Chapter 6

Unboundedness is Baire generic

In this section, we show that for plenty of subsets E of the phase space $\mathbb{R}^{2n}$, the operator $\operatorname{Op}_{\mathrm{w}}(\mathbf{1}_E)$ is not bounded on $L^2(\mathbb{R}^n)$.

6.1 Preliminaries

6.1.1 Prolegomena

Lemma 6.1.1. *Let $u, v \in L^2(\mathbb{R}^n)$ and let $\mathcal{W}(u,u)$, $\mathcal{W}(v,v)$, be their Wigner distributions. Then, we have*

$$\|\mathcal{W}(u,u)-\mathcal{W}(v,v)\|_{L^2(\mathbb{R}^{2n})} \le \|u-v\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)}+\|v\|_{L^2(\mathbb{R}^n)}\big).$$

As a consequence if a sequence (u_k) is converging in $L^2(\mathbb{R}^n)$, then the sequence $(\mathcal{W}(u_k,u_k))$ converges in $L^2(\mathbb{R}^{2n})$ towards $\mathcal{W}(u,u)$.

Proof. We have by sesquilinearity $\mathcal{W}(u,u)-\mathcal{W}(v,v)=\mathcal{W}(u-v,u)+\mathcal{W}(v,u-v)$, so that

$$\begin{aligned}\|\mathcal{W}(u,u)-\mathcal{W}(v,v)\|_{L^2(\mathbb{R}^{2n})} &\le \|\mathcal{W}(u-v,u)\|_{L^2(\mathbb{R}^{2n})}+\|\mathcal{W}(v,u-v)\|_{L^2(\mathbb{R}^{2n})}\\ &\underbrace{=}_{(1.1.6)} \|u-v\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)}+\|v\|_{L^2(\mathbb{R}^n)}\big),\end{aligned}$$

proving the lemma. ∎

Lemma 6.1.2. *Let (u_k) be a converging sequence in $L^2(\mathbb{R}^n)$ with limit u. Let us assume that there exists $C_0 \ge 0$ such that*

$$\forall k \in \mathbb{N}, \qquad \iint |\mathcal{W}(u_k,u_k)(x,\xi)|\,dx\,d\xi \le C_0.$$

Then, we have $\iint |\mathcal{W}(u,u)(x,\xi)|\,dx\,d\xi \le C_0$.

Proof. Let $R>0$ be given. We check

$$\begin{aligned}&\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi)-\mathcal{W}(u_k,u_k)(x,\xi)|\,dx\,d\xi\\ &\quad\le \iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u-u_k,u)(x,\xi)|\,dx\,d\xi\\ &\qquad+\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u_k,u-u_k)(x,\xi)|\,dx\,d\xi\\ &\quad\le \sqrt{|\mathbb{B}^{2n}|R^{2n}}\big(\|\mathcal{W}(u-u_k,u)\|_{L^2(\mathbb{R}^{2n})}+\|\mathcal{W}(u_k,u-u_k)\|_{L^2(\mathbb{R}^{2n})}\big)\\ &\quad= \sqrt{|\mathbb{B}^{2n}|R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)}+\|u_k\|_{L^2(\mathbb{R}^n)}\big),\end{aligned}$$

and thus

$$\begin{aligned}
&\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi)|dxd\xi \\
&\quad \le \iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u_k,u_k)(x,\xi)|dxd\xi \\
&\qquad + \sqrt{|\mathbb{B}^{2n}|R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)} + \|u_k\|_{L^2(\mathbb{R}^n)}\big) \\
&\quad \le C_0 + \sqrt{|\mathbb{B}^{2n}|R^{2n}}\|u-u_k\|_{L^2(\mathbb{R}^n)}\big(\|u\|_{L^2(\mathbb{R}^n)} + \|u_k\|_{L^2(\mathbb{R}^n)}\big),
\end{aligned}$$

implying for all $R > 0$,

$$\iint_{|x|^2+|\xi|^2\le R^2} |\mathcal{W}(u,u)(x,\xi)|dxd\xi \le C_0,$$

and thus the sought result. ∎

6.1.2 An explicit construction

We just calculate in this section $\mathcal{W}(v_0, v_0)$ for $v_0 = \mathbf{1}_{[-1/2,1/2]}$.

Remark 6.1.3. When u is supported in a closed convex set J, we have in the integral (1.1.4) defining $\mathcal{W}$, $x \pm \frac{z}{2} \in J \Rightarrow x \in J$, so that supp $\mathcal{W}(u,u) \subset J \times \mathbb{R}^n$.

We have

$$\mathcal{W}(v_0, v_0)(x,\xi) = \int_{\substack{-1/2\le x+z/2\le 1/2\\ -1/2\le x-z/2\le 1/2}} e^{2i\pi z\xi}dz,$$

and the integration domain is

$$-\min(1-2x, 1+2x) = \max(-1-2x, 2x-1) \le z \le \min(1-2x, 1+2x),$$

which is empty unless $1-2x, 1+2x \ge 0$, i.e., $x \in [-1/2, +1/2]$, and moreover we have the equivalence

$$1-2x \le 1+2x \Longleftrightarrow x \ge 0,$$

so that

$$\begin{aligned}
&\mathcal{W}(v_0, v_0)(x,\xi) \\
&\quad = H(x)\int_{-(1-2x)}^{1-2x} e^{2i\pi z\xi}dz + H(-x)\int_{-(1+2x)}^{1+2x} e^{2i\pi z\xi}dz \\
&\quad = H(x)\frac{e^{2i\pi\xi(1-2x)} - e^{-2i\pi\xi(1-2x)}}{2i\pi\xi} + H(-x)\frac{e^{2i\pi\xi(1+2x)} - e^{-2i\pi\xi(1+2x)}}{2i\pi\xi} \\
&\quad = \mathbf{1}_{[0,1/2]}(x)\frac{\sin(2\pi\xi(1-2x))}{\pi\xi} + \mathbf{1}_{[-1/2,0]}\frac{\sin(2\pi\xi(1+2x))}{\pi\xi}.
\end{aligned} \tag{6.1.1}$$

More generally for a, b, ω real numbers with $a < b$ and

$$u_{a,b,\omega}(x) = (b-a)^{-1/2}\mathbf{1}_{[a,b]}(x)e^{2i\pi\omega x}, \tag{6.1.2}$$

we have

$$\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi) = \frac{\big(\mathbf{1}_{[a,\frac{a+b}{2}]}(x)\sin[4\pi(\xi-\omega)(x-a)] + \mathbf{1}_{[\frac{a+b}{2},b]}(x)\sin[4\pi(\xi-\omega)(b-x)]\big)}{(b-a)\pi(\xi-\omega)}.$$

We check now, using (6.1.1), for $N > 0$,

$$\begin{aligned}\iint |\mathcal{W}(v_0, v_0)(x,\xi)|dxd\xi &\ge \int_{0\le x\le 1/4}\int_0^N \left|\frac{\sin(2\pi\xi(1-2x))}{\pi\xi}\right| d\xi dx \\ &= \int_{0\le x\le 1/4}\int_0^{N2\pi(1-2x)} \left|\frac{\sin\eta}{\pi\eta}\right| d\eta dx \\ &\ge \int_{0\le x\le 1/4}\int_0^{N\pi} \left|\frac{\sin\eta}{\pi\eta}\right| d\eta dx = \frac{1}{4}\int_0^{N\pi}\left|\frac{\sin\eta}{\pi\eta}\right| d\eta,\end{aligned}$$

so that

$$\iint |\mathcal{W}(v_0, v_0)(x,\xi)|dxd\xi = +\infty. \tag{6.1.3}$$

Proposition 6.1.4. *Let a, b, ω be real numbers with $a < b$ and let us define $u_{a,b,\omega}$ by* (6.1.2). *Then, we have*

$$\iint \big|\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)\big|dxd\xi = +\infty. \tag{6.1.4}$$

N.B. Since $u_{a,b,\omega}$ is a normalized $L^2(\mathbb{R})$ function, we also have from (1.1.6), (1.1.9) that the real-valued $\mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})$ does satisfy

$$\begin{aligned}\int\left|\int \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)dx\right|d\xi &= \int\left|\int \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)d\xi\right| dx \\ &= \|u_{a,b,\omega}\|^2_{L^2(\mathbb{R})} = 1,\\ \iint \mathcal{W}(u_{a,b,\omega}, u_{a,b,\omega})(x,\xi)^2 dxd\xi &= \|u_{a,b,\omega}\|^4_{L^2(\mathbb{R})} = 1.\end{aligned}$$

We shall see in the next sections that most of the time in the Baire Category sense, we have for $u \in L^2(\mathbb{R}^n)$, $\iint |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty$.

Proof. The proof is already given above for $v_0 = u_{-1/2,1/2,0}$. Moreover, we have with

$$\alpha = \frac{1}{b-a}, \quad \beta = \frac{b+a}{2(a-b)},$$

the formula

$$v_0(y) = e^{-2i\pi\omega(y-\beta)\alpha^{-1}} u_{a,b,\omega}((y-\beta)\alpha^{-1})\alpha^{-1/2},$$

so that $u_{a,b,\omega} = \mathcal{M} v_0$, where $\mathcal{M}$ belongs to the group $\mathrm{Mp}_a(n)$. (cf. Section 1.2.1) and the covariance property (1.2.49) shows that the already proven (6.1.3) implies (6.1.4). ∎

6.2 Modulation spaces

In this section, we use the Feichtinger algebra M^1, introduced in [10] (the terminology *Feichtinger algebra* goes back to the book [44]). The survey article [26] by M. S. Jakobsen is a good source for recent developments of the theory as well as Chapter 12 in the K. Gröchenig's book [16]. We refer the reader to the paper [18] by K. Gröchenig and M. Leinert as well as to J. Sjöstrand's article [48] for the use of modulation spaces to proving a non-commutative Wiener lemma.

6.2.1 Preliminary lemmas

The following lemmas in this subsection are well-known (see, e.g., [16, Theorem 11.2.5]). However, we provide a proof for the self-containedness of our survey.

Lemma 6.2.1. *Let ϕ_0 be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$. For $u \in \mathscr{S}'(\mathbb{R}^n)$ the following properties are equivalent.*

(i) $u \in \mathscr{S}(\mathbb{R}^n)$.

(ii) $\mathcal{W}(u, \phi_0) \in \mathscr{S}(\mathbb{R}^{2n})$.

(iii) $\forall N \in \mathbb{N}, \sup_{X\in\mathbb{R}^{2n}} |\mathcal{W}(u, \phi_0)(X)|(1 + |X|)^N < +\infty$.

Proof. Let us assume (i) holds true; with $\Omega(u, \phi_0)$ defined in (1.1.1), we find that $\Omega(u, \phi_0)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$, thus as well as its partial Fourier transform $\mathcal{W}(u, \phi_0)$, proving (ii). We have obviously that (ii) implies (iii). Let us now assume that (iii) holds true. Using (1.1.5), we find

$$u(x_1)\bar{\phi}_0(x_2) = \int \mathcal{W}(u, \phi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1-x_2)\xi} d\xi,$$

and thus

$$\begin{aligned} u(x_1)\|\phi_0\|^2_{L^2(\mathbb{R}^n)} &= \iint \mathcal{W}(u, \phi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big) e^{2i\pi(x_1-x_2)\xi} \phi_0(x_2) d\xi dx_2 \\ &= \iint \mathcal{W}(u, \phi_0)(y, \xi) e^{4i\pi(x_1-y)\xi} \phi_0(2y - x_1) d\xi dy 2^n, \end{aligned}$$

so that the latter equality, the fact that ϕ_0 belongs to $\mathscr{S}(\mathbb{R}^n)$ imply (i) by differentiation under the integral sign, concluding the proof of the lemma. ∎

Lemma 6.2.2. *Let ϕ_0, ϕ_1 be non-zero functions in $L^2(\mathbb{R}^n)$. Let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u, \phi_0)$ belongs to $L^1(\mathbb{R}^{2n})$. Then, $\mathcal{W}(u, \phi_1)$ belongs as well to $L^1(\mathbb{R}^{2n})$.*

Proof. According to Lemma 1.2.26 applied to $u_0 = u, u_1 = u_2 = \phi_0, u_3 = \phi_1$, we have

$$\|\phi_0\|_{L^2}^2 \mathcal{W}(u, \phi_1) \in L^1(\mathbb{R}^{2n}),$$

since $\mathcal{W}(u, \phi_0)$ belongs to $L^1(\mathbb{R}^{2n})$ as well as $\mathcal{W}(\check{\phi}_0, \phi_1)$. ∎

Lemma 6.2.3. *Let $u \in L^2(\mathbb{R}^n)$. The following properties are equivalent.*

(i) *For all $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have $\mathcal{W}(u, \phi) \in L^1(\mathbb{R}^{2n})$.*

(ii) *For a non-zero $\phi \in \mathscr{S}(\mathbb{R}^n)$, we have $\mathcal{W}(u, \phi) \in L^1(\mathbb{R}^{2n})$.*

(iii) *$\mathcal{W}(u, u)$ belongs to $L^1(\mathbb{R}^{2n})$.*

Proof. We have obviously (i)⇒(ii) and, conversely, Lemma 6.2.2 yields (ii)⇒(i). Assuming (i) and using Lemma 1.2.26 with $u_0 = u_3 = u, u_1 = u_2 = \phi \in \mathscr{S}(\mathbb{R}^n)$, we get

$$\|\phi\|_{L^2}^2 |\mathcal{W}(u, u)(X)| \le 2^n \big(|\mathcal{W}(u, \phi)| * |\mathcal{W}(\check{\phi}, u)|\big)(X),$$

so that choosing a non-zero ϕ in the Schwartz space, we obtain (iii). Conversely, assuming (iii) and using again Lemma 1.2.26 with $u_0 = u_2 = u, u_3 = \phi \in \mathscr{S}(\mathbb{R}^n)$, $u_1 = \psi \in \mathscr{S}(\mathbb{R}^n)$, we find

$$|\langle \psi, u \rangle_{L^2}| |\mathcal{W}(u, \phi)(X)| \le 2^n \big(\underbrace{|\mathcal{W}(u, u)|}_{\in L^1(\mathbb{R}^{2n})} * |\underbrace{\mathcal{W}(\check{\psi}, \phi)}_{\in \mathscr{S}(\mathbb{R}^{2n})}|\big)(X). \tag{6.2.1}$$

Assuming as we may $u \neq 0$, we can choose $\psi \in \mathscr{S}(\mathbb{R}^n)$ such that

$$\langle \psi, u \rangle_{L^2} \neq 0,$$

so that (6.2.1) implies (i). ∎

Lemma 6.2.4. *Let $u_1, u_2, u_3 \in L^2(\mathbb{R}^n)$. Then, we have the inversion formula,*

$$\operatorname{Op_w}(\mathcal{W}(u_1, u_2)) u_3 = \langle u_3, u_2 \rangle_{L^2(\mathbb{R}^n)} u_1.$$

Proof. It is an immediate consequence of Lemma 1.2.25. ∎

6.2.2 The space $M^1(\mathbb{R}^n)$

Definition 6.2.5. The space $M^1(\mathbb{R}^n)$ is defined as the set of $u \in L^2(\mathbb{R}^n)$ such that, for all $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{W}(u, \phi)$ belongs to $L^1(\mathbb{R}^{2n})$. According to Lemma 6.2.3, $M^1(\mathbb{R}^n)$ is also the set of $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u, u) \in L^1(\mathbb{R}^{2n})$ as well as the set of $u \in L^2(\mathbb{R}^n)$ such that, for a non-zero $\phi \in \mathscr{S}(\mathbb{R}^n)$, $\mathcal{W}(u, \phi)$ belongs to $L^1(\mathbb{R}^{2n})$.

Proposition 6.2.6. *Let ψ_0 be the standard fundamental state of the harmonic oscillator $\pi(D_x^2 + x^2)$ given by*

$$\psi_0(x) = 2^{n/4} e^{-\pi|x|^2}. \tag{6.2.2}$$

Then, $M^1(\mathbb{R}^n) \ni u \mapsto \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}$ is a norm on $M^1(\mathbb{R}^n)$. Let ψ be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$: then $M^1(\mathbb{R}^n) \ni u \mapsto \|\mathcal{W}(u, \psi)\|_{L^1(\mathbb{R}^{2n})}$ is a norm on $M^1(\mathbb{R}^n)$, equivalent to the previous norm.

Proof. The homogeneity and triangle inequality are immediate, let us check the separation: let $u \in L^2(\mathbb{R}^n)$ such that $\mathcal{W}(u, \psi) = 0$. Then, we have

$$0 = \langle \mathrm{Op_w}(\mathcal{W}(u, \psi))\psi, u\rangle_{L^2(\mathbb{R}^n)} = \|u\|^2_{L^2(\mathbb{R}^n)} \|\psi\|^2_{L^2(\mathbb{R}^n)},$$

proving the sought result. Let ψ be a non-zero function in $\mathscr{S}(\mathbb{R}^n)$; according to Lemma 1.2.26 applied to $u_0 = u, u_1 = u_2 = \psi_0, u_3 = \psi$, we find

$$|\mathcal{W}(u, \psi)(X)| \le 2^n \big(|\mathcal{W}(u, \psi_0)| * |\mathcal{W}(\psi_0, \psi)|\big)(X),$$

so that we have

$$\begin{aligned} \|\mathcal{W}(u, \psi)\|_{L^1(\mathbb{R}^{2n})} &\le 2^n \|\mathcal{W}(\psi_0, \psi)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}, \\ \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})} &\le 2^n \|\mathcal{W}(\psi, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \|\mathcal{W}(u, \psi)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned} \tag{6.2.3}$$

proving the equivalence of norms. ∎

Proposition 6.2.7. *The space $M^1(\mathbb{R}^n)$, equipped with the equivalent norms of Proposition 6.2.6, is a Banach space. The space $\mathscr{S}(\mathbb{R}^n)$ is dense in $M^1(\mathbb{R}^n)$.*

Proof. Let $(u_k)_{k\ge1}$ be a Cauchy sequence in $M^1(\mathbb{R}^n)$: it means that $(\mathcal{W}(u_k, \psi_0))_{k\ge1}$ is a Cauchy sequence in $L^1(\mathbb{R}^{2n})$, thus such that

$$\lim_k \mathcal{W}(u_k, \psi_0) = U \quad \text{in } L^1(\mathbb{R}^{2n}). \tag{6.2.4}$$

On the other hand, from Lemma 1.2.25, we have $u_k - u_l = \mathrm{Op_w}\,(\mathcal{W}(u_k - u_l, \psi_0))\psi_0$, so that

$$\begin{aligned} &\|u_k - u_l\|_{L^2(\mathbb{R}^n)} \\ &\quad \le \|\mathrm{Op_w}\,(\mathcal{W}(u_k - u_l, \psi_0))\|_{\mathcal{B}(L^2(\mathbb{R}^n))} \underbrace{\le}_{\text{cf. (1.2.5)}} 2^n \|\mathcal{W}(u_k - u_l, \psi_0)\|_{L^1(\mathbb{R}^{2n})}, \end{aligned}$$

implying that $(u_k)_{k\ge1}$ is a Cauchy sequence in $L^2(\mathbb{R}^n)$, thus converging towards a function u in $L^2(\mathbb{R}^n)$. Since from (1.1.6), we have $\|\mathcal{W}(u_k - u, \psi_0)\|_{L^2(\mathbb{R}^{2n})} = \|u_k - u\|_{L^2(\mathbb{R}^n)}$, we obtain as well that

$$\lim_k \mathcal{W}(u_k, \psi_0) = \mathcal{W}(u, \psi_0) \quad \text{in } L^2(\mathbb{R}^{2n}),$$

and this implies along with (6.2.4) that $U = \mathcal{W}(u, \psi_0)$ in $\mathscr{S}'(\mathbb{R}^{2n})$. As a result, we have $\mathcal{W}(u, \psi_0) \in L^1(\mathbb{R}^{2n})$, so that $u \in M^1(\mathbb{R}^n)$ and

$$\lim_k \mathcal{W}(u_k, \psi_0) = \mathcal{W}(u, \psi_0) \quad \text{in } L^1(\mathbb{R}^{2n}),$$

entailing convergence towards u for the sequence $(u_k)_{k\ge 1}$ in $M^1(\mathbb{R}^n)$ and the sought completeness. We are left with the density question and we start with a calculation.

Claim 6.2.8. With the phase symmetry $\sigma_{y,\eta}$ given by (1.2.6) and ψ_0 by (6.2.2) we have for $X, Y \in \mathbb{R}^{2n}$,

$$\mathcal{W}(\sigma_Y \psi_0, \psi_0)(X) = 2^n e^{-2\pi|X-Y|^2} e^{-4i\pi[X,Y]}, \tag{6.2.5}$$

where the symplectic form is given in (1.2.13).

Proof of the Claim. We have indeed

$$\begin{aligned}
\mathcal{W}(\sigma_{y,\eta}\psi_0, \psi_0)(x,\xi) &= \int (\sigma_{y,\eta}\psi_0)\Big(x + \frac{z}{2}\Big)\psi_0\Big(x - \frac{z}{2}\Big)e^{-2i\pi z\cdot\xi}dz \\
&= \int \psi_0\Big(2y - x - \frac{z}{2}\Big)e^{4i\pi\eta\cdot(x+\frac{z}{2}-y)}\psi_0\Big(x - \frac{z}{2}\Big)e^{-2i\pi z\cdot\xi}dz \\
&= 2^{n/2}\int e^{-\pi(|2y-x-\frac{z}{2}|^2+|x-\frac{z}{2}|^2)}e^{2i\pi z\cdot(\eta-\xi)}dz e^{4i\pi\eta\cdot(x-y)} \\
&= 2^{n/2}e^{4i\pi\eta\cdot(x-y)}\int e^{-\frac{\pi}{2}(|2y-z|^2+|2(y-x)|^2)}e^{2i\pi z\cdot(\eta-\xi)}dz \\
&= 2^{n/2}e^{4i\pi\eta\cdot(x-y)}e^{-2\pi|y-x|^2}e^{4i\pi y\cdot(\eta-\xi)}2^{n/2}e^{-2\pi|\eta-\xi|^2},
\end{aligned}$$

which is the sought formula. ∎

Let u be a function in $M^1(\mathbb{R}^n)$. For $\varepsilon > 0$ we define

$$u_\varepsilon(x) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y)e^{-\varepsilon|Y|^2}2^n(\sigma_Y\psi_0)(x)dY,$$

and we have

$$\mathcal{W}(u_\varepsilon, \psi_0)(X) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y)e^{-\varepsilon|Y|^2}2^n\mathcal{W}(\sigma_Y\psi_0, \psi_0)(X)dY,$$

so that Lemma 6.2.1 and (6.2.5) imply readily that u_ε belongs to the Schwartz space. Moreover, we have

$$u = \operatorname{Op_w}(\mathcal{W}(u, \psi_0))\psi_0,$$

from Lemma 6.2.4 and thus

$$\mathcal{W}(u, \psi_0)(X) = \int_{\mathbb{R}^{2n}} \mathcal{W}(u, \psi_0)(Y)2^n\mathcal{W}(\sigma_Y\psi_0, \psi_0)(X)dY,$$

so that

$$\int_{\mathbb{R}^{2n}} |\mathcal{W}(u_\varepsilon, \psi_0)(X) - \mathcal{W}(u, \psi_0)(X)| dX$$
$$\leq 2^n \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \underbrace{|\mathcal{W}(u, \psi_0)(Y)||\mathcal{W}(\sigma_Y \psi_0, \psi_0)(X)|}_{\in L^1(\mathbb{R}^{4n}) \text{ from (6.2.5) and } u \in M^1(\mathbb{R}^n)} \underbrace{\left(1 - e^{-\varepsilon|Y|^2}\right)}_{\in[0,1]} dY dX.$$

The Lebesgue dominated convergence theorem shows that the integral above tends to 0 when $\varepsilon \to 0_+$, proving the convergence in $M^1(\mathbb{R}^n)$ of the sequence (u_ε), which completes the proof of the density. ■

Theorem 6.2.9. *Let $\mathcal{M}$ be an element of the metaplectic group $Mp(n)$ (Definition 1.2.13). Then, $\mathcal{M}$ is an isomorphism of $M^1(\mathbb{R}^n)$ and we have for $u \in M^1(\mathbb{R}^n)$, $\phi \in \mathscr{S}(\mathbb{R}^n)$,*

$$\mathcal{W}(\mathcal{M}u, \mathcal{M}\phi) = \mathcal{W}(u, \phi) \circ S^{-1}, \tag{6.2.6}$$

where $\mathcal{M}$ is in the fiber of the symplectic transformation S. In particular, the space $M^1(\mathbb{R}^n)$ is invariant by the Fourier transformation and partial Fourier transformations, by the rescaling (1.2.31), by the transformations (1.2.30), (1.2.32) and also by the phase translations (1.2.51) and phase symmetries (1.2.6).

Proof. Formula (6.2.6) follows readily from (1.2.49) and if u belongs to $M^1(\mathbb{R}^n)$, we find that

$$\mathcal{W}(\mathcal{M}u, \underbrace{\mathcal{M}\psi_0}_{\in\mathscr{S}(\mathbb{R}^n)}) = \underbrace{\mathcal{W}(u, \psi_0)}_{\in L^1(\mathbb{R}^{2n})} \circ S^{-1},$$

and since $\det S = 1$, we have

$$\|\mathcal{W}(\mathcal{M}u, \mathcal{M}\psi_0)\|_{L^1(\mathbb{R}^{2n})} = \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})},$$

implying that $\mathcal{W}(\mathcal{M}u, \mathcal{M}\psi_0)$ belongs to $L^1(\mathbb{R}^{2n})$ so that, thanks to Definition 6.2.5, we get that $\mathcal{M}u$ belongs to $M^1(\mathbb{R}^n)$. The same properties are true for $\mathcal{M}^{-1}$. ■

Remark 6.2.10. From Definition 6.2.5, we see that, for $u \in M^1(\mathbb{R}^n)$, we have

$$\mathcal{W}(u, u) \in L^1(\mathbb{R}^{2n}),$$

and this implies, thanks to Theorem 1.2.24, that $M^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$. Moreover, we have

$$\mathcal{F}\left(M^1(\mathbb{R}^n)\right) \subset M^1(\mathbb{R}^n),$$

since for $u \in M^1(\mathbb{R}^n)$, we have

$$\mathcal{W}(\hat{u}, \psi_0) = \mathcal{W}(\hat{u}, \hat{\psi}_0)$$

and thanks to (6.2.6),

$$\|\mathcal{W}(\hat{u}, \hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})} = \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}.$$

As a consequence we find

$$\mathcal{F}\big(M^1(\mathbb{R}^n)\big) \subset M^1(\mathbb{R}^n) = \mathcal{F}^2\mathcal{C}(M^1(\mathbb{R}^n)) = \mathcal{F}^2(M^1(\mathbb{R}^n)) \subset \mathcal{F}\big(M^1(\mathbb{R}^n)\big),$$

and consequently

$$M^1(\mathbb{R}^n) = \mathcal{F}(M^1(\mathbb{R}^n)) \subset \mathcal{F}(L^1(\mathbb{R}^n)) \subset C_{(0)}(\mathbb{R}^n),$$

where the latter inclusion is due to the Riemann–Lebesgue lemma with $C_{(0)}(\mathbb{R}^n)$ standing for the space of continuous functions with limit 0 at infinity. Moreover, for $u \in M^1(\mathbb{R}^n)$ and ψ_0 given by (6.2.2), we get from (1.1.5),

$$u(x_1)\bar{\psi}_0(x_2) = \int \mathcal{W}(u, \psi_0)\Big(\frac{x_1 + x_2}{2}, \xi\Big)e^{2i\pi(x_1 - x_2)\cdot\xi}d\xi,$$

so that

$$u(x_1) = \iint \mathcal{W}(u, \psi_0)(y, \eta)e^{4i\pi(x_1 - y)\cdot\eta}\bar{\psi}_0(2y - x_1)dyd\eta 2^n,$$

implying

$$\|u\|_{L^1(\mathbb{R}^n)} \le \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}2^{\frac{5n}{4}}, \tag{6.2.7}$$

and similarly for $p \in [1, +\infty]$,

$$\|u\|_{L^p(\mathbb{R}^n)} \le \|\mathcal{W}(u, \psi_0)\|_{L^1(\mathbb{R}^{2n})}2^{\frac{5n}{4}}p^{-\frac{n}{2p}},$$

yielding the continuous injection of $M^1(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

Theorem 6.2.11. *The space $M^1(\mathbb{R}^n)$ is a Banach algebra for convolution and for pointwise multiplication.*

Proof. Let $u, v \in M^1(\mathbb{R}^n)$; then the convolution $u * v$ makes sense and belongs to all $L^p(\mathbb{R}^n)$ for $p \in [1, +\infty]$, since we have $u \in L^1(\mathbb{R}^n)$. We calculate

$$\mathcal{W}(u * v, \psi_0)(x, \xi) = \int_{\mathbb{R}^n} u(y)\mathcal{W}(\tau_y v, \psi_0)(x, \xi)dy, \quad (\tau_y v)(x) = v(x - y),$$

so that

$$\|\mathcal{W}(u * v, \psi_0)\|_{L^1(\mathbb{R}^{2n})} \le \int_{\mathbb{R}^n} |u(y)|\|\mathcal{W}(\tau_y v, \psi_0)\|_{L^1(\mathbb{R}^{2n})}dy,$$

and since we have

$$\mathcal{W}(\tau_y v, \psi_0)(x, \xi) = \mathcal{W}(v, \tau_y\psi_0)(x, \xi)e^{-4i\pi y\cdot\xi},$$

we get

$$\|\mathcal{W}(u*v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}\le\int_{\mathbb{R}^n}|u(y)|\|\mathcal{W}(v,\tau_y\psi_0)\|_{L^1(\mathbb{R}^{2n})}dy,$$

so that using (6.2.3), we obtain

$$\begin{aligned}&\|\mathcal{W}(u*v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}\\&\quad\le\int_{\mathbb{R}^n}|u(y)|2^n\|\mathcal{W}(\psi_0,\tau_y\psi_0)\|_{L^1(\mathbb{R}^{2n})}dy\|\mathcal{W}(v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}.\end{aligned}$$

We can check now that

$$\mathcal{W}(\psi_0,\tau_y\psi_0)(x,\xi)=2^ne^{-2\pi(\xi^2+(x-\frac{y}{2})^2)}e^{2i\pi\xi y},$$

so that

$$\begin{aligned}\|\mathcal{W}(u*v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}&\le 2^n\|u\|_{L^1(\mathbb{R}^n)}\|\mathcal{W}(v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}\\&\underbrace{\le}_{(6.2.7)}2^{\frac{9n}{4}}\|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})}\|\mathcal{W}(v,\psi_0)\|_{L^1(\mathbb{R}^{2n})},\end{aligned}\tag{6.2.8}$$

proving that $M^1(\mathbb{R}^n)$ is a Banach algebra for convolution when equipped with the norm

$$N(u)=2^{\frac{9n}{4}}\|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})}.\tag{6.2.9}$$

On the other hand, for $u,v\in M^1(\mathbb{R}^n)$, the pointwise product $u\cdot v$ makes sense and belongs to $L^1(\mathbb{R}^n)$ (since both functions are in $L^2(\mathbb{R}^n)$) and we have

$$u\cdot v=\mathcal{C}\mathcal{F}(\hat{u}*\hat{v}),$$

so that

$$\mathcal{W}(u\cdot v,\psi_0)(x,\xi)=\mathcal{W}(\mathcal{C}\mathcal{F}(\hat{u}*\hat{v}),\psi_0)(x,\xi)=\mathcal{W}(\mathcal{F}(\hat{u}*\hat{v}),\check{\psi}_0)(-x,-\xi),$$

and since $\psi_0=\hat{\psi}_0$ is also even, we get

$$\begin{aligned}\|\mathcal{W}(u\cdot v,\psi_0)\|_{L^1(\mathbb{R}^{2n})}&=\|\mathcal{W}(\mathcal{F}(\hat{u}*\hat{v}),\mathcal{F}\psi_0)\|_{L^1(\mathbb{R}^{2n})}\\&\underbrace{=}_{\text{cf. }(1.2.49)}\|\mathcal{W}\big(\hat{u}*\hat{v},\psi_0\big)\|_{L^1(\mathbb{R}^{2n})}\\&\underbrace{\le}_{(6.2.8)}2^{\frac{9n}{4}}\|\mathcal{W}(\hat{u},\hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})}\|\mathcal{W}(\hat{v},\hat{\psi}_0)\|_{L^1(\mathbb{R}^{2n})}\\&=2^{\frac{9n}{4}}\|\mathcal{W}(u,\psi_0)\|_{L^1(\mathbb{R}^{2n})}\|\mathcal{W}(v,\psi_0)\|_{L^1(\mathbb{R}^{2n})},\end{aligned}$$

proving as well that $M^1(\mathbb{R}^n)$ is a Banach algebra for pointwise multiplication with the norm (6.2.9). ∎

6.3 Most pulses give rise to a non-integrable Wigner distribution

In the sequel, n is an integer ≥ 1.

Lemma 6.3.1. *We have with ψ_0 given by (6.2.2),*

$$M^1(\mathbb{R}^n) = \Big\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi < +\infty\Big\}.$$

Then, $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$ with empty interior.

Proof. We have $M^1(\mathbb{R}^n) = \bigcup_{N\in\mathbb{N}} \Phi_N$ with

$$\Phi_N = \Big\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi \le N\Big\}.$$

The set Φ_N is a closed subset of $L^2(\mathbb{R}^n)$ since if $(u_k)_{k\ge 1}$ is a sequence in Φ_N which converges in $L^2(\mathbb{R}^n)$ with limit u, we get for $R \ge 0$,

$$\begin{aligned}
&\iint_{|(x,\xi)|\le R} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi \\
&\le \iint_{|(x,\xi)|\le R} |\mathcal{W}(u - u_k, \psi_0)(x, \xi)| dx d\xi + \iint_{|(x,\xi)|\le R} |\mathcal{W}(u_k, \psi_0)(x, \xi)| dx d\xi \\
&\le \|u - u_k\|_{L^2(\mathbb{R}^n)} (|\mathbb{B}^{2n}| R^{2n})^{1/2} + N,
\end{aligned}$$

implying $\iint_{|(x,\xi)|\le R} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi \le N$, and this for any R, so that we obtain $u \in \Phi_N$. The interior of Φ_N is empty, since if it were not the case, as Φ_N is also convex and symmetric, 0 would be an interior point of Φ_N in $L^2(\mathbb{R}^n)$ and we would find $\rho_0 > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^n)} \le \rho_0 \Longrightarrow \iint_{\mathbb{R}^{2n}} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi \le N,$$

and thus for any non-zero $u \in L^2(\mathbb{R}^n)$, we would have

$$\iint_{\mathbb{R}^{2n}} |\mathcal{W}(u, \psi_0)(x, \xi)| dx d\xi \|u\|^{-1}_{L^2(\mathbb{R}^n)} \rho_0 \le N$$

and thus

$$\|u\|_{M^1(\mathbb{R}^n)} \le N\rho_0^{-1} \|u\|_{L^2(\mathbb{R}^n)},$$

implying as well $L^2(\mathbb{R}^n) = M^1(\mathbb{R}^n)$ which is untrue, thanks to the examples of Section 6.1.2, e.g., (6.1.3), and this proves that the interior of Φ_N is actually empty. Now the Baire Category Theorem implies that the F_σ set $M^1(\mathbb{R}^n)$ is a subset of $L^2(\mathbb{R}^n)$ with empty interior. ∎

Let us give another decomposition of the space $M^1(\mathbb{R}^n)$.

Lemma 6.3.2. *According to Lemma 6.2.3, we have*

$$M^1(\mathbb{R}^n) = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi < +\infty\right\}.$$

Then, defining

$$\mathcal{F}_N = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi \le N\right\}, \tag{6.3.1}$$

each $\mathcal{F}_N$ is a closed subset of $L^2(\mathbb{R}^n)$ with empty interior.

Proof. We have $\mathscr{F} = M^1(\mathbb{R}^n) = \bigcup_{N\in\mathbb{N}} \mathcal{F}_N$. The set $\mathcal{F}_N$ is a closed subset of $L^2(\mathbb{R}^n)$ since if $(u_k)_{k\ge1}$ is a sequence in $\mathcal{F}_N$ which converges in $L^2(\mathbb{R}^n)$ with limit u, we have

$$\forall k \ge 1, \quad \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u_k,u_k)(x,\xi)|dxd\xi \le N,$$

so that we may apply Lemma 6.1.2 with $C_0 = N$, and readily get that u belongs to $\mathcal{F}_N$. We have also that $\text{interior}_{L^2(\mathbb{R}^n)}(\mathcal{F}_N) \subset \text{interior}_{L^2(\mathbb{R}^n)}(M^1(\mathbb{R}^n)) = \emptyset$. ∎

Theorem 6.3.3. *Defining*

$$\mathcal{G} = \left\{u \in L^2(\mathbb{R}^n), \iint_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty\right\} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n), \tag{6.3.2}$$

we obtain that the set $\mathcal{G}$ is a dense G_δ subset of $L^2(\mathbb{R}^n)$.

Proof. It follows immediately from Lemma 6.3.2 and formula $\{\mathring{A}\}^c = \overline{A^c}$, yielding for $\mathcal{F}_N$ defined in (6.3.1), $L^2(\mathbb{R}^n) = \{\text{interior}(\bigcup_{\mathbb{N}} \mathcal{F}_N)\}^c = \overline{\bigcap_{\mathbb{N}} \mathcal{F}_N^c}$. ∎

Remark 6.3.4. It is interesting to note that the space $M^1(\mathbb{R}^n)$ is not reflexive, as it can be identified to ℓ^1 via Wilson bases, but it is a dual space. It turns out that both properties are linked to the fact that $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$ as proven by Lemmas 6.3.1 and 6.3.2: if $\mathbb{X}$ is a reflexive Banach space continuously included in a Hilbert space $\mathbb{H}$, it is always an F_σ of $\mathbb{H}$, since we may write

$$\mathbb{X} = \bigcup_{N\in\mathbb{N}} N\mathbf{B}_{\mathbb{X}},$$

where $\mathbf{B}_{\mathbb{X}}$ is the closed unit ball of $\mathbb{X}$ and $N\mathbf{B}_{\mathbb{X}}$ is $\mathbb{H}$-closed since it is weakly compact (for the topology $\sigma(\mathbb{H},\mathbb{H})$); we cannot use that abstract argument in the case of the non-reflexive $M^1(\mathbb{R}^n)$, so we produced a direct elementary proof above. Also, it can be proven that if $\mathbb{X}$ is a Banach space continuously included in a Hilbert space $\mathbb{H}$, so that $\mathbb{X}$ is an F_σ of $\mathbb{H}$, then $\mathbb{X}$ must have a predual. As a result, the fact that $M^1(\mathbb{R}^n)$ has a predual appears as a consequence of the fact that $M^1(\mathbb{R}^n)$ is an F_σ of $L^2(\mathbb{R}^n)$.

6.4 Consequences on integrals of the Wigner distribution

Lemma 6.4.1. *Let $\mathscr{G}$ be defined in (6.3.2) and let $u \in \mathscr{G}$. Then, the positive and negative part of the real-valued $\mathcal{W}(u,u)$ are such that*

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty.$$

Proof. For $h \in (0,1]$, we define the symbol

$$a(x,\xi,h) = e^{-h(x^2+\xi^2)},$$

and we see that it is a semi-classical symbol in the sense (1.2.65). Let us start a *reductio ad absurdum* and assume $\iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi < +\infty$, (which implies since $u \in \mathscr{G}$, $\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = +\infty$). We note that

$$\langle \mathrm{Op_w}(a(x,\xi,h))u,u\rangle_{L^2(\mathbb{R}^n)} = \iint \underbrace{a(x,\xi,h)}_{\in L^2(\mathbb{R}^{2n})} \underbrace{\mathcal{W}(u,u)(x,\xi)}_{\in L^2(\mathbb{R}^{2n})} dxd\xi,$$

and thanks to Theorem 1.2.27, we have also

$$\sup_{h\in(0,1]} |\langle \mathrm{Op_w}(a(x,\xi,h))u,u\rangle_{L^2(\mathbb{R}^n)}| \le \sigma_n \|u\|^2_{L^2(\mathbb{R}^n)},$$

so that

$$\iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)(x,\xi)dxd\xi + \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_-(x,\xi)dxd\xi$$
$$= \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi,$$

and thus with $\theta_h \in [-1,1]$, we have

$$\theta_h\sigma_n\|u\|^2_{L^2(\mathbb{R}^n)} + \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_-(x,\xi)dxd\xi$$
$$= \iint e^{-h(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi. \tag{6.4.1}$$

Choosing $h = 1/m, m \in \mathbb{N}^*$, we note that

$$e^{-\frac{1}{m}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi) \le e^{-\frac{1}{m+1}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi).$$

From the Beppo–Levi Theorem (see, e.g., [34, Theorem 1.6.1]), we get that

$$\lim_{m\to+\infty} \iint e^{-\frac{1}{m}(x^2+\xi^2)}\mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = +\infty.$$

However, the left-hand side of (6.4.1) is bounded above by

$$\sigma_n \|u\|^2_{L^2(\mathbb{R}^n)} + \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi, \quad \text{which is finite,}$$

triggering a contradiction. We may now study the case where

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi < +\infty, \quad \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty.$$

The identity (6.4.1) still holds true with a left-hand side going to $+\infty$ when h goes to 0 whereas the right-hand side is bounded. This concludes the proof of the lemma. ∎

N.B. A shorter *heuristic* argument would be that the identity

$$\iint \mathcal{W}(u,u)(x,\xi)dxd\xi = \|u\|^2_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \iint |\mathcal{W}(u,u)(x,\xi)|dxd\xi = +\infty$$

should imply the lemma, but the former integral is not absolutely converging, so that argument fails to be completely convincing since we need to give a meaning to the first integral.

Theorem 6.4.2. *Defining $\mathscr{G} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n)$ (cf. (6.3.2)) we find that the set $\mathscr{G}$ is a dense G_δ set in $L^2(\mathbb{R}^n)$ and for all $u \in \mathscr{G}$, we have*[1]

$$\iint \mathcal{W}(u,u)_+(x,\xi)dxd\xi = \iint \mathcal{W}(u,u)_-(x,\xi)dxd\xi = +\infty, \tag{6.4.2}$$

Defining[2]

$$E_\pm(u) = \{(x,\xi) \in \mathbb{R}^{2n}, \pm\mathcal{W}(u,u)(x,\xi) > 0\}, \tag{6.4.3}$$

we have for all $u \in \mathscr{G}$,

$$\iint_{E_\pm(u)} \mathcal{W}(u,u)(x,\xi)dxd\xi = \pm\infty, \tag{6.4.4}$$

and both sets $E_\pm(u)$ are open subsets of $\mathbb{R}^{2n}$ with infinite Lebesgue measure.

Proof. The first statements follow from Theorem 6.3.3 and Lemma 6.4.1. As far as (6.4.4) is concerned, we note that $\mathcal{W}(u,u) > 0$ (resp., < 0) on $E_+(u)$ (resp., $E_-(u)$), so that Theorem 6.3.3 implies (6.4.4). Moreover, $E_\pm(u)$ are open subsets of $\mathbb{R}^{2n}$ since, thanks to Theorem 1.2.22, the function $\mathcal{W}(u,u)$ is continuous; also, both subsets have infinite Lebesgue measure from (6.4.2) since $\mathcal{W}(u,u)$ belongs to $L^2(\mathbb{R}^{2n})$. ∎

[1]Note that $\mathcal{W}(u,u)$ is real-valued.

[2]Thanks to Theorem 1.2.22, the function $\mathcal{W}(u,u)$ is a continuous function, so it makes sense to consider its pointwise values.

Remark 6.4.3. There are many other interesting properties and generalizations of the space M^1 and in particular a close link between the Bargmann transform, the Fock spaces and modulation spaces: we refer the reader to Remark 5 on page 243 in Section 11.4 of [16], to our Section 1.2.8 in this memoir and to Section 2.4 of [33].

Remark 6.4.4. As a consequence of the previous theorem, we could say that for any *generic* u in $L^2(\mathbb{R}^n)$ (i.e., any $u \in \mathscr{G} = L^2(\mathbb{R}^n)\backslash M^1(\mathbb{R}^n)$), we can find open sets E_+, E_- such that the real-valued $\pm\mathcal{W}(u,u)$ is positive on $E_\pm$ and

$$\iint_{E_\pm} \mathcal{W}(u,u)(x,\xi)dxd\xi = \pm\infty.$$

We shall see in the next section some results on polygons in the plane and for instance, we shall be able to prove that there exists a "universal number" $\mu_3^+ > 1$ such that for any triangle[3] $\mathcal{T}$ in the plane, we have

$$\forall u \in L^2(\mathbb{R}), \quad \iint_{\mathcal{T}} \mathcal{W}(u,u)(x,\xi)dxd\xi \le \mu_3^+ \|u\|^2_{L^2(\mathbb{R})}. \tag{6.4.5}$$

Note in particular that we will show that (6.4.5) holds true regardless of the area of the triangle (which could be infinite according to our definition of a triangle). Although that type of result may look pretty weak, it gets enhanced by Theorem 6.4.2 which proves that no triangle in the plane could be a set $E_+(u)$ (cf. (6.4.3)) for a generic u in $L^2(\mathbb{R})$.

[3] We define a triangle as the intersection of three half-planes, which includes of course the convex envelope of three points, but also the set with infinite area $\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x+\xi \ge \lambda\}$ for some $\lambda > 0$.