Chapter 5

Conics with eccentricity greater than 1

We want to consider now integrals of the Wigner distribution on “hyperbolic” convex subsets of the plane such as

$$\mathscr{C}_\sigma = \{(x,\xi) \in \mathbb{R}^2, x\xi \geq \sigma, x \geq 0\}, \tag{5.0.1}$$

where σ is a non-negative parameter. It is convenient to start with the limit-case where $\sigma = 0$ and $\mathscr{C}_0 = \{(x,\xi) \in \mathbb{R}^2, x \geq 0, \xi \geq 0\}$ (we will label $\mathscr{C}_0$ as the *quarter-plane*). The indicator function of $\mathscr{C}_0$ is $H(x)H(\xi)$ where $H = \mathbf{1}_{\mathbb{R}_+}$ is the Heaviside function.

N.B. The reader will see a great similarity between our calculations below in this section and the J. G. Wood and A. J. Bracken paper [55] (see also [4]). This article is very important for the problem at stake – Integrating the Wigner distribution on subsets of the phase space – and was a wealthy source of information for us, although as a mathematician, the author has a quite rigid relationship with calculations, and feels the need to justify formal manipulations; for instance, we may point out that the test functions used in [55] are homogeneous distributions of type

$$x_\pm^{-\frac{1}{2}+i\omega}, \quad \omega \in \mathbb{R},$$

which are not in $L^2(\mathbb{R})$ (not even in L^2_{loc}), a situation which raises some difficulties, first when you try to normalize in L^2 these test functions and also when trying to give a non-formal meaning to their images under the operator with Weyl symbol $H(x)H(\xi)$, images which are not clearly defined. In our joint paper [6] with B. Delourme and T. Duyckaerts, proving that Flandrin’s conjecture is not true, we followed numerical arguments which were quite apart from the arguments of [55]. However, in this memoir, we do follow many of the arguments of [55], along with avoiding formal calculations.

5.1 The quarter-plane, a counterexample to Flandrin’s conjecture

5.1.1 Preliminaries

We study in this section the operator

$$A_0 = \operatorname{Op}_{\mathrm{w}}(H(x)H(\xi)), \tag{5.1.1}$$

where $H = \mathbf{1}_{\mathbb{R}_+}$, that is the Weyl quantization of the characteristic function of the first quarter of the plane.

Lemma 5.1.1. *The operator A_0 given by* (5.1.1) *is bounded self-adjoint on* $L^2(\mathbb{R})$.

Proof. Since the Weyl symbol of A_0 is real-valued, A_0 is formally self-adjoint and it is enough to prove that A_0 is bounded on $L^2(\mathbb{R})$. Let us start with recalling the classical formulas

$$\begin{aligned}\widehat{H}(t) &= \frac{\delta_0(t)}{2} + \frac{1}{2i\pi}\mathrm{pv}\left(\frac{1}{t}\right),\\ \widehat{\mathrm{sign}} &= \frac{1}{i\pi}\mathrm{pv}\left(\frac{1}{t}\right),\end{aligned}$$

useful below. The kernel[1] of A_0 is

$$k_0(x,y) = H(x+y)\widehat{H}(y-x) = H(x+y)\frac{1}{2}\left(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\right). \tag{5.1.2}$$

For $\lambda > 0$, we define $A_{0,\lambda} = \big(H(x)\mathbf{1}_{[0,\lambda]}(\xi)\big)^w$, whose distribution-kernel is the $L^\infty(\mathbb{R}^{2n})$ function

$$k_{0,\lambda}(x,y) = H(x+y)e^{i\pi(x-y)\lambda}\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}.$$

We can thus notice that

$$\begin{aligned}k_{0,\lambda}(x,y) = {} & \overbrace{H(x)H(y)e^{i\pi(x-y)\lambda}\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}}^{k^\flat_{0,\lambda}(x,y)}\\ & + \underbrace{H(x+y)\big(H(-x)H(y)+H(x)H(-y)\big)\frac{\sin(\pi(x-y)\lambda)}{\pi(x-y)}e^{i\pi(x-y)\lambda}}_{k^\sharp_{0,\lambda}(x,y)},\end{aligned}$$

and the operator with distribution-kernel $k^\flat_{0,\lambda}$ is

$$H\mathrm{Op_w}(\mathbf{1}_{[0,\lambda]}(\xi))H, \text{ that is } H\mathbf{1}_{[0,\lambda]}(D)H,$$

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[1] There is no difficulty at defining the product $S\big((x+y)/2\big)T(x-y)$ for S, T tempered distributions on the real line since we may use the tensor product with

$$\begin{aligned}&\left\langle S\left(\frac{x+y}{2}\right)T(x-y), \Phi(x,y)\right\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)}\\ &= \left\langle S(x_1)\otimes T(x_2), \Phi\left(x_1+\frac{x_2}{2}, x_1-\frac{x_2}{2}\right)\right\rangle_{\mathscr{S}'(\mathbb{R}^2),\mathscr{S}(\mathbb{R}^2)}.\end{aligned}$$

However, we shall not use directly formula (5.1.2), since want to avoid formal manipulation involving for instance meaningless products such as $H(x)H(y)k_0(x,y)$. We refer the reader to Remark 5.1.2 for more details on this matter.

where H stands for the operator of multiplication by the Heaviside function H. On the other hand, the operator with distribution kernel $k^\sharp_{0,\lambda}$ is such that

$$\begin{aligned}|k^\sharp_{0,\lambda}(x,y)| &\le H(x+y)\frac{H(-x)H(y)+H(x)H(-y)}{\pi|x-y|}\\ &= H(x+y)\frac{H(-x)H(y)}{\pi(y-x)} + H(x+y)\frac{H(x)H(-y)}{\pi(x-y)}.\end{aligned}$$

According to Proposition A.5.1 in Appendix A.7, the Hardy operator and the modified Hardy operators are bounded on $L^2(\mathbb{R})$ and we obtain that, for $\phi,\psi\in\mathscr{S}(\mathbb{R}^n)$, with $H=H(x)$, $\check H = H(-x)$,

$$\begin{aligned}&\left|\iint H(x)\mathbf{1}_{[0,\lambda]}(\xi)W(\phi,\psi)(x,\xi)dxd\xi\right|\\ &\le \|H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})}+\frac12\|H\phi\|_{L^2(\mathbb{R})}\|\check H\psi\|_{L^2(\mathbb{R})}+\frac12\|\check H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})}\end{aligned}$$

so that

$$\begin{aligned}&|\langle A_0\phi,\psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})}|\\ &= \left|\iint H(x)H(\xi)\overbrace{W(\phi,\psi)(x,\xi)}^{\in\mathscr{S}(\mathbb{R}^2)}dxd\xi\right|\\ &= \lim_{\lambda\to+\infty}\left|\iint H(x)\mathbf{1}_{[0,\lambda]}(\xi)W(\phi,\psi)(x,\xi)dxd\xi\right|\\ &\le \|H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})}+\frac12\|H\phi\|_{L^2(\mathbb{R})}\|\check H\psi\|_{L^2(\mathbb{R})}\\ &\quad+\frac12\|\check H\phi\|_{L^2(\mathbb{R})}\|H\psi\|_{L^2(\mathbb{R})},\end{aligned}\tag{5.1.3}$$

yielding the L^2-boundedness of the operator A_0, and this concludes the proof of the lemma. ∎

Remark 5.1.2. That cumbersome detour with the operator $A_{0,\lambda}$ is useful to ensure that the operator A is indeed bounded on $L^2(\mathbb{R})$. The kernel k_0 of A_0 is a distribution of order 1 and the product $H(x)H(y)k_0(x,y)$ is not a priori meaningful, even when k is a Radon measure.

Even a wave-front-set approach, which would allow the product $H(x)\mathrm{pv}(1/(y-x))$, does not offer a meaning for the product $H(x)H(y)\mathrm{pv}(1/(y-x))$ since the wave-front-set of $\mathrm{pv}(1/(y-x))$ is located on the conormal of the first diagonal (i.e., $\{(x,x;\xi,-\xi)\}_{x\in\mathbb{R},\xi\in\mathbb{R}^*}$), whereas the wave-front set at $(0,0)$ of $H(x)H(y)$ contains all directions and in particular is antipodal to the conormal of the diagonal at $(0,0)$.

However, with the proven L^2-boundedness of A_0, then the products of operators HA_0H, $\check HA_0H$, $HA_0\check H$, $\check HA_0\check H$ make sense and for instance we may approximate in the strong-operator-topology the operator HA_0H by the operator $\chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon)$, where χ is a smooth function supported in $[1, +\infty)$ and equal to 1 on $[2, +\infty)$. We have indeed

$$HAH = (H - \chi(\cdot/\varepsilon))AH + \chi(\cdot/\varepsilon)A(H - \chi(\cdot/\varepsilon)) + \chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon),$$

so that for $u \in L^2(\mathbb{R})$, $HAHu = \lim_{\varepsilon\to 0_+} \chi(\cdot/\varepsilon)A\chi(\cdot/\varepsilon)u$. The operator with kernel

$$H(x+y)\chi(x/\varepsilon)\chi(y/\varepsilon)\mathrm{pv}\frac{1}{i\pi(y-x)} = \chi(x/\varepsilon)\chi(y/\varepsilon)\mathrm{pv}\frac{1}{i\pi(y-x)},$$

converges strongly towards the operator $H(\operatorname{sign} D)H$.

Proposition 5.1.3. *Let $A_0 = \mathrm{Op_w}\,(H(x)H(\xi))$ be the operator with Weyl symbol $H(x)H(\xi)$, a priori sending $\mathscr{S}(\mathbb{R})$ into $\mathscr{S}'(\mathbb{R})$. Then, A_0 can be uniquely extended to a self-adjoint bounded operator on $L^2(\mathbb{R})$ with*

$$\|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \frac{1+\sqrt{2}}{2} \approx 1.207 \tag{5.1.4}$$

N.B. The bound above can be significantly improved (see Proposition 5.4.4 for optimal bounds) and moreover we will show below that the spectrum of A_0 actually intersects $(1, +\infty)$. In fact, it is easier to start with the information that A_0 is indeed bounded on $L^2(\mathbb{R})$.

Proof. The $L^2(\mathbb{R})$-boundedness of A_0 is given by Lemma 5.1.1. We are left with proving the bound (5.1.4): we note that (5.1.3) implies

$$|\langle A_0u, u\rangle_{L^2(\mathbb{R})}| \le \|Hu\|^2_{L^2(\mathbb{R})} + \|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})},$$

proving the proposition, since the eigenvalues of the quadratic form $\mathbb{R}^2 \ni (x_1, x_2) \mapsto x_1^2 + x_1x_2$ are $(1 \pm \sqrt{2})/2$. ∎

We can do much better and actually diagonalise the operator A_0, using as in Proposition A.5.1 logarithmic coordinates on each half-line. We state a lemma on "diagonal" terms whose proof is already given above.

Lemma 5.1.4 (Diagonal terms). *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$. With H standing as well for the operator of multiplication by $H(x)$, we have*

$$HA_0H = HH(D)H = H\frac{(\mathrm{Id} + \operatorname{sign} D)}{2}H.$$

Lemma 5.1.5 (Off-diagonal terms). *Let $B_0 = 2\,\mathrm{Re}\,\check{H}A_0H = \check{H}A_0H + HA_0\check{H}$. Then, we have for all $u \in L^2(\mathbb{R})$,*

$$|\langle B_0u, u\rangle_{L^2(\mathbb{R})}| \le \frac{1}{2}\|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})}. \tag{5.1.5}$$

Proof of Lemma 5.1.5. For $u \in \mathscr{S}(\mathbb{R})$ such that $0 \notin \operatorname{supp} u$, we define for $t \in \mathbb{R}$,

$$\phi_1(t) = u(e^t)e^{t/2}, \quad \phi_2(t) = u(-e^t)e^{t/2}, \tag{5.1.6}$$

so that

$$\|Hu\|^2_{L^2(\mathbb{R})} = \|\phi_1\|^2_{L^2(\mathbb{R})},$$
$$\|\check{H}u\|^2_{L^2(\mathbb{R})} = \|\phi_2\|^2_{L^2(\mathbb{R})}.$$

We have

$$\begin{aligned}
\langle B_0 u, u\rangle_{L^2(\mathbb{R})} &= \iint \frac{H(x+y)(\check{H}(x)H(y) + H(x)\check{H}(y))}{2i\pi(y-x)} u(y)\bar{u}(x)dydx \\
&= \iint \frac{H(-e^s + e^t)e^{\frac{s+t}{2}}}{2i\pi(e^t + e^s)} \phi_1(t)\bar{\phi}_2(s)dsdt \\
&\quad - \iint \frac{H(e^s - e^t)e^{\frac{s+t}{2}}}{2i\pi(e^t + e^s)} \phi_2(t)\bar{\phi}_1(s)dsdt \\
&= \iint \frac{H(t-s)}{4i\pi\cosh(\frac{s-t}{2})} \phi_1(t)\bar{\phi}_2(s)dsdt \\
&\quad - \iint \frac{H(s-t)}{4i\pi\cosh(\frac{s-t}{2})} \phi_2(t)\bar{\phi}_1(s)dsdt,
\end{aligned}$$

so that

$$\langle B_0 u, u\rangle_{L^2(\mathbb{R})} = \langle \tilde{S}_0 * \phi_1, \phi_2\rangle_{L^2(\mathbb{R})} + \langle S_0 * \phi_2, \phi_1\rangle_{L^2(\mathbb{R})}$$

and

$$\tilde{S}_0(t) = \frac{\check{H}(t)}{4i\pi\cosh(t/2)}, \quad S_0(t) = \frac{iH(t)}{4\pi\cosh(t/2)}. \tag{5.1.7}$$

We calculate

$$\int_0^{+\infty} \frac{dt}{4\pi\cosh(t/2)} = \frac{1}{2\pi}[\arctan(\sinh(t/2))]_0^{+\infty} = \frac{1}{4} = \int_{-\infty}^0 \frac{dt}{4\pi\cosh(t/2)},$$

so that

$$|\langle B_0 u, u\rangle_{L^2(\mathbb{R})}| \le \frac{1}{2}\|\phi_1\|_{L^2(\mathbb{R})}\|\phi_2\|_{L^2(\mathbb{R})} = \frac{1}{2}\|Hu\|_{L^2(\mathbb{R})}\|\check{H}u\|_{L^2(\mathbb{R})},$$

proving the estimate of the lemma for $u \in \mathscr{S}(\mathbb{R})$ such that $0 \notin \operatorname{supp} u$. We use now that we already know that B_0 is a bounded self-adjoint operator on $L^2(\mathbb{R})$: let u be a function in $L^2(\mathbb{R})$ and let $(\phi_k)_{k\ge 1}$ be a sequence[2] in $\mathscr{S}(\mathbb{R})$ such that each ϕ_k vanishes in a neighborhood of 0 so that $\lim_k \phi_k = u$ in $L^2(\mathbb{R})$. We find that

$$\begin{aligned}
&|\langle B_0 u, u\rangle_{L^2(\mathbb{R})}|\\
&\le |\langle B_0(u-\phi_k), u\rangle_{L^2(\mathbb{R})}| + |\langle B_0\phi_k, u-\phi_k\rangle_{L^2(\mathbb{R})}| + |\langle B_0\phi_k, \phi_k\rangle_{L^2(\mathbb{R})}|\\
&\le \|B_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\big(\|u-\phi_k\|_{L^2(\mathbb{R})}\|u\|_{L^2(\mathbb{R})} + \|u-\phi_k\|_{L^2(\mathbb{R})}\|\phi_k\|_{L^2(\mathbb{R})}\big)\\
&\quad + \frac{1}{2}\|H\phi_k\|_{L^2(\mathbb{R})}\|\check{H}\phi_k\|_{L^2(\mathbb{R})},
\end{aligned}$$

providing readily the result of the lemma since the multiplication by H and $\check{H}$ are bounded operators on $L^2(\mathbb{R})$. ■

Remark 5.1.6. The estimate (5.1.5) and Lemma 5.1.4 are already improving (5.1.4), since the eigenvalues of the quadratic form $\mathbb{R}^2 \ni (x_1, x_2) \mapsto x_1^2 + \frac{1}{2}x_1x_2$ are $(2 \pm \sqrt{5})/4$, so that the right-hand side of (5.1.4) can be replaced by $(2+\sqrt{5})/4 \approx 1.059$. Anyhow, we shall provide below a diagonalisation of A_0 and optimal bounds.

N.B. We shall be a little faster in the sequel on the "cumbersome" detours to avoid formal multiplication of kernels by Heaviside functions but the reader should keep in mind that it is an important point to secure $L^2(\mathbb{R})$-boundedness *before* any further manipulation of the kernels.

5.1.2 An isometric isomorphism

Remark 5.1.7. The mapping Ψ defined by

$$\begin{aligned}
\Psi\colon L^2(\mathbb{R}) &\to L^2(\mathbb{R};\mathbb{C}^2)\\
u &\mapsto \big((Hu)(e^t)e^{t/2}, (\check{H}u)(-e^t)e^{t/2}\big)
\end{aligned} \tag{5.1.8}$$

is an isometric isomorphism of Hilbert spaces: indeed, we have

$$\|u\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |u(e^t)|^2 e^t\,dt + \int_{\mathbb{R}} |u(-e^t)|^2 e^t\,dt.$$

Moreover, if $(\phi_1, \phi_2) \in L^2(\mathbb{R};\mathbb{C}^2)$, we may define for $x \in \mathbb{R}^*$

$$u(x) = H(x)\phi_1(\ln x)x^{-1/2} + \check{H}(x)\phi_2(\ln|x|)|x|^{-1/2},$$

and we have

$$\Psi(u)(t) = \big(\phi_1(t), \phi_2(t)\big).$$

[2] Such a sequence is easy to find: a first step is to find a sequence $(\tilde{\phi}_k)_{k\ge1}$ in the Schwartz space converging in $L^2(\mathbb{R})$ towards u, then consider with a given $\omega \in C^\infty(\mathbb{R};[0,1])$ such that $\omega(t) = 0$ for $|t| \le 1$ and $\omega(t) = 1$ for $|t| \ge 2$, $\phi_k(x) = \omega(kx)\tilde{\phi}_k(x)$.

Remark 5.1.8. Using Lemma 5.1.4 and notations (5.1.6) we see that

$$\begin{aligned}\langle HA_0Hu, u\rangle_{L^2(\mathbb{R})} &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \iint \frac{1}{2i\pi}\mathrm{pv}\frac{e^{(s+t)/2}}{e^t - e^s}\phi_1(t)\bar{\phi}_1(s)dsdt\\ &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \iint \frac{1}{4i\pi}\mathrm{pv}\frac{1}{\sinh(\frac{t-s}{2})}\phi_1(t)\bar{\phi}_1(s)dsdt\\ &= \int_{\mathbb{R}} |\hat{\phi}_1(\tau)|^2\Big(\frac{1}{2} + \widehat{T}_0(\tau)\Big)d\tau,\end{aligned}$$

with

$$T_0(t) = \frac{t}{4\sinh(t/2)}\mathrm{pv}\frac{i}{\pi t}. \tag{5.1.9}$$

We have

$$\widehat{T}_0 = \mathrm{sign} * \rho_0 \quad \text{with } \rho_0(\tau) = \int \frac{t}{4\sinh(t/2)}e^{-2i\pi t\tau}dt, \tag{5.1.10}$$

and we note that the function ρ_0 belongs to $\mathscr{S}(\mathbb{R})$, as the Fourier transform of a function in $\mathscr{S}(\mathbb{R})$. Also, we have

$$\int \rho_0(\tau)d\tau = \hat{\rho}_0(0) = \frac{1}{2},$$

and this yields with $\frac{d}{d\tau}\{\frac{1}{2} + \widehat{T}_0\} = 2\rho_0$ (which follows from (5.1.10)) and

$$\frac{1}{2} + \widehat{T}_0(\tau) = 1 - \int_{\tau}^{+\infty} 2\rho_0(\tau')d\tau', \tag{5.1.11}$$

since

$$\frac{d}{d\tau}\Big\{\frac{1}{2} + \widehat{T}_0 + \int_{\tau}^{+\infty} 2\rho_0(\tau')d\tau'\Big\} = 0 \quad \text{and} \quad \lim_{\tau\to+\infty}(\mathrm{sign} * \rho_0)(\tau) = \frac{1}{2}.$$

Theorem 5.1.9. *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$. The operator A_0 is bounded self-adjoint on $L^2(\mathbb{R})$ so that we may define, with Ψ defined in (5.1.8),*

$$\widetilde{A}_0 = \Psi A_0 \Psi^{-1}.$$

The operator $\widetilde{A}_0$ is the Fourier multiplier on $L^2(\mathbb{R};\mathbb{C}^2)$ given by the matrix

$$\mathcal{M}_0(\tau) = \begin{pmatrix} \frac{1}{2} + \widehat{T}_0(\tau) & \widehat{S}_0(\tau)\\ \overline{\widehat{S}_0(\tau)} & 0\end{pmatrix}, \tag{5.1.12}$$

where T_0, S_0 are defined respectively in (5.1.9), (5.1.7). In particular, we have with $\Phi = (\phi_1, \phi_2) \in L^2(\mathbb{R};\mathbb{C}^2)$,

$$\langle \widetilde{A}_0\Phi, \Phi\rangle_{L^2(\mathbb{R};\mathbb{C}^2)} = \int_{\mathbb{R}} e^{2i\pi t\tau}\langle \mathcal{M}_0(\tau)\hat{\Phi}(\tau), \hat{\Phi}(\tau)\rangle_{\mathbb{C}^2}d\tau.$$

Remark 5.1.10. As a consequence of Theorem 5.1.9, we find that the spectrum of the self-adjoint bounded operator A_0 is the closure of the set of eigenvalues of the matrices $\mathcal{M}_0(\tau)$ when τ runs on the real line.

Proof. The proof follows readily from Remarks 5.1.7, 5.1.8 and Lemmas 5.1.4, 5.1.5. ■

Lemma 5.1.11. *Let $\mathcal{N}$ be a 2×2 Hermitian matrix*

$$\mathcal{N} = \begin{pmatrix} a_{11} & a_{12} \\ \overline{a_{12}} & 0 \end{pmatrix}.$$

Then, the eigenvalues $\lambda_- \leq \lambda_+$ of $\mathcal{N}$ are such that

$$\lambda_- < 0 < 1 < \lambda_+, \tag{5.1.13}$$

if and only if

$$a_{12} \neq 0 \quad \textit{and} \quad |a_{12}|^2 > 1 - a_{11}. \tag{5.1.14}$$

Proof. The characteristic polynomial of $\mathcal{N}$ is $p(\lambda) = \lambda^2 - a_{11}\lambda - |a_{12}|^2$ and since a_{11} is real-valued, has two real roots $\lambda_- \leq \lambda_+$. If (5.1.14) holds true, the roots are distinct and

$$p(0) = -|a_{12}|^2 < 0, \quad p(1) = 1 - a_{11} - |a_{12}|^2 < 0,$$

implying (5.1.13). Conversely, if (5.1.13) is satisfied, then $p(0)$, $p(1)$ are both negative, implying (5.1.14), completing the proof of the lemma. ■

Lemma 5.1.12. *Let us define for $\omega \in \mathbb{R}$,*

$$I(\omega) = \frac{1}{4\pi} \int_0^{+\infty} \frac{\sin(t\omega)}{\cosh(t/2)} dt.$$

Then, we have

$$I(\omega) = \frac{1}{4\pi\omega} + O(\omega^{-3}), \quad |\omega| \to +\infty.$$

Proof. Indeed, we have for $\omega \in \mathbb{R}^*$,

$$\begin{aligned} I(\omega) &= -\frac{1}{4\pi\omega} \int_0^{+\infty} \frac{\frac{d}{dt}\cos(t\omega)}{\cosh(t/2)} dt \\ &= \frac{1}{4\pi\omega}\left(1 - \int_0^{+\infty} \frac{\cos(t\omega)}{(\cosh(t/2))^2} \frac{1}{2} \sinh(t/2) dt\right) \\ &= \frac{1}{4\pi\omega}(1 + g(\omega)), \end{aligned}$$

with

$$\begin{aligned} g(\omega) &= -\int_0^{+\infty} \frac{d}{\omega dt}\{\sin(t\omega)\}\operatorname{sech}(t/2)\frac{1}{2}\tanh(t/2)dt \\ &= \frac{1}{2\omega}\int_0^{+\infty} \sin(t\omega)\frac{d}{dt}\{\operatorname{sech}(t/2)\tanh(t/2)\}dt \\ &= -\frac{1}{2\omega^2}\int_0^{+\infty} \frac{d}{dt}\{\cos(t\omega)\}\frac{d}{dt}\{\operatorname{sech}(t/2)\tanh(t/2)\}dt \\ &= \frac{1}{2\omega^2}\left\{\int_0^{+\infty} \cos(t\omega)\frac{d^2}{dt^2}\{\operatorname{sech}(t/2)\tanh(t/2)\}dt + \frac{1}{2}\right\} = O(\omega^{-2}), \end{aligned}$$

proving the lemma. ∎

Proposition 5.1.13. *The matrix $\mathcal{M}_0(\tau)$ defined in (5.1.12) is equal to*

$$\mathcal{M}_0(\tau) = \begin{pmatrix} a_{11}(\tau) & a_{12}(\tau) \\ \overline{a_{12}(\tau)} & 0 \end{pmatrix}, \tag{5.1.15}$$

with

$$1 - a_{11}(\tau) = \int_\tau^{+\infty} 2\rho_0(\tau')d\tau', \quad a_{12}(\tau) = \frac{i}{4\pi}\int_0^{+\infty} \frac{e^{-2i\pi\tau t}}{\cosh(t/2)}dt. \tag{5.1.16}$$

We have

$$1 - a_{11}(\tau) = O(\tau^{-N}) \quad \textit{for any } N \textit{ when } \tau \to +\infty, \tag{5.1.17}$$

$$\operatorname{Re}(a_{12}(\tau)) = \frac{1}{8\pi^2\tau} + O(\tau^{-3}) \quad \textit{when } \tau \to +\infty. \tag{5.1.18}$$

Proof. Formulas (5.1.15), (5.1.16) follow from Theorem 5.1.9, (5.1.11), and (5.1.7). The estimates (5.1.17) follow from the fact that ρ_0 belongs to the Schwartz class and (5.1.18) is a reformulation of Lemma 5.1.12. ∎

Theorem 5.1.14. *Let A_0 be the operator with Weyl symbol $H(x)H(\xi)$, where H is the Heaviside function. Then, A_0 is a bounded self-adjoint operator on $L^2(\mathbb{R})$ such that*

$$\inf\big(\operatorname{spectrum}(A_0)\big) < 0 < 1 < \sup\big(\operatorname{spectrum}(A_0)\big). \tag{5.1.19}$$

Proof. Using Remark 5.1.10 and Proposition 5.1.13 we find that for τ large enough, Conditions (5.1.14) are satisfied, proving readily (5.1.19). ∎

Corollary 5.1.15 (A counterexample to Flandrin's conjecture). *There exists a function $\phi_0 \in \mathscr{S}(\mathbb{R})$, with $L^2(\mathbb{R})$ norm equal to 1 such that*

$$\iint_{x\geq 0, \xi \geq 0} \mathcal{W}(\phi_0, \phi_0)(x,\xi)dxd\xi > 1.$$

There exists $a > 0$ such that $\iint_{0\leq x\leq a, 0\leq \xi\leq a} \mathcal{W}(\phi_0, \phi_0)(x,\xi)dxd\xi > 1$.

Remark 5.1.16. In [13, page 2178], we find the sentence *"it is conjectured that*

$$\forall u \in L^2(\mathbb{R}), \quad \iint_{\mathcal{C}} \mathcal{W}(u,u)(x,\xi)dxd\xi \le \|u\|^2_{L^2(\mathbb{R})}, \tag{5.1.20}$$

is true for any convex domain $\mathcal{C}$*"*, a quite mild commitment for the validity of (5.1.20), although that statement was referred to later on as *Flandrin's conjecture* in the literature. The second part of the above corollary is providing a disproof of that conjecture based upon an "abstract" argument used in the proof of Theorem 5.1.14; the result of that corollary was already known via a numerical analysis argument after our joint work [6] with B. Delourme and T. Duyckaerts.

Proof. From Theorem 5.1.14, we find $u_0 \in L^2(\mathbb{R})$ such that

$$\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0\rangle.$$

Let $\psi \in \mathscr{S}(\mathbb{R})$: we have

$$\begin{aligned}|\langle A_0 u_0, u_0\rangle - \langle A_0\psi, \psi\rangle| &= |\langle A_0(u_0-\psi), u_0\rangle + \langle A_0\psi, u_0-\psi\rangle| \\ &\le \|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\|u_0-\psi\|_{L^2(\mathbb{R})}\big(\|u_0\|_{L^2(\mathbb{R})} + \|\psi\|_{L^2(\mathbb{R})}\big),\end{aligned}$$

and thus if $(\psi_k)_{k\ge 1}$ is a sequence of $\mathscr{S}(\mathbb{R})$ converging towards u_0 in $L^2(\mathbb{R})$, we get

$$\begin{aligned}&\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0\rangle \\ &\quad\le \langle A_0\psi_k, \psi_k\rangle + \underbrace{\|A_0\|_{\mathcal{B}(L^2(\mathbb{R}))}\|u_0-\psi_k\|_{L^2(\mathbb{R})}\big(\|u_0\|_{L^2(\mathbb{R})} + \|\psi_k\|_{L^2(\mathbb{R})}\big)}_{=\sigma_k,\text{ goes to 0 when } k\to+\infty}.\end{aligned}$$

There exists $k_0 \ge 1$ such that for $k \ge k_0$, we have

$$0 \le \sigma_k \le \frac{1}{2}\big(\langle A_0 u_0, u_0\rangle - \|u_0\|^2_{L^2(\mathbb{R})}\big) = \frac{\varepsilon_0}{2}, \quad \varepsilon_0 > 0.$$

We obtain that for $k \ge k_0$,

$$\|u_0\|^2_{L^2(\mathbb{R})} < \langle A_0 u_0, u_0\rangle \le \langle A_0\psi_k, \psi_k\rangle + \frac{\varepsilon_0}{2},$$

and thus

$$\begin{aligned}\|\psi_k\|^2_{L^2(\mathbb{R})} &= \underbrace{\|\psi_k\|^2_{L^2(\mathbb{R})} - \|u_0\|^2_{L^2(\mathbb{R})}}_{=\theta_k,\text{ goes to 0 when } k\to+\infty} + \|u_0\|^2_{L^2(\mathbb{R})} \\ &= \theta_k + \langle A_0 u_0, u_0\rangle - \varepsilon_0 \le \theta_k + \langle A_0\psi_k, \psi_k\rangle + \frac{\varepsilon_0}{2} - \varepsilon_0 \\ &= \langle A_0\psi_k, \psi_k\rangle + \theta_k - \frac{\varepsilon_0}{2}.\end{aligned}$$

Choosing now $k \geq k_0$ and k large enough to have $\theta_k < \varepsilon_0/4$, we get

$$\|\psi_k\|^2_{L^2(\mathbb{R})} \leq \langle A_0\psi_k, \psi_k\rangle - \frac{\varepsilon_0}{4} < \langle A_0\psi_k, \psi_k\rangle,$$

and since for $\tilde{\phi} = \psi_k$, the Wigner distribution $\mathcal{W}(\tilde{\phi}, \tilde{\phi})$ belongs to $\mathscr{S}(\mathbb{R}^2)$, we have

$$\|\tilde{\phi}\|^2_{L^2(\mathbb{R})} < \langle A_0\tilde{\phi}, \tilde{\phi}\rangle = \iint H(x)H(\xi)\mathcal{W}(\tilde{\phi}, \tilde{\phi})(x, \xi)dxd\xi,$$

and noting that this strict inequality above implies that $\tilde{\phi} \neq 0$, we may set $\phi_0 = \tilde{\phi}/\|\tilde{\phi}\|$ and get the first statement in the corollary.

N.B. The proof above is complicated by the fact that the identity

$$\langle a^w u, u\rangle_{L^2(\mathbb{R}^n)} = \iint_{\mathbb{R}^{2n}} a(x, \xi)\mathcal{W}(u, u)(x, \xi)dxd\xi,$$

is valid a priori for $u \in \mathscr{S}(\mathbb{R}^n)$ (and in that case $\mathcal{W}(u, u)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$), but could be meaningless as a Lebesgue integral even for $\mathrm{Op_w}(a)$ bounded on $L^2(\mathbb{R}^n)$ and $u \in L^2(\mathbb{R}^n)$, since we shall have $\mathcal{W}(u, u) \in L^2(\mathbb{R}^{2n})$ but not in $L^1(\mathbb{R}^{2n})$ (we shall see in Chapter 6 that generically the Wigner distribution of a pulse u in $L^2(\mathbb{R}^n)$ does *not* belong to $L^1(\mathbb{R}^{2n})$).

Since $\mathcal{W}(\phi, \phi)$ belongs to the Schwartz space of $\mathbb{R}^2$, the Lebesgue dominated convergence theorem provides the last statement in the corollary. ∎

N.B. The reader will notice that the results of the incoming Section 5.2 in the special case $\sigma = 0$ imply the results of Section 5.1, which could be then erased, say at the second reading. However, as far as the first – and maybe only – reading is concerned, we checked that most of the computational arguments in the next section are much more involved and it seemed worthwhile to the author to avoid unnecessary complications for the disproof of Flandrin's conjecture via the quarter-plane example and set apart the more involved examples of the hyperbolic regions tackled in Section 5.2.

5.2 Hyperbolic regions

We consider in this section the (5.0.1) set $\mathcal{C}_\sigma$ with a non-negative σ.

5.2.1 A preliminary observation

We want to consider the operator A_σ with Weyl symbol $H(x)H(x\xi - \sigma)$ and as in Section 5.1.1, we would like to secure the fact that A_σ is bounded on $L^2(\mathbb{R})$.

Claim 5.2.1. For all $\sigma \geq 0$ the operator A_σ is bounded self-adjoint on $L^2(\mathbb{R})$.

Proof of the claim. Let us choose

$$\chi_0 \in C^\infty(\mathbb{R};[0,1]) \quad \text{with} \begin{cases} \chi_0(t) = 0 & \text{for } t \leq 1, \\ \chi_0(t) = 1 & \text{for } t \geq 2. \end{cases} \tag{5.2.1}$$

For $\phi, \psi \in \mathscr{S}(\mathbb{R})$, we have

$$\begin{aligned} &\langle (A_0 - A_\sigma)\phi, \psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})} \\ &= \iint H(x)H(\xi)H(\sigma - x\xi) \underbrace{\mathcal{W}(\phi,\psi)(x,\xi)}_{\in \mathscr{S}(\mathbb{R}^2)} dx d\xi \\ &= \lim_{\varepsilon \to 0_+} \iint \chi_0(x/\varepsilon)H(\xi)H(\sigma - x\xi)\mathcal{W}(\phi,\psi)(x,\xi) dx d\xi. \end{aligned} \tag{5.2.2}$$

The kernel $k_{\sigma,\varepsilon}$ of the operator with Weyl symbol $\chi_0(x/\varepsilon)H(\xi)H(\sigma - x\xi)$ is

$$\ell_{\sigma,\varepsilon}(x,y) = \chi_0\left(\frac{x+y}{2\varepsilon}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)},$$

and we have

$$\begin{aligned} &\iint \ell_{\sigma,\varepsilon}(x,y)\phi(y)\bar{\psi}(x) dy dx \\ &= \iint \chi_0\left(\frac{x+y}{2\varepsilon}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)} \phi(y)\bar{\psi}(x) dx dy \\ &= \iint \chi_0\left(\frac{x+y}{2}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi\varepsilon(x-y)} \phi(\varepsilon y)\bar{\psi}(\varepsilon x)\varepsilon^2 dx dy \\ &= \iint \underbrace{\chi_0\left(\frac{x+y}{2}\right) e^{2i\pi\sigma\frac{x-y}{x+y}} \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)}}_{m_\sigma(x,y)} \underbrace{\phi(\varepsilon y)\varepsilon^{1/2}}_{\phi_\varepsilon(y)} \underbrace{\bar{\psi}(\varepsilon x)\varepsilon^{1/2}}_{\bar{\psi}_\varepsilon(x)} dy dx. \end{aligned} \tag{5.2.3}$$

We note that, assuming as we may that $\sigma > 0$,

$$\begin{aligned} &|m_\sigma(x,y)H(x)H(y)| \\ &= \chi_0\left(\frac{x+y}{2}\right) \left| \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\frac{2\pi\sigma(x-y)}{x+y}} \right| \frac{2\sigma H(x)H(y)}{x+y} \leq \frac{2\sigma H(x)H(y)}{x+y}, \end{aligned} \tag{5.2.4}$$

and

$$|m_\sigma(x,y)\check{H}(x)H(y)| = \chi_0\left(\frac{x+y}{2}\right) \left| \frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)} \right| \check{H}(x)H(y) \leq \frac{\check{H}(x)H(y)}{\pi(y-x)}, \tag{5.2.5}$$

as well as

$$|m_\sigma(x,y)\check{H}(y)H(x)| = \chi_0\Big(\frac{x+y}{2}\Big)\left|\frac{\sin(\frac{2\pi\sigma(x-y)}{x+y})}{\pi(x-y)}\right|\check{H}(y)H(x) \le \frac{\check{H}(y)H(x)}{\pi(x-y)}. \tag{5.2.6}$$

As a consequence, since we have also $m_\sigma(x,y)\check{H}(x)\check{H}(y) \equiv 0$, the inequalities (5.2.4), (5.2.5), (5.2.6), the identities (5.2.3), (5.2.2) and Proposition A.5.1 imply that

$$|\langle (A_0 - A_\sigma)\phi, \psi\rangle_{\mathscr{S}^*(\mathbb{R}),\mathscr{S}(\mathbb{R})}| \le 2\pi\sigma \underbrace{\|H\phi_\varepsilon\|_{L^2(\mathbb{R})}}_{\|H\phi\|_{L^2(\mathbb{R})}} \|H\psi_\varepsilon\|_{L^2(\mathbb{R})}$$

$$+ \underbrace{\|\check{H}\phi_\varepsilon\|_{L^2(\mathbb{R})}}_{\|\check{H}\phi\|_{L^2(\mathbb{R})}}\|H\psi_\varepsilon\|_{L^2(\mathbb{R})} + \|H\phi_\varepsilon\|_{L^2(\mathbb{R})}\|\check{H}\psi_\varepsilon\|_{L^2(\mathbb{R})},$$

proving that $A_0 - A_\sigma$ is bounded on $L^2(\mathbb{R})$; with Proposition 5.1.3, this implies that A_σ is also bounded on $L^2(\mathbb{R})$, proving the claim. ∎

N.B. With that important piece of information in Claim 5.2.1, we shall be less strict in manipulations of kernels and accept below some abuse of language in these matters.

The Weyl quantization of $\mathbf{1}_{\mathscr{C}_\sigma}$ has the kernel

$$k_\sigma(x,y) = H(x+y)e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{1}{2}\Big(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\Big), \tag{5.2.7}$$

a formula to be compared to (5.1.2). Using the Schwartz function ϕ_0 of Corollary 5.1.15, we get from Lebesgue dominated convergence theorem that for σ small enough

$$\langle \mathrm{Op_w}(\mathbf{1}_{\mathscr{C}_\sigma})\phi_0, \phi_0\rangle_{L^2(\mathbb{R})} = \iint_{x\xi\ge\sigma, x>0} \mathcal{W}(\phi_0,\phi_0)(x,\xi)dxd\xi > 1.$$

However, this argument does not work for large positive σ and we must go back to a direct calculation.

5.2.2 Diagonal terms

Denoting by A_σ the operator with kernel (5.2.7) (and Weyl symbol $H(x\xi-\sigma)H(x)$), we find that for $u \in \mathscr{S}(\mathbb{R})$, $u_+ = Hu$, we have

$$\begin{aligned}
&\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})}\\
&= \iint e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{1}{2}\Big(\delta_0(y-x) + \frac{1}{i\pi}\mathrm{pv}\frac{1}{y-x}\Big)u_+(y)\bar{u}_+(x)dydx\\
&= \frac{1}{2}\|u_+\|^2_{L^2(\mathbb{R}_+)} + \iint_{\mathbb{R}^2} e^{4i\pi\sigma(\frac{e^s-e^t}{e^s+e^t})}\frac{1}{2i\pi}\mathrm{pv}\frac{1}{e^t-e^s}u_+(e^t)\bar{u}_+(e^s)e^{s+t}dsdt\\
&= \frac{1}{2}\|u_+\|^2_{L^2(\mathbb{R}_+)} + \iint_{\mathbb{R}^2} e^{4i\pi\sigma\tanh(\frac{s-t}{2})}\frac{1}{2i\pi}\mathrm{pv}\frac{e^{(s+t)/2}}{e^t-e^s}\phi_1(t)\bar{\phi}_1(s)dsdt,
\end{aligned}$$

with

$$\phi_1(t) = u_+(e^t)e^{t/2},$$

so that

$$\|\phi_1\|_{L^2(\mathbb{R})} = \|u_+\|_{L^2(\mathbb{R}_+)}.$$

We get

$$\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})} = \frac{1}{2}\|u\|^2_{L^2(\mathbb{R}_+)} + \frac{1}{4i\pi}\iint_{\mathbb{R}^2} \frac{e^{4i\pi\sigma\tanh(\frac{s-t}{2})}}{\sinh(\frac{t-s}{2})}\phi(t)\bar{\phi}(s)dsdt,$$

and noting that $\sinh x = xC(x)$, with C even such that $1/C \in \mathscr{S}(\mathbb{R})$, we find

$$\begin{aligned}\langle A_\sigma Hu, Hu\rangle_{L^2(\mathbb{R})} &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} - \frac{1}{2i\pi}\iint_{\mathbb{R}^2} \frac{e^{4i\pi\sigma\tanh(\frac{s-t}{2})}}{(s-t)C(\frac{s-t}{2})}\phi(t)\bar{\phi}(s)dsdt \\ &= \frac{1}{2}\|\phi_1\|^2_{L^2(\mathbb{R})} + \langle T_\sigma * \phi_1, \phi_1\rangle_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} |\hat{\phi}_1(\tau)|^2\Big(\frac{1}{2} + \widehat{T}_\sigma(\tau)\Big)d\tau, \end{aligned} \tag{5.2.8}$$

with

$$T_\sigma(t) = \frac{1}{4}\frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}\mathrm{pv}\frac{i}{\pi t}. \tag{5.2.9}$$

We note that

$$\widehat{T}_\sigma(\tau) = \mathrm{sign} * \rho_\sigma,$$

with

$$\rho_\sigma(\tau) = \frac{1}{4}\int \frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}e^{-2i\pi t\tau}dt, \quad \rho_\sigma \in \mathscr{S}(\mathbb{R}), \tag{5.2.10}$$

since the function

$$\mathbb{R} \ni t \mapsto \frac{te^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}$$

belongs to the Schwartz space[3]. Note also that the function ρ_σ is real-valued on the real line. This entails that

$$\frac{d}{d\tau}\Big\{\frac{1}{2} + \widehat{T}_\sigma\Big\} = 2\rho_\sigma, \tag{5.2.11}$$

and since

$$\rho_\sigma(\tau) = \frac{1}{4}\mathcal{F}\Big\{t \mapsto \frac{te^{4i\pi\sigma\tanh(t/2)}}{\sinh(t/2)}\Big\},$$

[3] Indeed, the iterated derivatives of tanh are polynomials of tanh (check this by induction on the order of derivatives) and thus bounded on the real line; since the function $t \mapsto t/\sinh(t/2)$ belongs to the Schwartz space, this proves that the above product is in $\mathscr{S}(\mathbb{R})$.

implying

$$\int_{\mathbb{R}} \rho_\sigma(\tau) d\tau = \frac{1}{2},$$

we get that

$$\lim_{\tau \to \pm\infty} \widehat{T}_\sigma(\tau) = \pm \frac{1}{2}. \tag{5.2.12}$$

This yields that

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) - 1 = \int_{+\infty}^{\tau} 2\rho_\sigma(\tau') d\tau' = -1 + \int_{-\infty}^{\tau} 2\rho_\sigma(\tau') d\tau',$$

where the last equality follows from (5.2.12): indeed, we have for $\tau > 0$, from (5.2.11),

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) - 1 = \int_{+\infty}^{\tau} 2\rho_\sigma(\tau') d\tau' = -1 + \int_{-\infty}^{\tau} 2\rho_\sigma(\tau') d\tau', \tag{5.2.13}$$

and for $\tau < 0$,

$$\frac{1}{2} + \widehat{T}_\sigma(\tau) = \int_{-\infty}^{\tau} 2\rho_\sigma(\tau') d\tau' = 1 + \int_{+\infty}^{\tau} 2\rho_\sigma(\tau') d\tau'.$$

We note that

$$\forall N \in \mathbb{N}, \quad \sup_{\tau \in \mathbb{R}} |\tau|^N \left| \frac{1}{2} + \widehat{T}_\sigma(\tau) - H(\tau) \right| < +\infty. \tag{5.2.14}$$

Indeed, for $\tau > 0$, we have, using $\rho_\sigma \in \mathscr{S}(\mathbb{R})$,

$$\left| \tau^N \int_{+\infty}^{\tau} \rho_\sigma(\tau') d\tau' \right| \le \int_{\tau}^{+\infty} |\rho_\sigma(\tau')| \tau'^N d\tau' \le \int_0^{+\infty} |\rho_\sigma(\tau')| \tau'^N d\tau < +\infty.$$

Also, for $\tau < 0$, we have

$$\left| \tau^N \int_{-\infty}^{\tau} \rho_\sigma(\tau') d\tau' \right| \le \int_{-\infty}^{\tau} |\rho_\sigma(\tau')| |\tau'|^N d\tau' \le \int_{-\infty}^{0} |\rho_\sigma(\tau')| |\tau'|^N d\tau < +\infty.$$

This means that the Fourier multiplier $\frac{1}{2} + \widehat{T}_\sigma(\tau)$ is somehow "exponentially close" to $H(\tau)$ for large values of $|\tau|$ and in particular close to 1 for large positive values of τ. We have also

$$\begin{aligned} \widehat{T}_\sigma(\tau) &= \frac{i}{4\pi} \int_{\mathbb{R}} e^{-2i\pi t\tau} \frac{e^{4i\pi\sigma \tanh(\frac{t}{2})}}{\sinh(t/2)} dt \\ &= \frac{1}{2\pi} \int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma \tanh(t/2))}{\sinh(t/2)} dt. \end{aligned} \tag{5.2.15}$$

The next lemma provides more precise estimates than (5.2.14).

Lemma 5.2.2. *Let $\tau > 0, \sigma \ge 0$. Defining $a_{11}(\tau,\sigma) = \frac{1}{2} + \widehat{T}_\sigma(\tau)$ as given by (5.2.9), we have*

$$|1 - a_{11}(\tau,\sigma)| \le 2e^{-\pi^2\tau}e^{4\pi\sigma}. \tag{5.2.16}$$

Proof. Using (5.2.13) and Lemma A.6.3, we find that for $\tau > 0$,

$$\begin{aligned} |1 - a_{11}(\tau,\sigma)| &\le 2\int_\tau^{+\infty} |\rho_\sigma(\tau')|d\tau' \\ &\le 2\int_\tau^{+\infty} |\rho_\sigma(\tau')|d\tau' \\ &\le 12e^{4\pi\sigma}\int_\tau^{+\infty} e^{-\pi^2\tau'}d\tau' \\ &= e^{4\pi\sigma}\frac{12}{\pi^2}e^{-\pi^2\tau}, \end{aligned}$$

entailing the sought result. ∎

5.2.3 Off-diagonal terms

We want now to check the off-diagonal terms: we have with $u \in \mathscr{S}(\mathbb{R})$,

$$\begin{gathered} u_+ = Hu, \quad u_- = \check{H}u, \\ \phi_1(t) = u_+(e^t)e^{t/2}, \quad \phi_2(t) = u_-(-e^t)e^{t/2}, \end{gathered}$$

and

$$\begin{aligned} &\langle A_\sigma \check{H}u, Hu\rangle_{L^2(\mathbb{R})} \\ &= \iint e^{4i\pi\sigma(\frac{x-y}{x+y})}\frac{H(x+y)\check{H}(y)H(x)}{2i\pi}\mathrm{pv}\frac{1}{y-x}u_-(y)\bar{u}_+(x)dydx \\ &= \iint e^{4i\pi\sigma(\frac{e^s+e^t}{e^s-e^t})}\frac{H(e^s-e^t)}{2i\pi}\mathrm{pv}\frac{1}{-e^t-e^s}\phi_2(t)\bar{\phi}_1(s)e^{\frac{t+s}{2}}dtds \\ &= \iint e^{4i\pi\sigma\coth(\frac{s-t}{2})}\frac{iH(s-t)}{4\pi}\frac{1}{\cosh(\frac{t-s}{2})}\phi_2(t)\bar{\phi}_1(s)dtds \\ &= \frac{i}{4\pi}\iint e^{4i\pi\sigma\coth(\frac{s-t}{2})}H(s-t)\frac{1}{\cosh(\frac{s-t}{2})}\phi_2(t)\bar{\phi}_1(s)dtds \\ &= \langle S_\sigma * \phi_2, \phi_1\rangle_{L^2(\mathbb{R})}, \end{aligned} \tag{5.2.17}$$

with

$$S_\sigma(t) = \frac{i}{4\pi}H(t)\frac{e^{4i\pi\sigma\coth(\frac{t}{2})}}{\cosh(\frac{t}{2})}. \tag{5.2.18}$$

We have also that

$$
\begin{aligned}
\widehat{S}_\sigma(\tau) &= \frac{i}{4\pi}\int H(t)\frac{e^{4i\pi\sigma\coth(\frac{t}{2})}}{\cosh(\frac{t}{2})}e^{-2i\pi t\tau}dt \\
&= \frac{i}{4\pi}\int_0^{+\infty}\frac{\cos(4\pi\sigma\coth(t/2)-2\pi t\tau)}{\cosh(\frac{t}{2})}dt \\
&\quad -\frac{1}{4\pi}\int_0^{+\infty}\frac{\sin(4\pi\sigma\coth(t/2)-2\pi t\tau)}{\cosh(\frac{t}{2})}dt \\
&= \frac{i}{4\pi}\int_0^{+\infty}\frac{\cos(2\pi t\tau-4\pi\sigma\coth(t/2))}{\cosh(\frac{t}{2})}dt \\
&\quad +\frac{1}{4\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau-4\pi\sigma\coth(t/2))}{\cosh(t/2)}dt. \qquad (5.2.19)
\end{aligned}
$$

Note that from (5.2.9), (5.2.10), we have

$$
\widehat{T}_\sigma(\tau) = \frac{i}{4\pi}\int\frac{e^{4i\pi\sigma\tanh(\frac{t}{2})}}{\sinh(t/2)}e^{-2i\pi t\tau}dt = \frac{1}{2\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau-4\pi\sigma\tanh(t/2))}{\sinh(t/2)}dt.
$$

5.2.4 An isometric isomorphism

Theorem 5.2.3. *Let $\sigma \ge 0$ be given, let $\mathscr{C}_\sigma$ be the set defined by (5.0.1) and let A_σ be the operator with Weyl symbol $\mathbf{1}_{\mathscr{C}_\sigma}$, (whose kernel is given by (5.2.7)). The operator A_σ is bounded self-adjoint on $L^2(\mathbb{R})$ so that we may define, with Ψ defined in (5.1.8),*

$$
\widetilde{A}_\sigma = \Psi A_\sigma \Psi^{-1}.
$$

The operator $\widetilde{A}_\sigma$ is the Fourier multiplier on $L^2(\mathbb{R};\mathbb{C}^2)$ given by the matrix

$$
\mathcal{M}_\sigma(\tau) = \begin{pmatrix} \frac{1}{2}+\widehat{T}_\sigma(\tau) & \widehat{S}_\sigma(\tau) \\ \overline{\widehat{S}_\sigma(\tau)} & 0 \end{pmatrix}, \qquad (5.2.20)
$$

where T_σ, S_σ are defined respectively in (5.2.9), (5.2.15), (5.2.18). In particular, we have with $\Phi = (\phi_1, \phi_2) \in L^2(\mathbb{R};\mathbb{C}^2)$,

$$
\langle \widetilde{A}_\sigma \Phi, \Phi\rangle_{L^2(\mathbb{R};\mathbb{C}^2)} = \int_{\mathbb{R}} e^{2i\pi t\tau}\langle \mathcal{M}_\sigma(\tau)\hat{\Phi}(\tau), \hat{\Phi}(\tau)\rangle_{\mathbb{C}^2}d\tau. \qquad (5.2.21)
$$

Proof. We have

$$
\begin{aligned}
&\operatorname{kernel}(HA_\sigma H) = e^{4i\pi\sigma\frac{x-y}{x+y}}H(x)H(y)\widehat{H}(y-x), \\
&\operatorname{kernel}(\check{H}A_\sigma H + HA_\sigma\check{H}) \\
&\quad = e^{4i\pi\sigma\frac{x-y}{x+y}}H(x+y)\big(\check{H}(x)H(y)+H(x)\check{H}(y)\big)\frac{1}{2i\pi(y-x)}, \\
&\check{H}A_\sigma\check{H} = 0.
\end{aligned}
$$

Proposition A.5.1 in Appendix A.7 is readily giving the L^2-boundedness (and self-adjointness) of

$$\check{H} A_\sigma H + H A_\sigma \check{H}.$$

We find also that $HA_\sigma H - \frac{H}{2}$ has kernel

$$e^{4i\pi\sigma \frac{x-y}{x+y}} H(x)H(y) \frac{1}{2i\pi(y-x)},$$

and thus it is enough to study the operator with kernel

$$e^{4i\pi\sigma \frac{e^s-e^t}{e^s+e^t}} \frac{e^{\frac{s+t}{2}}}{2i\pi(e^t - e^s)} = e^{4i\pi\sigma \tanh(\frac{s-t}{2})} \frac{1}{4i\pi \sinh(\frac{t-s}{2})},$$

which is a convolution operator by

$$T_\sigma(t) = e^{4i\pi\sigma \tanh(\frac{t}{2})} \frac{t}{4\sinh(\frac{t}{2})} \operatorname{pv} \frac{i}{\pi t},$$

given by (5.2.9). Formula (5.2.10) implies in particular that $\widehat{T_\sigma}$ is bounded (and real-valued) on the real line, entailing eventually the boundedness and self-adjointness of A_σ. Formulas (5.2.8), (5.2.17), and (5.2.18) are providing (5.2.21), completing the proof of the theorem. ∎

5.2.5 The main result on hyperbolic regions

Theorem 5.2.4. *Let $\sigma \ge 0$ be given and let A_σ be the operator defined in Theorem 5.2.3. Then, A_σ is a bounded self-adjoint operator on $L^2(\mathbb{R})$ such that*

$$\inf\bigl(\text{spectrum}(A_\sigma)\bigr) < 0 < 1 < \sup\bigl(\text{spectrum}(A_\sigma)\bigr).$$

The spectrum of A_σ is the closure of the set of eigenvalues of $\mathcal{M}_\sigma(\tau)$ for τ running on the real line.

Remark 5.2.5. It is enough to prove that, with a given $\sigma \ge 0$, there exists $\tau \in \mathbb{R}$ such that $\mathcal{M}_\sigma(\tau)$ satisfies (5.1.14).

Proof. We have from (5.2.20), (5.2.15), and (5.2.19),

$$\mathcal{M}_\sigma(\tau) = \begin{pmatrix} \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma \tanh(t/2))}{\sinh(t/2)} dt & \cdot & \frac{i}{4\pi} \int_0^{+\infty} \frac{e^{-2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)} dt \\ \frac{1}{4i\pi} \int_0^{+\infty} \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)} dt & \cdot & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}(\tau,\sigma) & a_{12}(\tau,\sigma) \\ a_{21}(\tau,\sigma) & a_{22}(\tau,\sigma) \end{pmatrix}. \tag{5.2.22}$$

On the other hand, we have

$$\overline{a_{12}} = a_{21} = \frac{1}{4i\pi}\int_0^{+\infty} \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)}dt, \tag{5.2.23}$$

so that

$$\operatorname{Re} a_{12}(\tau,\sigma) = \frac{1}{4\pi}\int_0^{+\infty} \frac{\sin[2\pi(t\tau - \frac{2\sigma}{\tanh(\frac{t}{2})})]}{\cosh(\frac{t}{2})}dt. \tag{5.2.24}$$

We note that the function

$$t \mapsto \frac{e^{2i\pi(t\tau - \frac{2\sigma}{\tanh(t/2)})}}{\cosh(t/2)},$$

is holomorphic on $\mathbb{C}\backslash i\pi\mathbb{Z}$, with simple poles at $(2\mathbb{Z}+1)i\pi$ (zeroes of $\cosh(t/2)$) and essential singularities at $2\mathbb{Z}i\pi$ (zeroes of $\sinh(t/2)$). We shall need a more explicit quantitative expression for a_{21} to obtain a precise asymptotic result which could be compared to the estimate (5.2.16). The next lemma is proven in [55]; we provide a proof here for the convenience of the reader.

Lemma 5.2.6. *Let $\tau > 0, \sigma \geq 0$ be given and let $a_{21}(\tau,\sigma)$ be given by (5.2.23). We have*

$$\begin{aligned}\operatorname{Re} a_{21}(\tau,\sigma) = \frac{e^{-2\pi^2\tau}}{4\pi}\Bigg\{&\int_0^{\pi}\Big(\frac{e^{2\pi(t\tau - 2\sigma\tan(t/2))}-1}{\sin(t/2)} + \frac{\sinh(t/2)-\sin(t/2)}{\sinh(t/2)\sin(t/2)}\Big)dt\\ &+\int_0^{\pi}\frac{1-\cos 2\pi(t\tau - 2\sigma\tanh(t/2))}{\sinh(t/2)}dt\\ &-\int_{\pi}^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt\Bigg\}. \end{aligned}\tag{5.2.25}$$

Proof of Lemma 5.2.6. Let $0 < \varepsilon < \pi/2 < \pi < R$ be given. We consider the closed path $\gamma_{\varepsilon,R}$ of $\mathbb{C}\backslash i\pi\mathbb{Z}$ with $\operatorname{index}_{\gamma_{\varepsilon,R}}(i\pi\mathbb{Z}) \equiv 0$,

$$\begin{aligned}\gamma_{\varepsilon,R} = {}&[\varepsilon, R]\cup[R, R+i\pi]\cup[R+i\pi, \varepsilon+i\pi]\\ &\cup\{i\pi+\varepsilon e^{i\theta}\}_{0\geq\theta\geq-\pi/2}\cup i[\pi-\varepsilon,\varepsilon]\cup\{\varepsilon e^{i\theta}\}_{\pi/2\geq\theta\geq 0},\end{aligned}\tag{5.2.26}$$

and we have

$$\oint_{\gamma_{\varepsilon,R}} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)}dz = 0. \tag{5.2.27}$$

We note as well that

$$\begin{aligned} I_2 &= \oint_{[R,R+i\pi]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)}dz = i\int_0^{\pi} \frac{e^{2i\pi((R+it)\tau - \frac{2\sigma}{\tanh(\frac{R+it}{2})})}}{\cosh(\frac{R+it}{2})}dt\\ &= ie^{2i\pi R\tau}\int_0^{\pi} e^{-2\pi t\tau}e^{-4i\pi\sigma\frac{1+e^{-R-it}}{1-e^{-R-it}}}\frac{2dt}{e^{\frac{R+it}{2}}(1+e^{-R-it})}, \end{aligned}\tag{5.2.28}$$

so that

$$|I_2| \le 2e^{-R/2} \int_0^{\pi} e^{4\pi\sigma \operatorname{Im}\left(\frac{1+e^{-R-it}}{1-e^{-R-it}}\right)} \frac{dt}{|1-e^{-R}|},$$

and since

$$\operatorname{Im}\left(\frac{1+e^{-R-it}}{1-e^{-R-it}}\right) = \operatorname{Im} \frac{(1+e^{-R-it})(1-e^{-R+it})}{|1-e^{-R-it}|^2} = \frac{-2e^{-R}\sin t}{|1-e^{-R-it}|^2} \le 0,$$

we get

$$|I_2| \le e^{-R/2} \frac{2\pi}{1-e^{-R}}, \quad \text{where } I_2 \text{ is defined in (5.2.28).} \tag{5.2.29}$$

We note for future reference the standard formulas,

$$\cosh\left(\frac{i\pi}{2}+z\right) = i\sinh z, \quad \sinh\left(\frac{i\pi}{2}+z\right) = i\cosh z, \quad \tanh\left(\frac{i\pi}{2}+z\right) = \coth z, \tag{5.2.30}$$

and we check now

$$\begin{aligned} I_4 &= -\int_{-\pi/2}^{0} \frac{e^{2i\pi((i\pi+\varepsilon e^{i\theta})\tau - 2\sigma\coth(\frac{i\pi+\varepsilon e^{i\theta}}{2}))}}{\cosh \frac{i\pi+\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta} d\theta \\ &= -e^{-2\pi^2\tau} \int_{-\pi/2}^{0} \frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\tanh(\frac{\varepsilon e^{i\theta}}{2}))}}{i\sinh \frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta} d\theta, \end{aligned} \tag{5.2.31}$$

and since

$$\left| \frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\tanh(\frac{\varepsilon e^{i\theta}}{2}))}}{i\sinh \frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta} \right| \le 2 \max_{|z|\le\pi/2} \left| \frac{z}{\sinh z} \right| e^{\pi^2\tau} e^{4\pi\sigma \sup_{|z|\le\pi/4} \left|\frac{\sinh z}{\cosh z}\right|},$$

the Lebesgue dominated convergence theorem gives

$$\lim_{\varepsilon\to 0_+} I_4 = -\pi e^{-2\pi^2\tau}. \tag{5.2.32}$$

Defining now

$$I_6 = -\int_0^{\pi/2} \frac{e^{2i\pi(\varepsilon e^{i\theta}\tau - 2\sigma\coth(\frac{\varepsilon e^{i\theta}}{2}))}}{\cosh \frac{\varepsilon e^{i\theta}}{2}} i\varepsilon e^{i\theta} d\theta, \tag{5.2.33}$$

and noting that

$$\begin{aligned} 4\pi\sigma \operatorname{Im}\coth\left(\frac{\varepsilon e^{i\theta}}{2}\right) &= 4\pi\sigma \operatorname{Im} \frac{1+e^{-\varepsilon e^{i\theta}}}{1-e^{-\varepsilon e^{i\theta}}} = 4\pi\sigma \operatorname{Im} \frac{(1+e^{-\varepsilon e^{i\theta}})(1-e^{-\varepsilon e^{-i\theta}})}{|1-e^{-\varepsilon e^{i\theta}}|^2} \\ &= 4\pi\sigma \operatorname{Im} \frac{e^{-\varepsilon e^{i\theta}} - e^{-\varepsilon e^{-i\theta}}}{|1-e^{-\varepsilon e^{i\theta}}|^2} = 4\pi\sigma \operatorname{Im} \frac{e^{-\varepsilon\cos\theta}(e^{-i\varepsilon\sin\theta} - e^{i\varepsilon\sin\theta})}{|1-e^{-\varepsilon e^{i\theta}}|^2} \\ &= 4\pi\sigma e^{-\varepsilon\cos\theta} \operatorname{Im} \frac{(-2i)\sin(\varepsilon\sin\theta)}{|1-e^{-\varepsilon e^{i\theta}}|^2} = -4\pi\sigma e^{-\varepsilon\cos\theta} \frac{2\sin(\varepsilon\sin\theta)}{|1-e^{-\varepsilon e^{i\theta}}|^2} \le 0, \end{aligned}$$

we get that

$$|I_6| \le \int_0^{\pi/2} \frac{e^{-2\pi\varepsilon\tau\sin\theta}}{\min_{|z|\le\pi/4}|\cosh z|} d\theta\varepsilon \le \varepsilon \frac{\pi/2}{\min_{|z|\le\pi/4}|\cosh z|},$$

entailing

$$\lim_{\varepsilon\to 0_+} I_6 = 0. \tag{5.2.34}$$

With

$$I_1 = \oint_{[\varepsilon,R]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz, \tag{5.2.35}$$

we have from (5.2.23)

$$\lim_{\substack{\varepsilon\to 0_+ \\ R\to+\infty}} I_1 = 4i\pi a_{21}. \tag{5.2.36}$$

We define now

$$\begin{aligned} I_5 &= -\oint_{[i\varepsilon, i(\pi-\varepsilon)]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz = -\int_\varepsilon^{\pi-\varepsilon} \frac{e^{2i\pi(it\tau - \frac{2\sigma}{\tanh(it/2)})}}{\cosh(it/2)} i\,dt \\ &= -\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi t\tau} \frac{e^{\frac{-4i\pi\sigma}{i\tan(t/2)}}}{\cos(t/2)} i\,dt = -i\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi t\tau} \frac{e^{\frac{-4\pi\sigma}{\tan(t/2)}}}{\cos(t/2)} dt \\ &= -i\int_\varepsilon^{\pi-\varepsilon} e^{-2\pi(\pi-s)\tau} \frac{e^{-\frac{4\pi\sigma}{\tan((\pi-s)/2)}}}{\cos((\pi-s)/2)} ds \\ &= -ie^{-2\pi^2\tau}\int_\varepsilon^{\pi-\varepsilon} e^{2\pi s\tau} \frac{e^{-\frac{4\pi\sigma\sin(s/2)}{\cos(s/2)}}}{\sin(s/2)} ds, \end{aligned}$$

so that

$$I_5 = -ie^{-2\pi^2\tau}\int_\varepsilon^{\pi-\varepsilon} e^{2\pi s\tau} \frac{e^{-4\pi\sigma\tan(s/2)}}{\sin(s/2)} ds. \tag{5.2.37}$$

We have also

$$\begin{aligned} I_3 &= \oint_{[R+i\pi, \varepsilon+i\pi]} \frac{e^{2i\pi(z\tau - \frac{2\sigma}{\tanh(z/2)})}}{\cosh(z/2)} dz \\ &= -\int_\varepsilon^R \frac{e^{2i\pi((t+i\pi)\tau - \frac{2\sigma}{\tanh((t+i\pi)/2)})}}{\cosh((t+i\pi)/2)} dt, \end{aligned} \tag{5.2.38}$$

so that using Formulas (5.2.30), we get

$$I_3 = -e^{-2\pi^2\tau}\int_\varepsilon^R \frac{e^{2i\pi(t\tau - 2\sigma\tanh(t/2))}}{i\sinh(t/2)} dt,$$

and

$$I_3 + I_5 = ie^{-2\pi^2\tau}\left(\int_{\varepsilon}^{R} \frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt - \int_{\varepsilon}^{\pi-\varepsilon} e^{2\pi t\tau}\frac{e^{-4\pi\sigma\tan(t/2)}}{\sin(t/2)}dt\right)$$
$$= ie^{-2\pi^2\tau}\left\{\int_{\varepsilon}^{\pi-\varepsilon}\left(\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} - \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt + \int_{\pi-\varepsilon}^{R}\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\}. \tag{5.2.39}$$

From (5.2.27), (5.2.26), (5.2.28), (5.2.31), (5.2.33), (5.2.35), and (5.2.37), (5.2.38), we find that

$$I_1 = -I_2 - (I_3 + I_5) - I_4 - I_6,$$

so that taking the limit of both sides[4] when $\varepsilon \to 0_+$, $R \to +\infty$ we get, thanks to (5.2.36), (5.2.29), (5.2.39), (5.2.32), and (5.2.34),

$$4i\pi a_{21} = -ie^{-2\pi^2\tau}\left\{\int_0^{\pi}\left(\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} - \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt + \int_{\pi}^{+\infty}\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\} + \pi e^{-2\pi^2\tau},$$

implying that

$$a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\left\{\int_0^{\pi}\left(-\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)} + \frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)}\right)dt - \int_{\pi}^{+\infty}\frac{e^{2i\pi(t\tau-2\sigma\tanh(t/2))}}{\sinh(t/2)}dt\right\} - \frac{i}{4}e^{-2\pi^2\tau}$$

that is

$$\begin{aligned} a_{21} = {} & \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\left(\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)} - \frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}\right)dt \\ & - \frac{e^{-2\pi^2\tau}}{4\pi}\int_{\pi}^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt \\ & - i\frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt - \frac{i}{4}e^{-2\pi^2\tau} \\ & - i\frac{e^{-2\pi^2\tau}}{4\pi}\int_{\pi}^{+\infty}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt, \end{aligned} \tag{5.2.40}$$

[4] $I_1, I_2, I_4, I_6, I_3 + I_5$ do have limits when $\varepsilon \to 0_+$, $R \to +\infty$.

yielding

$$\operatorname{Re} a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\left(\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}}{\sin(t/2)} - \frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}\right)dt$$
$$- \frac{e^{-2\pi^2\tau}}{4\pi}\int_{\pi}^{+\infty}\frac{\cos 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt,$$

completing the proof of Lemma 5.2.6. ∎

Remark 5.2.7. Formula (5.2.40) also yields

$$\operatorname{Im} a_{12} = -\operatorname{Im} a_{21} = \frac{e^{-2\pi^2\tau}}{4\pi}\left\{\int_0^{\pi}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt + \pi + \int_{\pi}^{+\infty}\frac{\sin 2\pi(t\tau-2\sigma\tanh(t/2))}{\sinh(t/2)}dt\right\},$$

and since from (5.2.22), we have

$$a_{11} = \frac{1}{2} + \frac{1}{2\pi}\int_0^{+\infty}\frac{\sin(2\pi t\tau - 4\pi\sigma\tanh(t/2))}{\sinh(t/2)}dt,$$

this gives

$$\operatorname{Im} a_{12} = \frac{e^{-2\pi^2\tau}}{4\pi}\left(2\pi\left(a_{11}-\frac{1}{2}\right)+\pi\right) = \frac{e^{-2\pi^2\tau}}{2}a_{11}. \tag{5.2.41}$$

To complete the proof of Theorem 5.2.4, it will be enough, according to Lemma 5.1.11, to prove that, for $\tau \to +\infty$, $|a_{12}|^2 \gg 1 - a_{11}$. To achieve that, we note from (5.2.41) that the imaginary part of a_{12} is useless and we shall prove simply that

$$(\operatorname{Re} a_{12})^2 \gg 1 - a_{11}.$$

To get this we are going to use (5.2.16) and a precise asymptotic behavior for $(\operatorname{Re} a_{12})^2$ displayed in the next lemma and issued from the explicit formula (5.2.25).

Lemma 5.2.8. *Let $\tau \geq 1, \sigma \geq 0$ be given and let $a_{21}(\tau,\sigma)$ be given by (5.2.23). We have then*

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau} - \frac{1}{2\pi}e^{-2\pi^2\tau}. \tag{5.2.42}$$

Proof of Lemma 5.2.8. Since for $t \geq 0$ we have $\sinh(t/2) - \sin(t/2) \geq 0$, we get from (5.2.25),

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-2\pi^2\tau}}{4\pi}\left\{\int_0^{\pi}\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}-1}{\sin(t/2)}dt - \int_{\pi}^{+\infty}\frac{1}{\sinh(t/2)}dt\right\}$$
$$= \frac{e^{-2\pi^2\tau}}{4\pi}\int_0^{\pi}\frac{e^{2\pi(t\tau-2\sigma\tan(t/2))}-1}{\sin(t/2)}dt - \frac{e^{-2\pi^2\tau}}{2\pi}\ln\left(\coth\frac{\pi}{4}\right).$$

Let us define

$$\omega = 2\pi\tau, \quad \kappa = 2\pi\sigma, \quad \nu = \kappa^{1/2}\omega^{-1/2}, \quad \phi_\nu(s) = s - \nu^2 \tan s. \tag{5.2.43}$$

We have

$$2\pi\big(t\tau - 2\sigma\tan(t/2)\big) = 2\pi\tau(t - 2\nu^2\tan(t/2)) = 4\pi\tau\left(\frac{t}{2} - \nu^2\tan\frac{t}{2}\right) = 2\omega\phi_\nu(t/2).$$

We have thus

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega\phi_\nu(s)}-1}{\sin s}ds - \frac{e^{-\pi\omega}}{2\pi}\underbrace{\ln\left(\coth\frac{\pi}{4}\right)}_{\approx 0.421908}. \tag{5.2.44}$$

Defining

$$\psi_\nu(\omega) = \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega\phi_\nu(s)}-1}{\sin s}ds, \tag{5.2.45}$$

we can use (5.2.43), (5.2.44), and (A.6.13) to get whenever $\tau > 0$,

$$2\pi\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{\pi^2\tau}\left(\frac{1}{2} - \frac{1}{4\tau}\right) - e^{-2\pi^2\tau},$$

so that for $\tau \geq 1$ we find

$$2\pi\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{4\pi^2\tau} - e^{-2\pi^2\tau},$$

yielding the lemma. ■

We eventually go back to the proof of Theorem 5.2.4: let $\sigma > 0$ be given. From Lemma 5.2.8 and (5.2.16), we have for $\tau \geq 1$,

$$|1 - a_{11}(\tau,\sigma)| \leq 2e^{-\pi^2\tau}e^{4\pi\sigma},$$

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau} - \frac{1}{2\pi}e^{-2\pi^2\tau} = \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{8\pi^3\tau}\left(1 - \frac{4\pi^2\tau e^{8\pi\sqrt{\tau}\sqrt{\sigma}}}{e^{2\pi^2\tau}}\right).$$

This entails that for $\tau \geq \tau_0(\sigma)$, we have

$$\operatorname{Re} a_{21}(\tau,\sigma) \geq \frac{e^{-8\pi\sqrt{\tau}\sqrt{\sigma}}}{16\pi^3\tau}, \tag{5.2.46}$$

and thus $a_{21} \neq 0$ and

$$|a_{21}(\sigma,\tau)|^2 \geq \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2} > |1 - a_{11}(\tau,\sigma)|, \tag{5.2.47}$$

where the last inequality above holds true (thanks to (5.2.16)) whenever

$$2e^{-\pi^2\tau}e^{4\pi\sigma} < \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2},$$

which is indeed true for $\tau \ge \tau_1(\sigma)$. As a result for $\tau \ge \max(4\sigma, 4, \tau_0(\sigma), \tau_1(\sigma))$, we obtain that (5.2.47) is satisfied so that Remark 5.2.5 implies the result of Theorem 5.2.4, completing our proof. ∎

Remark 5.2.9. The functions $\tau_0(\sigma), \tau_1(\sigma)$ can be determined rather easily, the first one by the condition

$$\tau \ge \tau_0(\sigma) \Longrightarrow \frac{4\pi^2\tau e^{8\pi\sqrt{\tau}\sqrt{\sigma}}}{e^{2\pi^2\tau}} \le \frac{1}{2},$$

whereas the second one should satisfy

$$\tau \ge \tau_1(\sigma) \Longrightarrow e^{4\pi\sigma}2^9\pi^6\tau^2 e^{16\pi\sqrt{\tau}\sqrt{\sigma}} < e^{\pi^2\tau}.$$

5.3 Comments and further results

5.3.1 Qualitative explanations on the various computations

We would like to go back to our proofs that

$$|a_{12}(\tau,\sigma)|^2 \gg |1 - a_{11}(\tau,\sigma)|, \quad \tau \to +\infty, \tag{5.3.1}$$

which is our key argument via Lemma 5.1.11 and give a couple of qualitative explanations which may enlighten the calculations. It is of course much simpler to begin with the case $\sigma = 0$: in that case, according to Proposition 5.1.13 and (5.1.10), we have

$$1 - a_{11}(\tau, 0) = \int_\tau^{+\infty} 2\rho_0(\tau')d\tau', \quad 2\rho_0(\tau) = \int \underbrace{\left(\frac{t/2}{\sinh(t/2)}\right)}_{\substack{=f_0(t),\ f_0 \in \mathscr{S}(\mathbb{R}) \\ \text{holomorphic} \\ \text{on } |\operatorname{Im} t| < 2\pi.}} e^{-2i\pi t\tau}ds,$$

so that $2\rho_0(\tau) = \hat{f}_0(\tau)$. We get thus readily that ρ_0 belongs to the Schwartz space, as the Fourier transform of a function in the Schwartz space and this implies in particular that $1 - a_{11}(\tau, 0)$ has fast decay towards 0 when $\tau \to +\infty$, as proven in Proposition 5.1.13. We note also that (5.2.41) gives $\operatorname{Im} a_{12}(\tau, 0)^2 = e^{-4\pi^2\tau}a_{11}(\tau, 0)^2/4$, and since the limit of a_{11} is 1, we do not expect any help from the imaginary part of a_{12} to proving (5.3.1). Turning our attention to $\operatorname{Re} a_{12}$ in (5.1.18), we have

$$4\pi \operatorname{Re} a_{21}(\tau, 0) = \int_0^{+\infty} \frac{\sin(2\pi t\tau)}{\cosh(t/2)}dt, \tag{5.3.2}$$

which is the sine-Fourier transform of the function $t \mapsto H(t)\,\mathrm{sech}(t/2) = g_0(t)$, which has a singularity at $t = 0$: as a consequence, thanks to Lemma A.1.1, the Fourier transform $\widehat{g_0}$ cannot be rapidly decreasing, cannot even belong to $L^1(\mathbb{R})$ (that would imply that g_0 is continuous). Moreover, the sine-Fourier transform above is the Fourier transform of the odd part of g_0, $g_{\mathrm{odd}}(t) = \mathrm{sech}(t/2)\,\mathrm{sign}\, t$, which is also singular at 0, thus $\widehat{g_{\mathrm{odd}}}$ cannot be rapidly decreasing and is an odd function, which is enough to prove, without more calculations, that (5.3.1) holds true. In Section 5.1, we used a more explicit argument, with providing an equivalent of (5.3.2) equal to $1/(2\pi\tau)$ near $+\infty$. Summing-up, (5.3.1) in the case $\sigma = 0$ follows from the existence of a singularity of the function g_0 above, which is discontinuous at 0.

Let us now take a look at the case $\sigma > 0$, which turns out to be more computationally involved. We have from (5.2.23)

$$4\pi i a_{21}(\tau, \sigma) = \int_{\mathbb{R}} H(t)\,\mathrm{sech}(t/2) e^{-i4\pi\sigma\coth(t/2)} e^{2i\pi t\tau} dt = \check{\widehat{g_\sigma}}(\tau),$$
$$g_\sigma(t) = H(t)\,\mathrm{sech}(t/2) e^{-i4\pi\sigma\coth(t/2)}.$$

The single discontinuity at $t = 0$ of g_σ when $\sigma > 0$ is much wilder than for $\sigma = 0$: in the latter case, we had only a jump discontinuity with different limits on both sides, whereas when $\sigma > 0$, we have an essential discontinuity with an oscillatory behaviour in $(-1, +1)$ when $t \to 0_+$ for the real and imaginary parts of a_{12}. However, g_σ belongs to all $L^p(\mathbb{R})$, $p \in [1, +\infty]$, so that its Fourier transform belongs to $L^p(\mathbb{R})$, $p \in [2, +\infty]$: we expect then that both sides of (5.3.1) have limit 0 for $\tau \to +\infty$ and we must prove that $1 - a_{11}$ decays much faster than a_{12}. Looking at a slightly simplified model and using the notations (5.2.43), we define for ω, ν positive, a function α presumably close to $4\pi i a_{21}$, given by

$$\alpha(\omega, \nu) = \int_0^{+\infty} e^{i2\omega\mu_\nu(s)}\,\mathrm{sech}(s) ds, \quad \mu_\nu(s) = s - \frac{\nu^2}{s}, \quad \mu'_\nu(s) = 1 + \frac{\nu^2}{s^2}.$$

Trying our hand with the stationary phase method, we look at

$$\begin{aligned}
\alpha(\omega, \nu) &= \frac{1}{2i\omega} \int_0^{+\infty} \frac{d}{ds}\{e^{i2\omega\mu_\nu(s)}\} \frac{\mathrm{sech}(s)}{\mu'_\nu(s)} ds \\
&= \frac{1}{2i\omega} \int_0^{+\infty} \frac{d}{ds}\{e^{i2\omega\mu_\nu(s)}\} \frac{s^2\,\mathrm{sech}(s)}{s^2 + \nu^2} ds \\
&= \frac{i}{2\omega} \int_0^{+\infty} e^{i2\omega\mu_\nu(s)} \frac{d}{ds}\left\{\frac{s^2\,\mathrm{sech}(s)}{s^2 + \nu^2}\right\},
\end{aligned}$$

since the boundary term vanishes. Iterating that computation shows that $\alpha(\omega, \nu) = O_\sigma(\omega^{-N})$ for all N when $\omega \to +\infty$, meaning that the information of fast decay for $1 - a_{11}$ will not suffice to get (5.3.1). Also, it is worth noticing that no fast decay of the function α occurs when $\omega \to -\infty$, otherwise Lemma A.1.1 would give smoothness for the function $s \mapsto e^{-2i\kappa/s} H(s) \operatorname{sech} s$: in fact, we see also that for $\sigma > 0$, $\tau = -\lambda$, $\lambda > 0$, we have

$$2\pi i a_{21}(-\lambda, \sigma) = \int_0^{+\infty} \operatorname{sech}(s) e^{-i4\pi\sigma \coth(s)} e^{-4i\pi s\lambda} ds,$$

and the phase function is $\tilde{\mu}(s) = -4i\pi(s\lambda + \sigma \coth(s))$ and we have

$$\frac{d}{ds}\{s\lambda + \sigma \coth(s)\} = \lambda - \frac{\sigma(1 - \tanh^2 s)}{\tanh^2 s} = \frac{(\lambda + \sigma)\tanh^2 s - \sigma}{\tanh^2 s},$$

which does vanish at $\tanh s = \sigma/(\lambda + \sigma)$. As a result we could say that, for $\sigma > 0$, the C^∞ wave-front-set (see, e.g., [23, Section 8.1]) of the function g_σ is reduced to $\{0\} \times (-\infty, 0)$. It turns out that we can show that the Gevrey-2 wave-front-set of g_σ is $\{0\} \times \mathbb{R}^*$, and it is expressed via the lowerbound estimate (5.2.42); the route that we took for proving this was an explicit calculation of $\operatorname{Re} a_{12}$, following the paper [55]. Finally, the upper bound (5.2.16) can be improved as

$$|1 - a_{11}(\tau, \sigma)| \le C_{\sigma,\varepsilon} e^{-(\pi-\varepsilon)2\pi\tau}, \quad \varepsilon > 0,$$

and is expressing the fact that function

$$t \mapsto \frac{t e^{4i\pi\sigma \tanh(\frac{t}{2})}}{\sinh(\frac{t}{2})}$$

is analytic on the real line, with a radius of convergence on the real line bounded below by π (cf. Proposition A.1.2).

5.3.2 More results and examples: ℓ^p balls, corners

For a, ϕ_0 like in Corollary 5.1.15, defining

$$\Omega_p = \left\{(x, \xi) \in \mathbb{R}^2, \left|x - \frac{a}{2}\right|^p + \left|\xi - \frac{a}{2}\right|^p < \left(\frac{a}{2}\right)^p\right\},$$

since $\mathcal{W}(\phi_0, \phi_0) \in \mathscr{S}(\mathbb{R}^2)$, we get

$$\lim_{p \to +\infty} \iint_{\Omega_p} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi = \iint_{[0,a]^2} \mathcal{W}(\phi_0, \phi_0)(x, \xi) dx d\xi > \|\phi_0\|^2_{L^2(\mathbb{R})},$$

proving that the spectrum of $\operatorname{Op_w}(\mathbf{1}_{\Omega_p})$ intersects $(1, +\infty)$ for p large enough, showing that a counterexample to Flandrin's conjecture can be a convex analytic open bounded set. Moreover, defining

$$Q_a = \{(x, \xi) \in \mathbb{R}^2, |x| + |\xi| \le a/\sqrt{2}\},$$

we note that Q_a is obtained by rotation and translation of $[0,a]^2$ so that we can find ϕ_1 in the Schwartz space such that

$$\iint_{Q_a} \mathcal{W}(\phi_1,\phi_1)(x,\xi)dxd\xi > \|\phi_1\|^2_{L^2(\mathbb{R})}.$$

Since we have

$$\lim_{p\to 1}\iint_{|x|^p+|\xi|^p\le(a/\sqrt{2})^p} \mathcal{W}(\phi_1,\phi_1)(x,\xi)dxd\xi$$
$$= \iint_{Q_a} \mathcal{W}(\phi_1,\phi_1)(x,\xi)dxd\xi > \|\phi_1\|^2_{L^2(\mathbb{R})},$$

we get that for $p-1$ small enough we have

$$\iint_{|x|^p+|\xi|^p\le(a/\sqrt{2})^p} \mathcal{W}(\phi_1,\phi_1)(x,\xi)dxd\xi > \|\phi_1\|^2_{L^2(\mathbb{R})},$$

proving that ℓ^p balls are counterexamples to Flandrin's conjecture for $p-1$ or $1/p$ small enough.

Convex affine cones with aperture strictly less than π *of* $\mathbb{R}^2$ *are translations and rotations of*

$$\Sigma_{\theta_0} = \big\{(x,\xi)\in\mathbb{R}^2\backslash(\mathbb{R}_-\times\{0\}), \arg(x+i\xi)\in(0,\theta_0)\big\} \quad \text{for some } \theta_0\in(0,\pi). \tag{5.3.3}$$

The vertex of Σ_{θ_0} and its rotations is defined as 0 and the vertex of the translation of vector T_0 of Σ_{θ_0} is defined as T_0. We note that all convex affine cones with aperture strictly less than π are symplectically equivalent in $\mathbb{R}^2$, since Σ_{θ_0} is symplectically equivalent to (the interior of) the quarter plane $\Sigma_{\pi/2}$: indeed, let θ_0 be in $(0,\pi)$; the symplectic matrix M_{θ_0} defined by

$$M_{\theta_0} = \begin{pmatrix} 1 & -\operatorname{cotan}\theta_0 \\ 0 & 1 \end{pmatrix},$$

is such that $M_{\theta_0}\binom{1}{0} = \binom{1}{0}$, $M_{\theta_0}\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix} = \binom{0}{\sin\theta_0}$, proving that

$$M_{\theta_0}\Sigma_{\theta_0} = \Sigma_{\pi/2}.$$

The next result follows from [6, Theorem 1.3] and shows that many counterexamples to Flandrin's conjecture can be obtained.

Theorem 5.3.1. *Let* K *be a subset of the closure of a convex affine cone with aperture strictly less than* π *and vertex* X_0 *such that* K *contains a neighborhood of the vertex* *in the cone*[5]*. Then, there exists* $\lambda > 0$ *such that, with*

$$K_\lambda = X_0 + \lambda(K - X_0),$$

there exists $\phi \in \mathscr{S}(\mathbb{R})$ *such that*

$$\iint_{K_\lambda} \mathcal{W}(\phi,\phi)(x,\xi)dxd\xi > \|\phi\|^2_{L^2(\mathbb{R})}. \tag{5.3.4}$$

N.B. Note that (5.3.4) implies that ϕ is not the zero function. Also, taking K convex produces another counterexample to Flandrin's conjecture since K_λ will be then convex, but we do not need that assumption to proving the result.

Proof. There is no loss of generality at assuming $X_0 = 0$ and

$$[0, \rho_0]^2 \subset K \subset \overline{\Sigma}_{\pi/2}, \quad \rho_0 > 0.$$

Using Corollary 5.1.15, we find $\phi_0 \in \mathscr{S}(\mathbb{R})$ (so that $\mathcal{W}(\phi_0,\phi_0) \in \mathscr{S}(\mathbb{R}^2)$) such that

$$\lim_{\lambda\to+\infty} \iint_{K_\lambda} \mathcal{W}(\phi_0,\phi_0)(x,\xi)dxd\xi = \iint_{\Sigma_{\pi/2}} \mathcal{W}(\phi_0,\phi_0)(x,\xi)dxd\xi > \|\phi_0\|^2_{L^2(\mathbb{R})},$$

implying for λ large enough that $\iint_{K_\lambda} \mathcal{W}(\phi_0,\phi_0)(x,\xi)dxd\xi > \|\phi_0\|^2_{L^2(\mathbb{R})}$, which is the sought result. ∎

5.4 Numerics

Definition 5.4.1. Let $\sigma \geq 0$ be given. With the 2×2 Hermitian matrix $\mathcal{M}_\sigma$ given by (5.2.22), we define for $\tau \in \mathbb{R}$,

$$\lambda_+(\tau,\sigma) = \frac{1}{2}\Big(a_{11}(\tau,\sigma) + \sqrt{a_{11}^2(\tau,\sigma) + 4|a_{12}(\tau,\sigma)|^2}\Big),$$
$$\lambda_-(\tau,\sigma) = \frac{1}{2}\Big(a_{11}(\tau,\sigma) - \sqrt{a_{11}^2(\tau,\sigma) + 4|a_{12}(\tau,\sigma)|^2}\Big).$$

Remark 5.4.2. According to (5.2.41), we have

$$\lambda_+(\tau,\sigma) = \frac{1}{2}\Big(a_{11}(\tau,\sigma) + \sqrt{a_{11}^2(\tau,\sigma)(1 + e^{-4\pi^2\tau}) + 4(\operatorname{Re} a_{12}(\tau,\sigma))^2}\Big), \tag{5.4.1}$$
$$\lambda_-(\tau,\sigma) = \frac{1}{2}\Big(a_{11}(\tau,\sigma) - \sqrt{a_{11}^2(\tau,\sigma)\big(1 + e^{-4\pi^2\tau}\big) + 4(\operatorname{Re} a_{12}(\tau,\sigma))^2}\Big), \tag{5.4.2}$$

so that the knowledge of a_{11} and $\operatorname{Re} a_{12}$ suffices for expressing $\lambda_\pm$.

[5]We shall say that the set K has a corner.

An immediate consequence of Theorem 5.2.4 is the following theorem.

Theorem 5.4.3. *Let $\sigma \ge 0$ be given and let A_σ be the self-adjoint operator bounded in $L^2(\mathbb{R})$ defined in Theorem 5.2.4. With the notations of Definition 5.4.1, we have*

$$M_\sigma := \sup\{\text{spectrum}(A_\sigma)\} = \sup_{\tau\in\mathbb{R}} \lambda_+(\tau, \sigma), \tag{5.4.3}$$

$$m_\sigma := \inf\{\text{spectrum}(A_\sigma)\} = \inf_{\tau\in\mathbb{R}} \lambda_-(\tau, \sigma). \tag{5.4.4}$$

Moreover, for all $\sigma \ge 0$ we have

$$m_\sigma < 0 < 1 < M_\sigma.$$

5.4.1 The quarter-plane: $\sigma = 0$

Of course, as shown by the respective calculations of Sections 5.1 and 5.2, the case $\sigma = 0$, dealing with the quarter-plane is much simpler than the cases where $\sigma > 0$. Nonetheless, we know explicitly a spectral decomposition of the operator with Weyl symbol $H(x)H(\xi)$ from Theorem 5.2.3, but we can calculate without difficulty numerical expressions of M_0, m_0 as defined in (5.4.3), (5.4.4).

Proposition 5.4.4. *We have from (A.6.22), (5.2.24),*

$$a_{11}(\tau, 0) = \frac{1}{1 + e^{-4\pi^2\tau}}, \quad \operatorname{Re} a_{12}(\tau, 0) = \frac{1}{4\pi}\int_0^{+\infty} \sin(2\pi t\tau)\,\operatorname{sech}(t/2)dt,$$

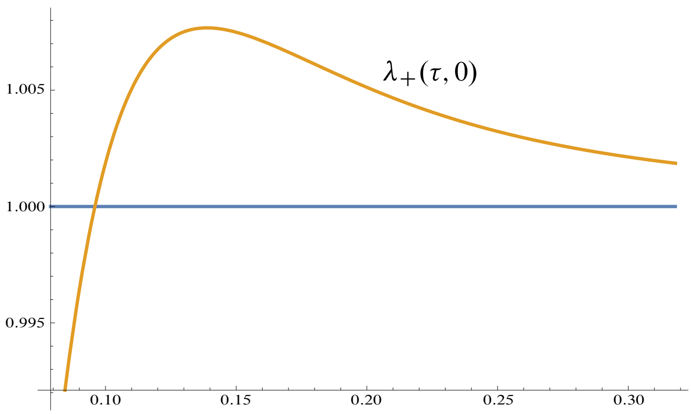

Figure 5.1. The function $\tau \mapsto \lambda_+(\tau, 0)$ near its maximum, well above 1.

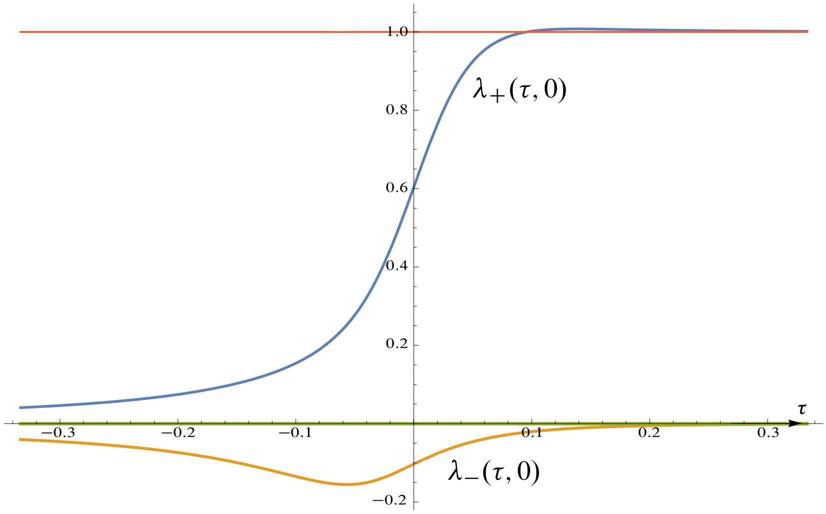

Figure 5.2. The functions $\tau \mapsto \lambda_+(\tau,0), \lambda_-(\tau,0)$.

and we can use these formulas and (5.4.1)*,* (5.4.2)*,* (5.4.3)*, and* (5.4.4) *to calculate numerically*

$$M_0 \approx \mathbf{1.00767997007003}, \quad (\lambda_+(\tau,0) \; at \; \tau \approx 0.138815397930141),$$
$$m_0 \approx -0.155939843191243, \quad (\lambda_-(\tau,0) \; at \; \tau \approx -0.0566304954736227).$$

5.4.2 On hyperbolic regions

We want now to tackle the case $\sigma > 0$. In order to use the expressions (A.6.22), (5.2.25) respectively for a_{11} and a_{12}, we need first to evaluate the residue term in (A.6.22). The mapping $z \mapsto \tanh z$ is a biholomorphism of neighborhoods of 0 in the complex plane, so that we have for z near the origin,

$$\zeta = \tanh z, \quad d\zeta = (1-\zeta^2)dz, \quad z = \operatorname{arcth} \zeta = \frac{1}{2} \ln\left(\frac{1+\zeta}{1-\zeta}\right),$$
$$\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z} dz = \left(\frac{1+\zeta}{1-\zeta}\right)^{i\omega} e^{-2i\frac{\kappa}{\zeta}} \frac{2}{\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2} + \left(\frac{1-\zeta}{1+\zeta}\right)^{1/2}} \frac{d\zeta}{(1-\zeta^2)}$$
$$= (1+\zeta)^{-\frac{1}{2}+i\omega}(1-\zeta)^{-\frac{1}{2}-i\omega} e^{-2i\frac{\kappa}{\zeta}} d\zeta,$$

so that

$$\mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\right) = \mathtt{Res}((1+\zeta)^{-\frac{1}{2}+i\omega}(1-\zeta)^{-\frac{1}{2}-i\omega} e^{-2i\frac{\kappa}{\zeta}}, 0). \quad (5.4.5)$$

Proposition 5.4.5. *Let $\sigma \geq 0$ be given. Then, for any $\tau \in \mathbb{R}$, using the notations, $\omega = 2\pi\tau$, $\kappa = 2\pi\sigma$, we have, for any $\rho \in (0,1)$,*

$$a_{11}(\tau,\sigma) = \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{1+e^{-2\pi\omega}}\frac{\rho}{2\pi} \times \operatorname{Im}\left\{\int_{-\pi}^{\pi} \exp\left(i\omega \operatorname{Log}\left(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}}\right)\right)\frac{e^{-\frac{2i\kappa e^{-i\theta}}{\rho}} e^{i\theta}}{\sqrt{1-\rho^2 e^{2i\theta}}} d\theta\right\}. \quad (5.4.6)$$

$$\operatorname{Re} a_{21}(\tau,\sigma) = \frac{e^{-\pi\omega}}{2\pi}\left\{2\int_0^{\pi/2} \frac{e^{(s\omega-\kappa \tan s)}\sinh(s\omega-\kappa\tan s)}{\sin s} ds + \ln\left(\coth\frac{\pi}{4}\right) + 2\int_0^{\pi/2}\frac{\sin^2(s\omega-\kappa\tanh s)}{\sinh s} ds - \int_{\pi/2}^{+\infty}\frac{\cos 2(s\omega-\kappa\tanh s)}{\sinh s} ds\right\}, \quad (5.4.7)$$

$$\operatorname{Im} a_{12}(\tau,\sigma) = \frac{e^{-\pi\omega}}{2} a_{11}(\tau,\sigma). \quad (5.4.8)$$

Proof. Formula (5.4.6) follows from (5.4.5) and (A.6.22) whereas (5.4.7) is (5.2.25) after a change of variable $t = 2s$, where the second integral term inside the brackets is evaluated (cf. Lemma A.6.1); formula (5.4.8) is a reminder of (5.2.41). ∎

N.B. Our choice for ρ in the numerical calculations of (5.4.6) is $\rho = 3/4$, which is a good compromise between using a value of ρ clearly away from 1 (to avoid singularities coming from small denominators in the Log term) and minimize the oscillations and size coming from the term $\exp(-2i\kappa\rho^{-1}e^{-i\theta})$; note that the modulus of the latter is

$$\exp(-2\kappa\rho^{-1}\sin\theta),$$

which is a smooth function of ρ (flat at 0) when $\theta \in [0,\pi]$, but is unbounded for $\rho \to 0_+$ when $\theta \in (-\pi, 0)$. There is no surprise here since although the residue does not depend on the choice of $\rho \in (0,1)$, we cannot get the value of that residue by letting ρ go to 0 because of the part of the path in the lower half-plane. The argument of $\exp(-2i\kappa\rho^{-1}e^{-i\theta})$ is $-2\kappa\rho^{-1}\cos\theta$ and taking ρ too small would be devastating for the calculations because of the strong oscillations triggered by the term $\exp(-2i\kappa\rho^{-1}\cos\theta)$ all over the circle. Of course for the evaluation of $\operatorname{Log}(\frac{1+\rho e^{i\theta}}{1-\rho e^{i\theta}})$ is easier for ρ small, but we have to take into account the constraints in that direction mentioned above.

Remark 5.4.6. It seems easier numerically for the evaluation of a_{11} to use (5.4.6) rather than any other expression (see, e.g., Lemma 5.2.2, (5.2.22), (A.6.14)). However, the following formula could be interesting, theoretically and numerically: recall-

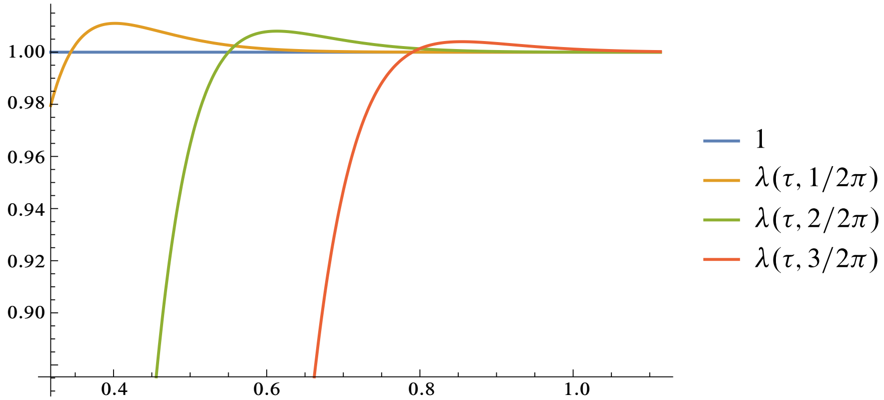

Figure 5.3. Functions $\lambda_+(\tau, \kappa/2\pi)$ with $\kappa = 1, 2, 3$: their maxima are strictly greater than 1.

ing that $\operatorname{sinc} x = \frac{\sin x}{x}$, we have from (5.2.22)

$$\begin{aligned} a_{11}(\tau,\sigma) = \frac{1}{2} + \frac{2\omega}{\pi}\int_0^{+\infty} \operatorname{sinc}(2\omega s)\frac{s}{\sinh s}\cos(2\kappa\tanh s)ds \\ - \frac{2\kappa}{\pi}\int_0^{+\infty} \operatorname{sinc}(2\kappa s)\frac{1}{\cosh s}\cos(2\omega s)ds, \end{aligned} \tag{5.4.9}$$

but it turns out that numerical calculations involving (5.4.9) seem to be less reliable than the methods using (5.4.6).

We can also take a look at the following curves.

Remark 5.4.7. In the above figure, in order to put the three curves on the same picture, we have used three different logarithmic scales on the vertical axis, namely, we have drawn

$$\tau \mapsto 1 + \alpha_j \operatorname{Log}(\lambda_+(\tau, \sigma_j)), \quad 1 \le j \le 3, \sigma_j = j/2\pi, \alpha_1 = 20, \alpha_2 = 100, \alpha_3 = 500.$$

Of course, we have

$$1 + \alpha_j \operatorname{Log}(\lambda_+(\tau,\sigma_j)) > 1 \Longleftrightarrow \operatorname{Log}(\lambda_+(\tau,\sigma_j)) > 0 \Longleftrightarrow \lambda_+(\tau,\sigma_j) > 1,$$

so that the piece of curves in Figure 5.3 which are above 1 are indeed corresponding to curves of $\tau \mapsto \lambda_+(\tau, \sigma_j)$ which go strictly above the threshold 1. We have also

$$\begin{aligned} &\max_\tau \lambda_+(\tau,\sigma_1) \approx 1 + 55 \times 10^{-5} && \text{at } \tau \approx 0.402030, \\ &\max_\tau \lambda_+(\tau,\sigma_2) \approx 1 + 8 \times 10^{-5} && \text{at } \tau \approx 0.613262, \\ &\max_\tau \lambda_+(\tau,\sigma_3) \approx 1 + 10^{-5} && \text{at } \tau \approx 0.854746. \end{aligned}$$

We are glad to have a theoretical proof of Theorem 5.2.4 since the numerical analysis of cases where σ is large, say larger than 10, seems to be very difficult to achieve, at least through a standard use of `Mathematica`. The reason for that is quite clear since using our Lemma 5.1.11, we did study the function β defined by

$$\beta(\tau,\sigma) = |a_{12}(\tau,\sigma)|^2 + a_{11}(\tau,\sigma) - 1, \tag{5.4.10}$$

and proved that for each $\sigma \geq 0$ there exists $T_0(\sigma)$ such that for all $\tau \geq T_0(\sigma)$ we have $\beta(\tau,\sigma) > 0$ and $a_{12}(\tau,\sigma) \neq 0$. Thanks to Lemma 5.2.2 and (5.2.46) we knew that for $\tau \geq T_0(\sigma)$, we had

$$|1 - a_{11}| \leq 2e^{-\pi^2\tau}e^{4\pi\sigma} \ll \frac{e^{-16\pi\sqrt{\tau}\sqrt{\sigma}}}{2^8\pi^6\tau^2} \leq (\operatorname{Re} a_{21})^2 \leq |a_{12}|^2,$$

where the second inequality $\ll$ is in fact comparing for σ fixed two exponential decays. The numerical analysis of that inequality is certainly quite difficult when σ and τ are large since both sides are converging to zero quite fast for σ fixed and $\tau \to +\infty$; of course taking the logarithm of both sides looks quite reasonable, but in practice does not seem really easy numerically. When $\sigma = 0$, the situation is much better, since we had to compare (cf. Section 5.3.1) an exponential decay $|1 - a_{11}| \leq 2e^{-\pi^2\tau}$ to a polynomial decay

$$|\operatorname{Re} a_{12}|^2 \sim \frac{1}{2^6\pi^4\tau^2}, \quad \tau \to +\infty,$$

and this could be an *a posteriori* explanation for which our numerical argument in [6] worked smoothly to disprove Flandrin's conjecture. So to pick up the quarter-plane ((5.0.1) with $\sigma = 0$) to produce a counterexample to that conjecture was indeed a very wise choice: if you choose instead $\mathscr{C}_\sigma$ for σ large, our Theorem 5.2.4 shows that it is also a counterexample to Flandrin's conjecture[6], but we have a theoretical proof for that Theorem and if we depended on a numerical analysis, it is quite likely that checking numerically the positivity of the function β defined in (5.4.10) could be rather difficult, even say for $\sigma = 10$.

[6]As a convex subset of the plane on which the integral of the Wigner distribution of some normalized pulse is strictly larger than 1.