 \\
&= e^{-a_1} + e^{-a_2}\frac{e^{a_2-a_1} - 1}{\frac{a_2}{a_1} - 1} = e^{-a_1} + e^{-a_2} a_1 \frac{e^{a_2-a_1} - 1}{a_2 - a_1} \\
&\ge e^{-a_1} + e^{-a_2} a_1.
\end{aligned}$$

Let us consider for some $n \geq 3$, $0 < a_1 < \cdots < a_n$ and inductively,

$$
\begin{aligned}
&K_{0\in\mathbb{N}^n}(a_1,\dots,a_n)\\
&\quad= e^{-a_1}P_0(a_1) + \int_0^{a_1} e^{-t_1}K_{0\in\mathbb{N}^{n-1}}\big(a_2(1-t_1/a_1),\dots,a_n(1-t_1/a_1)\big)dt_1\\
&\quad= e^{-a_1}P_0(a_1) + a_1\int_0^{1} e^{-a_1\theta}K_{0\in\mathbb{N}^{n-1}}\big(a_2(1-\theta),\dots,a_n(1-\theta)\big)d\theta\\
&\quad\geq e^{-a_1} + a_1\int_0^1 e^{-a_1\theta}\sum_{2\leq j\leq n} e^{-a_j(1-\theta)}\frac{\prod_{2\leq l<j} a_l}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad= e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\underbrace{\Big(a_1\prod_{2\leq l<j} a_l\Big)}_{\prod_{1\leq k<j} a_k}\int_0^1 e^{(a_j-a_1)\theta}\frac{1}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad\geq e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\Big(\prod_{1\leq k<j} a_k\Big)\int_0^1 \frac{1}{(j-2)!}(1-\theta)^{j-2}d\theta\\
&\quad= e^{-a_1} + \sum_{2\leq j\leq n} e^{-a_j}\Big(\prod_{1\leq k<j} a_k\Big)\frac{1}{(j-1)!},
\end{aligned}
$$

concluding the proof of the lemma. ■

Remark 3.3.13. The reader may have noticed that it is not obvious on formula (3.3.7)

$$
K_{0,\dots,0}(a_1,\dots,a_n) = \sum_{1\leq j\leq n} e^{-a_j}\frac{\prod_{k\neq j} a_k}{\prod_{k\neq j}(a_k-a_j)},
$$

that K_0 is an entire function. Let us start with taking a look at

$$
\begin{aligned}
&K_{0,0}(a_1,a_2)\\
&= \frac{e^{-a_1}a_2}{a_2-a_1} + \frac{e^{-a_2}a_1}{a_1-a_2} = \frac{a_2e^{-a_1}-a_1e^{-a_2}}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\frac{a_2e^{-\frac{a_1}{2}+\frac{a_2}{2}}-a_1e^{-\frac{a_2}{2}+\frac{a_1}{2}}}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\frac{a_2(\cosh\frac{a_2-a_1}{2}+\sinh\frac{a_2-a_1}{2})-a_1(\cosh\frac{a_1-a_2}{2}+\sinh\frac{a_1-a_2}{2})}{a_2-a_1}\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac{(a_2+a_1)\sinh(\frac{a_2-a_1}{2})}{a_2-a_1}\Big]\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac{\frac12(a_2+a_1)\sinh(\frac{a_2-a_1}{2})}{\frac{a_2-a_1}{2}}\Big]\\
&= e^{-\frac{(a_1+a_2)}{2}}\Big[\cosh\Big(\frac{a_2-a_1}{2}\Big)+\frac12(a_2+a_1)\operatorname{shc}\Big(\frac{a_2-a_1}{2}\Big)\Big],
\end{aligned}
\tag{3.3.10}
$$

where shc stands for the even entire function defined by

$$\operatorname{shc} t = \frac{\sinh t}{t}.$$

We have also from Lemma 3.3.5

$$F_\alpha(a) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau,$$

and defining the function $F_\alpha(a,\lambda)$ as the absolutely converging integral,

$$F_\alpha(a,\lambda) = \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau, \quad F_\alpha(a) = F_\alpha(a,1),$$

we get

$$\begin{aligned}
\frac{\partial F_\alpha}{\partial\lambda}(a,\lambda) &= \frac{1}{\pi}\int_{\mathbb{R}} \cos(\lambda\tau)\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda\tau}\Big(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}} + \prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}\Big) d\tau \\
&= i\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} \frac{(1-i\tau\mu_j)^{\alpha_j}}{(1+i\tau\mu_j)^{\alpha_j+1}}; \tau = i/\mu_j = ia_j\Big) \\
&= i\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} \frac{(-i\mu_j)^{\alpha_j}(ia_j+\tau)^{\alpha_j}}{(i\mu_j)^{\alpha_j+1}(-ia_j+\tau)^{\alpha_j+1}}; \tau = ia_j\Big) \\
&= \frac{1}{i^{n-1}}\sum_{1\le j\le n} \operatorname{Res}\Big(e^{i\lambda\tau}\prod_{1\le j\le n} (-1)^{\alpha_j}\frac{a_j(ia_j+\tau)^{\alpha_j}}{(\tau-ia_j)^{\alpha_j+1}}; \tau = ia_j\Big),
\end{aligned}$$

so that assuming that the a_j are positive and distinct, we get

$$\begin{aligned}
\frac{\partial F_\alpha}{\partial\lambda}(a,\lambda) &= \frac{1}{i^{n-1}}\Big(\prod_{1\le k\le n} a_k\Big)\sum_{1\le j\le n} \frac{1}{\alpha_j!} \\
&\times \Big(\frac{d}{d\tau}\Big)^{\alpha_j}\Big(e^{i\lambda\tau}(-1)^{\alpha_j}(ia_j+\tau)^{\alpha_j}\prod_{1\le k\le n, k\ne j} (-1)^{\alpha_k}\frac{(ia_k+\tau)^{\alpha_k}}{(\tau-ia_k)^{\alpha_k+1}}\Big)_{|\tau=ia_j}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{i^{n-1}} \Big(\prod_{1 \le k \le n} a_k \Big) \sum_{1 \le j \le n} \frac{1}{\alpha_j !} \\
&\times \Big(\frac{d}{i d\sigma} \Big)^{\alpha_j} \Big(e^{-\lambda\sigma} (-1)^{\alpha_j} (i a_j + i\sigma)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} (-1)^{\alpha_k} \frac{(i a_k + i\sigma)^{\alpha_k}}{(i\sigma - i a_k)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} \\
&= (-1)^{n-1+|\alpha|} \Big(\prod_{1 \le k \le n} a_k \Big) \sum_{1 \le j \le n} \frac{1}{\alpha_j !} \\
&\times \Big(\frac{d}{d\sigma} \Big)^{\alpha_j} \Big(e^{-\lambda\sigma} (a_j + \sigma)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(\sigma - a_k)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} \\
&= \Big(\prod_{1 \le k \le n} a_k \Big) \sum_{1 \le j \le n} \frac{(-1)^{\alpha_j}}{\alpha_j !} \\
&\times \Big(\frac{d}{d\sigma} \Big)^{\alpha_j} \Big(e^{-\lambda\sigma} (a_j + \sigma)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} .
\end{aligned}
$$

Since $F_\alpha(a, +\infty) = 1$, thanks to Lemma A.1.7, we find eventually that

$$
\begin{aligned}
&F_\alpha(a) = F_\alpha(a, 1) = \int_{+\infty}^{1} \frac{\partial F_\alpha}{\partial \lambda}(a, \lambda) d\lambda + 1 = 1 - K_\alpha(a), \\
&K_\alpha(a) = \Big(\prod_{1 \le k \le n} a_k \Big) \sum_{1 \le j \le n} \frac{(-1)^{\alpha_j}}{\alpha_j !} \\
&\times \int_1^{+\infty} \Big(\frac{d}{d\sigma} \Big)^{\alpha_j} \Big(e^{-\lambda\sigma} (a_j + \sigma)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} d\lambda \\
&= \sum_{1 \le j \le n} \frac{(-1)^{\alpha_j}}{\alpha_j !} \\
&\times \int_1^{+\infty} e^{-\lambda a_j} \Big(\frac{d}{d\sigma} - \lambda \Big)^{\alpha_j} \Big((a_j + \sigma)^{\alpha_j} a_j \prod_{1 \le k \le n, k \ne j} \frac{(a_k + \sigma)^{\alpha_k} a_k}{(a_k - \sigma)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} d\lambda \\
&= \sum_{1 \le j \le n} \frac{(-1)^{\alpha_j}}{\alpha_j !} \\
&\times \int_1^{+\infty} e^{-\lambda a_j} \Big(\frac{d}{d\sigma} - \lambda \Big)^{\alpha_j} \Big((a_j + \sigma)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} \frac{(a_k + \sigma)^{\alpha_k}}{(a_k - \sigma)^{\alpha_k + 1}} \Big)_{|\sigma = a_j} d\lambda \\
&= \sum_{1 \le j \le n} \frac{(-1)^{\alpha_j}}{\alpha_j !} \\
&\times \int_{a_j}^{+\infty} e^{-t_j} \Big(\frac{d}{d a_j s} - \frac{t_j}{a_j} \Big)^{\alpha_j} \Big((a_j + a_j s)^{\alpha_j} \prod_{1 \le k \le n, k \ne j} \frac{a_k (a_k + a_j s)^{\alpha_k}}{(a_k - a_j s)^{\alpha_k + 1}} \Big)_{|s = 1} d t_j
\end{aligned}
$$

$$= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}$$
$$\times \int_{a_j}^{+\infty} e^{-t}\Big(\frac{d}{ds}-t\Big)^{\alpha_j}\Big((1+s)^{\alpha_j}\prod_{1\le k\le n,k\ne j}\frac{a_k(a_k+a_js)^{\alpha_k}}{(a_k-a_js)^{\alpha_k+1}}\Big)_{|s=1}dt$$
$$= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}$$
$$\times \int_{a_j}^{+\infty} e^{-t}\Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big((t+s)^{\alpha_j}\prod_{1\le k\le n,k\ne j}\frac{a_k(a_k+a_js/t)^{\alpha_k}}{(a_k-a_js/t)^{\alpha_k+1}}\Big)_{|s=t}dt$$
$$= \sum_{1\le j\le n} \frac{(-1)^{\alpha_j}}{\alpha_j!}\int_{a_j}^{+\infty} e^{-t}$$
$$\times \Big(\frac{d}{d(s+t)}-1\Big)^{\alpha_j}\Big((t+s)^{\alpha_j}\prod_{1\le k\le n,k\ne j}\frac{ta_k(t(a_k-a_j)+a_j(s+t))^{\alpha_k}}{(t(a_k+a_j)-a_j(s+t))^{\alpha_k+1}}\Big)_{|s+t=2t}dt$$
$$= \sum_{1\le j\le n} (-1)^{\alpha_j}\int_{a_j}^{+\infty} e^{-t}$$
$$\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!}\prod_{1\le k\le n,k\ne j}\frac{ta_k(t(a_k-a_j)+a_js)^{\alpha_k}}{(t(a_k+a_j)-a_js)^{\alpha_k+1}}\Big)_{|s=2t}dt$$
$$= \sum_{1\le j\le n} (-1)^{\alpha_j}e^{-a_j}\int_{0}^{+\infty} e^{-t}$$
$$\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!}\prod_{1\le k\le n,k\ne j}\frac{(t+a_j)a_k\big((t+a_j)(a_k-a_j)+a_js\big)^{\alpha_k}}{\big((t+a_j)(a_k+a_j)-a_js\big)^{\alpha_k+1}}\Big)_{|s=2t+2a_j}dt.$$

We have also to deal with

$$\prod_{1\le k\le n,k\ne j}\frac{(t+a_j)a_k\big((t+a_j)(a_k-a_j)+a_js\big)^{\alpha_k}}{\big((t+a_j)(a_k+a_j)-a_js\big)^{\alpha_k+1}}$$

and

$$\big((t+a_j)(a_k+a_j)-a_j(2t+2a_j)\big) = a_j(a_k+a_j)-2a_j^2+t(a_k-a_j) = (t+a_j)(a_k-a_j)$$
$$(t+a_j)(a_k+a_j)-a_js = (t+a_j)(a_k-a_j)+a_j(2t+2a_j-s)$$

so that

$$K_\alpha(a) = \sum_{1\le j\le n}(-1)^{\alpha_j}e^{-a_j}\int_0^{+\infty}e^{-t}$$
$$\times \Big(\frac{d}{ds}-1\Big)^{\alpha_j}\Big(\frac{s^{\alpha_j}}{\alpha_j!}\prod_{1\le k\le n,k\ne j}\frac{(t+a_j)a_k((t+a_j)(a_k+a_j)+a_j(s-2t-2a_j))^{\alpha_k}}{((t+a_j)(a_k-a_j)-a_j(s-2t-2a_j))^{\alpha_k+1}}\Big)_{|s=2t+2a_j}dt. \tag{3.3.11}$$

3.4 A conjecture on integrals of products of Laguerre polynomials

We formulate in this section a conjecture on the behaviour of the functions $K_\alpha(a)$; as displayed in the previous sections, we know several useful elements for the analysis of these functions, including some quite explicit expression. However, in the non-isotropic case, we were not able to prove the estimate $F_\alpha(a) \le 1$, equivalent to $K_\alpha(a) \ge 0$, except for the case $\alpha = 0$. We are thus reduced to conjectural statements.

Conjecture 3.4.1. Let $n \ge 1$ be an integer and let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$. For $a = (a_1, \dots, a_n) \in (0, +\infty)^n$, we define

$$K_\alpha(a) = \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\le j\le n} t_j/a_j \ge 1}} e^{-(t_1+\dots+t_n)} \prod_{1\le j\le n} (-1)^{\alpha_j} L_{\alpha_j}(2t_j)dt,$$

where L_k stands for the classical Laguerre polynomial

$$L_k(X) = \left(\frac{d}{dX} - 1\right)^k \frac{X^k}{k!}.$$

Then, we conjecture that, assuming $0 < a_1 \le \dots \le a_n$, we have

$$K_\alpha(a) \ge \sum_{1\le j\le n} e^{-a_j} \frac{\prod_{1\le l<j} a_l}{(j-1)!}. \tag{3.4.1}$$

Remark 3.4.2. A slightly stronger and more symmetrical version of the above conjecture is that for n, α, a, K_α as above, we have

$$K_\alpha(a) \ge K_0(a). \tag{3.4.2}$$

It is indeed stronger since we have proven in Lemma 3.3.11 that $K_0(a)$ is greater than the right-hand side of (3.4.1).

Theorem 3.4.3. *The previous conjecture is a proven theorem in the following cases.*

(1) *When* $n = 1$.
(2) *For all* $n \ge 1$, *when all the* a_j *are equal.*
(3) *For all* $n \ge 1$, *when* $\alpha = 0_{\mathbb{N}^n}$.
(4) *When* $n = 2$ *and* $\min(\alpha_1, \alpha_2) = 0$.

Proof. (1) When $n = 1$, we have proven above (in Proposition 3.1.3) that for $\alpha \in \mathbb{N}$, $a > 0$,

$$K_\alpha(a) = e^{-a} P_\alpha(a) \ge e^{-a},$$

which is indeed (3.4.2) in that case. With the notations of Theorem 3.1.5 (and in particular where D_a is defined in (3.1.11)) this implies

$$\mathrm{Op_w}(\mathbf{1}_{D_a}) \le 1 - e^{-a},$$

an inequality due to P. Flandrin in 1988 paper [13].

(2) Assuming that all the a_j are equal to $a > 0$, we have proven in Theorem 3.2.2 that for $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{1\le j\le n} \alpha_j$,

$$K_\alpha(a,\dots,a) \ge \frac{\Gamma(n,a)}{\Gamma(n)} = e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!} = K_0(a,\dots,a),$$

since from (3.3.3), we have

$$\begin{aligned}
K_0(a,\dots,a) &= \int_{\substack{\sum t_j \ge a\\ t_j\ge 0}} e^{-(t_1+\dots+t_n)} dt \\
&= \int_{\substack{t_n \ge a\\ t_j\ge 0}} e^{-(t_1+\dots+t_n)} dt + \int_0^a e^{-t_n} \int_{\sum t_j \ge a - t_n} e^{-(t_1+\dots+t_{n-1})} dt \\
\text{(inductively)} \quad &= e^{-a} + \int_0^a e^{-t_n} e^{-(a-t_n)} \sum_{1\le j\le n-1} \frac{(a-t_n)^{j-1}}{(j-1)!} dt_n \\
&= e^{-a}\left(1 + \sum_{1\le j\le n-1} \frac{a^j}{j!}\right) = e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!},
\end{aligned}$$

proving (3.4.2) in that case. With

$$D_{(a)} = \left\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi \frac{|x|^2 + |\xi|^2}{a} \le 1\right\},$$

this implies that

$$\operatorname{Op_w}(\mathbf{1}_{D_{(a)}}) \le 1 - e^{-a} \sum_{1\le j\le n} \frac{a^{j-1}}{(j-1)!},$$

an inequality proven in the 2010 article [39] by E. Lieb and Y. Ostrover.

(3) When $\alpha = 0_{\mathbb{N}^n}$, we have proven (3.4.1) in Lemma 3.3.11.

(4) When $n = 2$, from the case $n = 1$ we have $K_{\alpha_2}(a_2) = e^{-a_2} P_{\alpha_2}(a_2)$, so that from Lemma 3.3.9, we obtain

$$\begin{aligned}
&K_{\alpha_1,\alpha_2}(a_1,a_2) \\
&\quad = e^{-a_1} P_{\alpha_1}(a_1) + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} (-1)^{\alpha_1} L_{\alpha_1}(2\theta a_1) P_{\alpha_2}(a_2(1-\theta)) d\theta,
\end{aligned}$$

and if α_1=0, it means that

$$\begin{aligned}
K_{0,\alpha_2}(a_1,a_2) &= e^{-a_1} + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} P_{\alpha_2}(a_2(1-\theta)) d\theta \\
&\ge e^{-a_1} + a_1 \int_0^1 e^{-\theta a_1 - (1-\theta) a_2} d\theta = K_{0,0}(a_1,a_2),
\end{aligned}$$

and the reasoning is identical for $\alpha_2 = 0$, concluding the proof of the theorem. ∎

We are interested in the Weyl quantization of the indicatrix of

$$D_{a_1,\dots,a_n} = \left\{(x,\xi) \in \mathbb{R}^{2n}, 2\pi \sum_{1\le j\le n} \frac{x_j^2+\xi_j^2}{a_j} \le 1\right\}, \quad a_j > 0,$$

and we have a weaker conjecture.

Conjecture 3.4.4 (A weak form of Conjecture 3.4.1). With n, α, a, K_α as in Conjecture 3.4.1, we conjecture that

$$K_\alpha(a) \ge 0. \tag{3.4.3}$$

Note that inequality (3.4.3) is equivalent to

$$\operatorname{Op}_{\mathrm{w}}(\mathbf{1}_{D_{a_1,\dots,a_n}}) \le 1.$$

Remark 3.4.5. In the first place, although the second conjecture is much weaker than the first, there is no reason to believe that the weak conjecture should be easier to prove than the first: in particular, in the known cases, it is indeed the proof of the precise statement (3.4.1) which leads to (3.4.3) and we are not aware of a direct proof of (3.4.3), even in one dimension.

A summary of our knowledge on the functions K_α. As proven in Remarks 3.3.3 and 3.3.4, the functions K_α are entire functions given on the open subset (3.3.4) by formula (3.3.3) (see also formula (3.3.10)). Moreover, the function $F_\alpha(a) = 1 - K_\alpha(a)$ can be expressed as a simple integral for $a_j > 0$,

$$F_\alpha(a_1,\dots,a_n) = \int_{\mathbb{R}} \frac{\sin\tau}{\pi\tau}\left(\prod_{1\le j\le n} \frac{(1+i\tau\mu_j)^{\alpha_j}}{(1-i\tau\mu_j)^{\alpha_j+1}}\right)d\tau, \quad \mu_j = \frac{1}{a_j},$$

and we have an explicit expression of the function K_α as a sum of simple integrals in (3.3.11). However, having an explicit expression does not mean much and for instance, we do have several explicit expressions for the Laguerre polynomials but inequality (A.4.2) remains very hard work, requiring a deep understanding of these polynomials. We have also an induction formula in Lemma 3.3.9. As a further remark, we have the following

Lemma 3.4.6. *Let n, α, a, K_α as in Conjecture 3.4.1. Then, we have*

$$\lim_{a_n\to+\infty} K_{\alpha_1,\dots,\alpha_{n-1},\alpha_n}(a_1,\dots,a_{n-1},a_n) = K_{\alpha_1,\dots,\alpha_{n-1}}(a_1,\dots,a_{n-1}), \tag{3.4.4}$$

$$\lim_{a_1\to0_+} K_{\alpha_1,\alpha_2,\dots,\alpha_n}(a_1,a_2,\dots,a_n) = 1. \tag{3.4.5}$$

Proof. Formula (3.3.3) and the Lebesgue dominated convergence theorem imply the first equality (3.4.4). Lemma 3.3.9, in which we may swap the variables a_1 and a_n

gives for $a_1 > 0$

$$K_{\alpha_1,\alpha_2,\dots,\alpha_n}(a_1,a_2,\dots,a_n) = e^{-a_1}P_{\alpha_1}(a_1) + a_1\int_0^1 e^{-\theta a_1}(-1)^{\alpha_1}L_{\alpha_1}(2a_1\theta)K_{\alpha_2,\dots,\alpha_n}\big(a_2(1-\theta),\dots,a_n(1-\theta)\big)d\theta,$$

and since P_{α_1} is a polynomial such that $P_{\alpha_1}(0) = 1$, we get (3.4.5). ■

Reasons to believe in the conjecture. This is true in one dimension, also in n dimensions for spheres and it is a quadratic problem in the sense that ellipsoids are convex subsets of $\mathbb{R}^{2n}$ characterized by an inequality

$$\{X \in \mathbb{R}^{2n}, p(X) \le 0\},$$

where p is a polynomial of degree 2 with a positive-definite quadratic part. We shall see below in this memoir that convexity of a set A does not guarantee that the quantization $\operatorname{Op_w}(\mathbf{1}_A)$ is smaller than 1 as an operator and that Flandrin's conjecture is not true, but it is hard to believe that such a phenomenon could occur for ellipsoids. We must point out a specific feature of anisotropy related to Mehler's formula (2.2.1): if all the μ_j are equal to the same $\mu > 0$ (this is the isotropic case), then, with $q_\mu(x,\xi) = \mu(|x|^2 + |\xi|^2)$, we have

$$\operatorname{Op_w}(e^{2i\pi\tau q_\mu(x,\xi)}) = \phi(\tau\mu)e^{2i\arctan(\tau\mu)\sum_{1\le j\le n}\pi(x_j^2+D_j^2)},$$

where $\phi(\tau\mu)$ is a scalar quantity. As a consequence, if we quantize $F(q_\mu(x,\xi))$, we get

$$\operatorname{Op_w}\big(F\big(q_\mu(x,\xi)\big)\big) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu)e^{2i\frac{\arctan(\tau\mu)}{\mu}\pi\operatorname{Op_w}(q_\mu)}d\tau,$$

and thus

$$\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big) = \widetilde{F}(\operatorname{Op_w}(q_\mu)), \quad \widetilde{F}(\lambda) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu)e^{2i\pi\frac{\arctan(\tau\mu)}{\mu}\lambda}d\tau,$$

and $\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big)$ appears as a function of the self-adjoint operator $\operatorname{Op_w}(q_\mu)$. Following the same route in the anisotropic case, we get, with

$$q_\mu(x,\xi) = \sum_{1\le j\le n}\mu_j(x_j^2+\xi_j^2),$$

$$\operatorname{Op_w}\big(F(q_\mu(x,\xi))\big) = \int_{\mathbb{R}} \widehat{F}(\tau)\phi(\tau\mu)e^{2i\pi\sum_{1\le j\le n}\left(\frac{\arctan(\tau\mu_j)}{\mu_j}\right)\mu_j(x_j^2+D_j^2)}d\tau,$$

and since $\frac{1}{\mu_j}\arctan(\tau\mu_j)$ does depend on μ_j (and not only on τ), the operator $\operatorname{Op_w}$ $(F(q_\mu(x,\xi)))$ is not a function of the self-adjoint operator $\operatorname{Op_w}(q_\mu)$.

As a final comment on the strongest form of the Conjecture (3.4.2), we would say that it could be seen as a property of the Laguerre polynomials, known in the case $n = 1$, where it stands as follows: we define for $k \in \mathbb{N}$, the polynomial P_k by

$$P_k(x) = \int_0^{+\infty} e^{-t}(-1)^k L_k(2x + 2t)dt,$$

and we have $P_k(0) = 1$ from (A.4.4). Moreover, we have the inequality (equivalent to (3.4.2) for $n = 1$)

$$\forall x \geq 0, \quad P_k(x) \geq P_k(0). \tag{3.4.6}$$

We note that $e^{-x} P_k(x) = \int_x^{+\infty} e^{-s}(-1)^k L_k(2s)ds$, so that the unique solution P_k of the Initial Value Problem for the ODE

$$P_k(x) - P_k'(x) = (-1)^k L_k(2x), \quad P_k(0) = 1,$$

does satisfy (3.4.6). We note that from Lemma 3.1.2, we have

$$P_k'(X) = 2 \sum_{0 \leq l < k} (-1)^l L_l(2X),$$

so that (3.4.6) is a consequence of Feldheim inequality (A.4.2). Let us reformulate (3.4.2), using the polynomials P_k: for $a_j \geq 0$,

$$\begin{aligned} K_\alpha(a) &= \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\leq j\leq n} t_j/a_j \geq 1}} \prod_{1\leq j\leq n} \frac{\partial}{\partial t_j}\{-e^{-t_j} P_{\alpha_j}(t_j)\}dt \\ &\geq K_0(a) = \int_{\substack{t=(t_1,\dots,t_n)\in\mathbb{R}_+^n \\ \sum_{1\leq j\leq n} t_j/a_j \geq 1}} \prod_{1\leq j\leq n} \frac{\partial}{\partial t_j}\{-e^{-t_j}\}dt, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} H(s_j)\frac{\partial}{\partial s_j}\{-e^{-a_j s_j} P_{\alpha_j}(a_j s_j)\}ds \\ &\leq \int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} a_j H(s_j) e^{-a_j s_j} ds, \end{aligned}$$

where $H = \mathbf{1}_{\mathbb{R}_+}$ (Heaviside function). This is equivalent to

$$\begin{aligned} &\int H\Big(1 - \sum_{1\leq j\leq n} s_j\Big) \prod_{1\leq j\leq n} H(s_j)e^{-a_j s_j}\Big(a_j - \frac{\partial}{\partial s_j}\Big)\{P_{\alpha_j}(a_j s_j)\}ds \\ &\leq \int H(1 - \sum_{1\leq j\leq n} s_j) \prod_{1\leq j\leq n} a_j H(s_j) e^{-a_j s_j} ds, \end{aligned}$$

i.e., to

$$\int H\Big(1 - \sum_{1\le j\le n} s_j\Big) \prod_{1\le j\le n} H(s_j) e^{-a_j s_j} \times \Big(\prod_{1\le j\le n} a_j - \prod_{1\le j\le n} \Big(a_j - \frac{\partial}{\partial s_j}\Big)\big\{P_{\alpha_j}(a_j s_j)\big\}\Big) ds \geq 0.$$

Note that for $n = 1$, it means for $a \geq 0$,

$$\begin{aligned}\int_0^1 & e^{-as}(a - aP_k(as) + aP_k'(as))ds \\ &= 1 - e^{-a} + \int_0^1 \frac{d}{ds}\{e^{-as}P_k(as)\} \\ &= 1 - e^{-a} + e^{-a}P_k(a) - P_k(0) = e^{-a}(P_k(a) - 1) \geq 0,\end{aligned}$$

which holds true from (3.4.6).

Remark 3.4.7. There are several classical results on products of Laguerre polynomials, in particular, the article [7], *On some expansions in Laguerre polynomials* by A. Erdélyi and also the paper [40], *Linearization of the products of the generalized Lauricella polynomials and the multivariate Laguerre polynomials via their integral representations* by Shuoh-Jung Liu, Shy-Der Lin, Han-Chun Lu and H. M. Srivastava. However, it seems that the non-negativity of the polynomials $P_{\alpha;1}$, $P'_{\alpha;1}$, do not suffice to tackle the conjecture in two dimensions and more.