Chapter 4

Parabolas

4.1 Preliminary remarks

We start with a picture, demonstrating that the epigraph of a parabola is an increasing union of ellipses (see Figure 4.1). It is easy to see that the epigraph of a parabola, i.e., the set $\{(x,\xi)\in\mathbb{R}^2, \xi > x^2\}$ is a countable increasing union of ellipses in the sense that

$$\mathcal{P} = \{(x,\xi)\in\mathbb{R}^2, \xi > x^2\} = \bigcup_{k\ge1} \underbrace{\{(x,\xi)\in\mathbb{R}^2, \xi > x^2 + k^{-2}\xi^2\}}_{\mathcal{E}_k}. \tag{4.1.1}$$

Note that for $k \ge 1$ we have $\mathcal{E}_k \subset \mathcal{E}_{k+1} \subset \mathcal{P}$ since $x^2 + k^{-2}\xi^2 \ge x^2 + (k+1)^{-2}\xi^2 > x^2$, from the fact that $\xi > 0$ on $\mathcal{E}_k$. Moreover, if $\xi > x^2$ and $k > \xi/\sqrt{\xi - x^2}$, we get $(x,\xi)\in\mathcal{E}_k$.

Remark 4.1.1. The ellipse $\mathcal{E}_k$ is symplectically equivalent to a circle with area $\frac{\pi k^3}{4}$ since

$$\begin{aligned} x^2+k^{-2}\xi^2-\xi &= x^2 + k^{-2}\Big(\xi-\frac{k^2}{2}\Big)^2-\frac{k^2}{4} = (\lambda^{-1}y)^2+k^{-2}\Big(\lambda\eta-\frac{k^2}{2}\Big)^2-\frac{k^2}{4} \\ &= \lambda^{-2}y^2+\lambda^2k^{-2}\Big(\eta-\frac{k^2}{2\lambda}\Big)^2-\frac{k^2}{4}, \end{aligned}$$

so that choosing λ such that $\lambda^{-2}=\lambda^2k^{-2}$, e.g., $\lambda=\sqrt{k}$, we get

$$x^2+k^{-2}\xi^2-\xi = k^{-1}\Big(y^2+\Big(\eta-\frac{k^2}{2\lambda}\Big)^2\Big)-\frac{k^2}{4},$$

and $\mathcal{E}_k = \{(y,\zeta)\in\mathbb{R}^2, y^2+\zeta^2<\frac{k^3}{4}\}$, where (y,ζ) are the affine symplectic coordinates

$$y = xk^{1/2}, \quad \zeta = \xi k^{-1/2} - \frac{k^{3/2}}{2}.$$

Lemma 4.1.2. *Let $u\in\mathscr{S}(\mathbb{R})$. Then, $\mathcal{W}(u,u)$ belongs to $\mathscr{S}(\mathbb{R}^2)$ and with $\mathcal{E}, \mathcal{E}_k$ defined by (4.1.1), we have*

$$\iint_{\xi>x^2}\mathcal{W}(u,u)(x,\xi)dxd\xi = \lim_{k\to+\infty}\iint_{\mathcal{E}_k}\mathcal{W}(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R})}.$$

Proof. Since $\mathcal{W}(u,u)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})\subset L^1(\mathbb{R}^{2n})$, we may apply the Lebesgue dominated convergence theorem and (4.1.1) to obtain the equality in the lemma. On

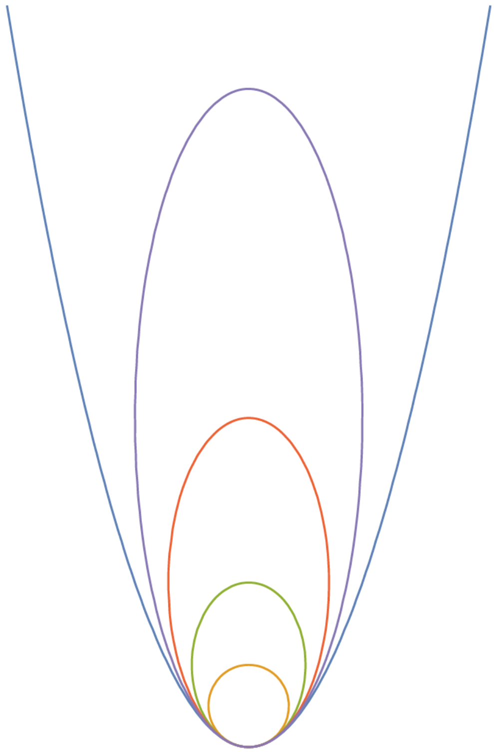

Figure 4.1. The epigraph of a parabola is an increasing union of ellipses.

the other hand, Theorem 3.1.5 and Remark 4.1.1 imply

$$\iint_{\mathcal{E}_k} \mathcal{W}(u,u)(x,\xi)dxd\xi = \langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{E}_k})u,u\rangle \le \big(1-e^{-\frac{\pi k^3}{2}}\big)\|u\|^2_{L^2(\mathbb{R})} \le \|u\|^2_{L^2(\mathbb{R})},$$

and the sought result. ■

Remark 4.1.3. Moreover, Theorem 3.1.5 and the expression of $F_0(a) = 1 - e^{-a}$ imply that with ψ_0 defined in (A.1.16), we have

$$\iint_{\mathcal{E}_k} \mathcal{W}(\psi_0,\psi_0)(x,\xi)dxd\xi = \langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{E}_k})\psi_0,\psi_0\rangle = \|\psi_0\|^2_{L^2(\mathbb{R})}(1-e^{-\pi k^3/3}),$$

so that from Lemma 4.1.2, we have $\iint_{\mathcal{P}} \mathcal{W}(\psi_0,\psi_0)(x,\xi)dxd\xi = \|\psi_0\|^2_{L^2(\mathbb{R})}$, entailing

$$\sup_{\phi\in\mathscr{S}(\mathbb{R}),\|\phi\|_{L^2(\mathbb{R})}=1} \iint_{\mathcal{P}} \mathcal{W}(\phi,\phi)(x,\xi)dxd\xi = 1.$$

Remark 4.1.4. We want to study the operator with Weyl symbol $H(\xi - x^2)$ ($H = \mathbf{1}_{\mathbb{R}_+}$ is the Heaviside function) and since $\xi - x^2$ is a polynomial with degree less than 2, see from (1.2.3) that $\operatorname{Op_w}\big(H(\xi - x^2)\big)$ commutes with

$$D_x - x^2 = e^{2\pi i x^3/3} D_x e^{-2\pi i x^3/3},$$

and the latter has (continuous) spectrum $\mathbb{R}$: we expect thus that $\operatorname{Op_w}\big(H(\xi - x^2)\big)$ should have continuous spectrum and be conjugated to a Fourier multiplier.

4.2 Calculation of the kernel

The Weyl symbol of the operator $\operatorname{Op_w}(\mathbf{1}_{\mathscr{P}})$ is

$$H(\xi - x^2),$$

($\mathscr{P}$ is defined in (4.1.1), H is the Heaviside function $H = \mathbf{1}_{\mathbb{R}_+}$), corresponding to the distribution kernel $k_{\mathscr{P}}(x, y)$ obtained from Proposition 1.2.5 by (we use freely integrals meaning only Fourier transform in the distributional sense),

$$\begin{aligned} k_{\mathscr{P}}(x, y) &= \int e^{2i\pi(x-y)\xi} H\left(\xi - \left(\frac{x+y}{2}\right)^2\right) d\xi = \int e^{2i\pi(x-y)(\xi + (\frac{x+y}{2})^2)} H(\xi) d\xi \\ &= e^{2i\pi(x-y)(\frac{x+y}{2})^2} \frac{1}{2}\left(\delta_0(y-x) + \frac{1}{i\pi(y-x)}\right) \\ &= \frac{\delta_0(y-x)}{2} + \frac{e^{2i\pi(x-y)(\frac{x+y}{2})^2}}{2i\pi(y-x)}. \end{aligned}$$

We have

$$\begin{aligned} 4(x-y)\left(\frac{x+y}{2}\right)^2 &= (x^2 - y^2)(x+y) = x^3 - y^3 + x^2 y - y^2 x \\ &= \frac{4}{3}(x^3 - y^3) + \frac{1}{3}(y-x)^3, \end{aligned}$$

so that

$$k_{\mathscr{P}}(x, y) = e^{i\frac{2\pi}{3}x^3}\left(\frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{2}\frac{1}{3}(y-x)^3}}{2i\pi(y-x)}\right) e^{-i\frac{2\pi}{3}y^3},$$

and the operator $\operatorname{Op_w}(\mathbf{1}_{\mathscr{P}})$ is unitarily equivalent to the operator with kernel

$$\tilde{k}(x, y) = \frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{6}(y-x)^3}}{2i\pi(y-x)}.$$

We have proven the following result.

Lemma 4.2.1. *The operator with Weyl symbol* $\mathbb{R}^2 \ni (x,\xi) \mapsto \mathbf{1}_{\mathbb{R}_+}(\xi - x^2)$ *has the distribution kernel*

$$k_{\mathscr{P}}(x,y) = e^{i\frac{2\pi}{3}x^3}\left(\frac{\delta_0(y-x)}{2} + \frac{e^{i\frac{\pi}{6}(y-x)^3}}{2i\pi(y-x)}\right)e^{-i\frac{2\pi}{3}y^3},$$

and is thus unitarily equivalent to

$$\frac{\mathrm{Id}}{2} + \textit{convolution with}\quad \frac{ie^{-i\pi t^3/6}}{2\pi}\mathrm{pv}\frac{1}{t}. \tag{4.2.1}$$

Lemma 4.2.2. *The distribution* $\frac{ie^{-i\pi t^3/6}}{2\pi}\mathrm{pv}\frac{1}{t}$ *has the Fourier transform*

$$\frac{1}{2\pi}\int \frac{\sin(2\pi a s\tau + \frac{s^3}{3})}{s}ds, \quad a = (2/\pi)^{1/3}.$$

The operator (4.2.1) *is the Fourier multiplier* $\omega(D_t)$ *with*

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right), \quad \eta = 2^{4/3}\pi^{2/3}\tau.$$

Proof. We calculate in the distribution sense ($t = as, a = (2/\pi)^{1/3}$),

$$\begin{aligned}\int e^{-2i\pi t\tau} i\frac{e^{-i\pi t^3/6}}{2\pi t}dt &= \frac{i}{2\pi}\int e^{-2i\pi a s\tau}\frac{e^{-i\pi a^3 s^3/6}}{s}ds\\ &= \frac{i}{2\pi}\int \frac{(-i)\sin(\frac{s^3}{3} + 2\pi a s\tau)}{s}ds\\ &= \frac{1}{2\pi}\int \frac{\sin(2\pi a s\tau + \frac{s^3}{3})}{s}ds,\end{aligned}$$

so that with $\eta = 2\pi a\tau$, we get

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right) = \frac{1}{2}\big(1 - F(\eta)\big) = G(\eta),$$

proving the lemma. ∎

Lemma 4.2.3. *We have, with* $\eta = 2^{4/3}\pi^{2/3}\tau$,

$$\omega(\tau) = \frac{1}{2}\left(1 + \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(s\eta + \frac{s^3}{3})}{s}ds\right) = G(\eta), \quad \omega(0) = \frac{2}{3} = G(0), \tag{4.2.2}$$

$$G'(\eta) = \frac{1}{2\pi}\int_{\mathbb{R}} \cos\left(s\eta + \frac{s^3}{3}\right)ds = \mathrm{Re}\,\frac{1}{2\pi}\int_{\mathbb{R}} \exp i\left(s\eta + \frac{s^3}{3}\right)ds = \mathtt{Ai}(\eta), \tag{4.2.3}$$

$$G(\eta) = \frac{2}{3} + \int_0^{\eta} \mathtt{Ai}(\xi)d\xi, \tag{4.2.4}$$

where $\mathtt{Ai}$ *is Airy function defined as the inverse Fourier transform of* $t \mapsto e^{i(2\pi t)^3/3}$.

Proof. We have

$$\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\frac{s^3}{3})}{s}ds=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\sin(\sigma)}{3^{1/3}\sigma^{1/3}}3^{1/3}\frac{1}{3}\sigma^{-2/3}d\sigma=\frac{1}{3\pi}\int_{-\infty}^{+\infty}\frac{\sin\sigma}{\sigma}d\sigma=\frac{1}{3}, \tag{4.2.5}$$

proving (4.2.2). We have also

$$G(\eta)=\frac{1}{2}+\operatorname{Im}\Big[\text{Inverse Fourier Transform}\Big\{y\mapsto e^{i(2\pi y)^3/3}\operatorname{pv}\Big(\frac{1}{2\pi y}\Big)\Big\}\Big],$$

and thus

$$\begin{aligned}G'(\eta)&=\operatorname{Im}\Big[\text{Inverse Fourier Transform}\big\{y\mapsto e^{i(2\pi y)^3/3}i\big\}\Big]\\&=\operatorname{Im}\Big(\int e^{2i\pi y\eta}e^{i(2\pi y)^3/3}i\,dy\Big)=\operatorname{Im}\Big(\frac{1}{2\pi}\int e^{it\eta}e^{it^3/3}i\,dt\Big)=\mathtt{Ai}(\eta),\end{aligned}$$

which is (4.2.3), implying (4.2.4). ∎

Lemma 4.2.4. *With G defined in Lemma 4.2.3, we get that G is an entire function, real-valued on the real line such that*

$$\lim_{\eta\to+\infty}G(\eta)=1,\qquad\lim_{\eta\to-\infty}G(\eta)=0, \tag{4.2.6}$$

and moreover with η_0 the largest zero of the Airy function ($\eta_0\approx-2.33811$), the function G has an absolute minimum at η_0 with $G(\eta_0)\approx-0.274352$,

$$\forall\eta\in\mathbb{R},\quad G(\eta_0)\le G(\eta)<1. \tag{4.2.7}$$

Proof. The first statements follow from Lemma 4.2.3 and (4.2.6) is implied by (4.2.4) and (A.7.18), (A.7.22). The strict inequality in (4.2.7) follows for $\eta\ge0$ from (4.2.3) since Ai is positive on $[0,+\infty)$ so that G is strictly increasing there from $G(0)=2/3$ to $G(+\infty)=1$. The other statements are proven in Section A.7 of the appendix. ∎

4.3 The main result

Collecting the results of Lemmas 4.2.1, 4.2.2, 4.2.3, 4.2.4, and of Section A.7 in the appendix, we have proven the following theorem.

Theorem 4.3.1. *Let $H(\xi-x^2)=\mathbf{1}\{(x,\xi)\in\mathbb{R}^2,\xi\ge x^2\}$ be the indicatrix of the epigraph of the parabola with equation $\xi=x^2$. Then, the operator with Weyl symbol $H(\xi-x^2)$ is unitary equivalent to the Fourier multiplier $G(2^{4/3}\pi^{2/3}\tau)$, where*

$$G(\eta)=\frac{2}{3}+\int_0^\eta\mathtt{Ai}(\xi)d\xi=\int_{-\infty}^\eta\mathtt{Ai}(\xi)d\xi,\quad(\mathtt{Ai}\text{ is the Airy function}).$$

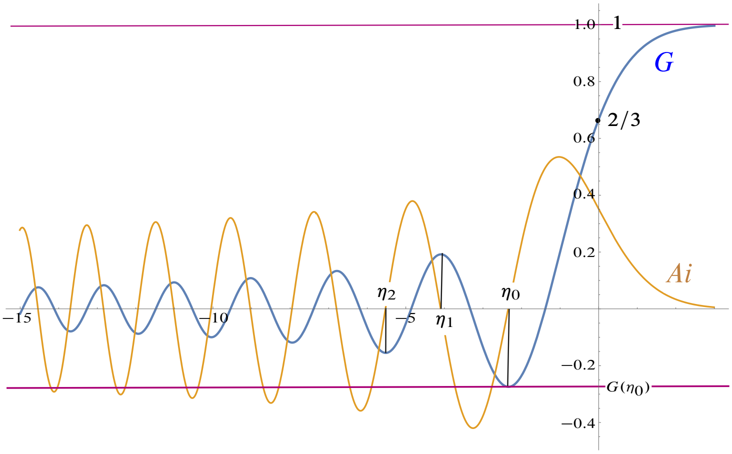

Figure 4.2. The function G and its derivative `Ai`. More details on G are given in Appendix A.7, Figure A.1.

The function G is entire on $\mathbb{C}$, real-valued on the real line (see Figure 4.2) and such that

$$G(\mathbb{R}) = [G(\eta_0), 1),$$

where η_0 is the largest zero of the Airy function. We have

$$\eta_0 \approx -2.338107410,$$
$$G(\eta_0) \approx -0.2743520591.$$

The operator with Weyl symbol $H(\xi - x^2)$ is self-adjoint bounded on $L^2(\mathbb{R})$ with norm 1*, with spectrum equal to* $[G(\eta_0), 1]$ *(continuous spectrum) and*

$$\forall u \in L^2(\mathbb{R}), \quad G(\eta_0)\|u\|^2_{L^2(\mathbb{R})} \le \iint_{\xi \ge x^2} \mathcal{W}(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R})}.$$

4.4 Paraboloids, a conjecture

We are interested now in multi-dimensional versions of the previous results, namely, we would like to find a bound for integrals of the Wigner distribution on paraboloids of $\mathbb{R}^{2n}$ for $n \ge 2$. Let us start with recalling in [24, Theorem 21.5.3], a version of which was given in our Theorem 3.3.1 in the positive-definite case.

4.4.1 On non-negative quadratic forms

Theorem 4.4.1 (Symplectic reduction of quadratic forms, [24, Theorem 21.5.3]). *Let q be a non-negative quadratic form on $\mathbb{R}^n \times \mathbb{R}^n$ equipped with the canonical* *symplectic form* (1.2.13). *Then, there exists* S *in symplectic group* $\mathrm{Sp}(n, \mathbb{R})$ *of* $\mathbb{R}^{2n}$,

$$r \in \{0, \dots, n\},\ \mu_1, \dots, \mu_r \text{ positive, and } s \in \mathbb{N} \text{ such that } r + s \le n,$$

so that for all $X = (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$q(SX) = \sum_{1 \le j \le r} \mu_j (x_j^2 + \xi_j^2) + \sum_{r+1 \le j \le r+s} x_j^2.$$

Definition 4.4.2. Let $n \in \mathbb{N}^*$ and let $\mathbb{R}^{2n}$ be equipped with the canonical symplectic form (1.2.13). Let q be a non-negative quadratic form on $\mathbb{R}^{2n}$ with rank $2n - 1$ and T be a non-zero vector in $\mathbb{R}^{2n}$ such that $q(\sigma T) = 0$. A paraboloid $\mathscr{P}$ of $\mathbb{R}^{2n}$ with vertex 0 and shape (q, T) is defined by

$$\mathscr{P} = \left\{ X \in \mathbb{R}^{2n}, q(X) \le [X, T] \right\}.$$

A paraboloid $\mathscr{Q}$ with vertex $m \in \mathbb{R}^{2n}$ and shape (q, T) is defined as

$$\mathscr{Q} = \mathscr{P} + m,$$

where $\mathscr{P}$ is a paraboloid with vertex 0 and shape (q, T).

Remark 4.4.3. We can find some symplectic coordinates such that

$$q(X) - [X, T] = \sum_{1 \le j \le r} \mu_j (x_j^2 + \xi_j^2) + \sum_{r+1 \le j \le r+s} x_j^2 + \sum_{1 \le j \le n} (x_j \tau_j - \xi_j t_j),$$

with $2r + s = 2n - 1$. We can get rid of the linear terms $x_j \tau_j - \xi_j t_j$ when $1 \le j \le r$ by writing

$$\mu_j (x_j^2 + \xi_j^2) + x_j \tau_j - \xi_j t_j = \mu_j \left(x_j + \frac{\tau_j}{2\mu_j} \right)^2 + \mu_j \left(\xi_j - \frac{t_j}{2\mu_j} \right)^2 - \frac{1}{4\mu_j} (t_j^2 + \tau_j^2),$$

and also of $x_j \tau_j$ for $r + 1 \le j \le r + s$, since

$$x_j^2 + x_j \tau_j = \left(x_j + \frac{\tau_j}{2} \right)^2 - \frac{\tau_j^2}{4}.$$

We are left with using affine symplectic coordinates (y, η) so that

$$\begin{aligned} q(X) - [X, T] = &\sum_{1 \le j \le r} \mu_j (y_j^2 + \eta_j^2) + \sum_{r+1 \le j \le r+s} y_j^2 - \sum_{r+1 \le j \le r+s} \eta_j t_j \\ &+ \sum_{r+s+1 \le j \le n} (y_j \tau_j - \eta_j t_j) - a. \end{aligned}$$

Since we have $2r + s = 2n - 1$, we get $r + s + 1 = 2n - r$: we cannot have $r + s + 1 \le n$ since it would imply that $2n - r \le n$ and thus $r \ge n$, which is incompatible with $2r+s=2n-1, r,s\ge 0$. We get then that $s=2l+1, r=n-1-l$ and since $r+s\le n, 1\le s$, we have $l=0, s=1, r=n-1$, and

$$q(X)-[X,T]=\sum_{1\le j\le n-1}\mu_j(y_j^2+\eta_j^2)+y_n^2-\eta_n t_n-a,$$

and $t_n\in\mathbb{R}^*$. With $y_n=t^{1/3}\tilde y_n, \eta_n=t^{-1/3}\tilde\eta_n$, we get

$$q(X)-[X,T]=\sum_{1\le j\le n-1}\mu_j(y_j^2+\eta_j^2)+t^{2/3}(\tilde y_n^2-\tilde\eta_n-at^{-2/3}),$$

and the inequality $q(X)-[X,T]\le 0$ is equivalent to

$$\sum_{1\le j\le n-1}t^{-2/3}\mu_j(y_j^2+\eta_j^2)+\tilde y_n^2\le\tilde\eta_n+at^{-2/3}.$$

We can thus assume *ab initio* that our paraboloid is given by the inequality

$$\sum_{1\le j\le n-1}\nu_j(x_j^2+\xi_j^2)+x_n^2\le\xi_n.$$

4.4.2 On the kernel for the paraboloid

We shall consider the paraboloid

$$\mathscr{P}_n=\Big\{(x,\xi)\in\mathbb{R}^{2n}, x_n^2+\sum_{1\le j\le n-1}(x_j^2+\xi_j^2)\le\xi_n\Big\}.$$

We have with $X'=(x';\xi')=(x_1,\dots,x_{n-1};\xi_1,\dots,\xi_{n-1})$,

$$\begin{aligned}P&=\operatorname{Op_w}\big(H(\xi_n-x_n^2-|X'|^2)\big)=\int_{\mathbb{R}}\widehat H(\tau)\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\operatorname{Op_w}(e^{-2i\pi\tau|X'|^2})d\tau\\&=\sum_{k\ge0}\int_{\mathbb{R}}\widehat H(\tau)\mathbb{P}_{k;n-1}\otimes\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})e^{-i(\arctan\tau)(2k+n-1)}(1+\tau^2)^{-\frac{(n-1)}{2}}d\tau\\&=\frac12\operatorname{Id}+\frac{1}{2i\pi}\sum_{k\ge0}\mathbb{P}_{k;n-1}\\&\quad\otimes\int_{\mathbb{R}}\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\frac1\tau\left(\frac{1-i\tau}{(1+\tau^2)^{1/2}}\right)^{2k+n-1}(1+\tau^2)^{-\frac{(n-1)}{2}}d\tau\\&=\frac12\operatorname{Id}+\frac12\sum_{k\ge0}\mathbb{P}_{k;n-1}\otimes\int_{\mathbb{R}}\operatorname{Op_w}(e^{2i\pi\tau(\xi_n-x_n^2)})\frac{(1-i\tau)^k}{i\pi\tau(1+i\tau)^{k+n-1}}d\tau.\end{aligned}$$

Let $k(x_n,y_n)$ be the kernel of the operator in the integral, we have

$$k(x_n,y_n)=e^{\frac{2i\pi}{3}(x_n^3-y_n^3)}e^{-\frac{i\pi}{6}(x_n-y_n)^3}\frac{i}{\pi(x_n-y_n)}\frac{(1+i(x_n-y_n))^k}{(1-i(x_n-y_n))^{k+n-1}}.$$

As a result, we find that P is unitarily equivalent to $\tilde{P}$, with

$$2\tilde{P} = \sum_{k\ge 0} \mathbb{P}_{k;n-1} \otimes \left(I_n + \text{convolution with } \frac{ie^{-\frac{i\pi}{6}x_n^3}}{\pi x_n} \frac{(1+ix_n)^k}{(1-ix_n)^{k+n-1}} \right).$$

We define

$$\begin{aligned}\omega_{k,n-1}(\tau) &= \frac{1}{2} + \int \frac{ie^{-\frac{i\pi}{6}t^3}}{2\pi t} \frac{(1+it)^k}{(1-it)^{k+n-1}} e^{-2i\pi t\tau} dt \\ &= \frac{1}{2} + \int \frac{e^{\frac{i\pi}{6}t^3}}{2i\pi t} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2i\pi t\tau} dt,\end{aligned}$$

and we get that

$$\tilde{P} = \sum_{k\ge 0} \mathbb{P}_{k;n-1} \otimes \omega_{k,n-1}(D_{x_n}).$$

We note that for $n = 1$, the sum is reduced to $k = 0$ with $\mathbb{P}_{0;0} = I$, so that we recover formula (4.2.2) with $\omega_{0,0} = \omega$. We find also that

$$\omega'_{k,n-1}(\tau) = \int e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2i\pi t\tau} dt, \tag{4.4.1}$$

in the sense that the inverse Fourier transform of $t \mapsto e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}}$ is the distribution derivative of $\omega_{k,n-1}$. Going back to the normalization of Lemma 4.2.3, we have, with $\eta = 2^{4/3}\pi^{2/3}\tau$,

$$\begin{aligned}G_{k,n-1}(\eta) &= \omega_{k,n-1}(\tau), \\ G'_{k,n-1}(\eta) &= 2^{-4/3}\pi^{-2/3} \int e^{\frac{i\pi}{6}t^3} \frac{(1-it)^k}{(1+it)^{k+n-1}} e^{2^{-\frac{1}{3}} i\pi^{\frac{1}{3}} t\eta} dt, \\ &\underbrace{=}_{t=\pi^{-\frac{1}{3}}2^{\frac{1}{3}}s} \frac{1}{2\pi} \int e^{\frac{is^3}{3}} \frac{(1-i\pi^{-1/3}2^{1/3}s)^k}{(1+i\pi^{-1/3}2^{1/3}s)^{k+n-1}} e^{is\eta} ds := A_{k,n-1}(\eta).\end{aligned}$$

We have $A_{0,0} = \mathtt{Ai}$ and $A_{k,n-1}$ is an entire function, real-valued on the real line; we have

$$G_{k,n-1}(\eta) = \int_{-\infty}^{\eta} A_{k,n-1}(\xi) d\xi, \quad G_{k,n-1}(+\infty) = 1.$$

Remark 4.4.4. We claim that the asymptotic properties of the functions $A_{k,n-1}$ are analogous to the properties of the standard Airy function and we have indeed from (4.4.1),

$$\omega'_{k,n-1}(\tau) = (1-iD)^k (1+iD)^{-k-n+1} \mathcal{F}^{-1}\big(e^{\frac{i\pi}{6}t^3}\big).$$

We claim as well that

$$-\frac{1}{2} < \inf_{k\ge 0, \eta\in\mathbb{R}} G(\eta) < 0, \qquad \sup_{k\ge 0, \eta\in\mathbb{R}} G(\eta) = 1,$$

so that $\tilde{P}$ is bounded on $L^2(\mathbb{R}^n)$ and

$$\int_{\xi_n \ge x_n^2 + \sum_{1\le j\le n-1}(x_j^2+\xi_j^2)} \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi \le \|u\|^2_{L^2(\mathbb{R}^n)}.$$