Chapter 7

Convex polygons of the plane

7.1 Convex cones

We have seen in Proposition 5.4.4 and Theorem 5.2.4 that the self-adjoint bounded operator with Weyl symbol $H(x)H(\xi)$ does satisfy

$$\begin{aligned}\mu_2^- = m_0 = \lambda_{\min}\big(\operatorname{Op_w}(H(x)H(\xi))\big) &\le \operatorname{Op_w}(H(x)H(\xi)) \\ &\le \lambda_{\max}\big(\operatorname{Op_w}(H(x)H(\xi))\big) = M_0 = \mu_2^+, \\ [\mu_2^-, \mu_2^+] &= \operatorname{spectrum}\big(\operatorname{Op_w}(H(x)H(\xi))\big),\end{aligned} \tag{7.1.1}$$

with

$$\mu_2^- \approx -0.155939843191243, \quad \mu_2^+ \approx 1.00767997007003. \tag{7.1.2}$$

This result is true as well for the characteristic function of any convex cone (which is not a half-plane nor the full plane) in the plane since we can map it to the quarter plane by a transformation in $\mathrm{Sl}(2,\mathbb{R}) = \mathrm{Sp}(1,\mathbb{R})$. On the other hand, a concave cone is the complement of a convex cone and the diagonalisation offered by Theorem 5.2.3 proves that the spectrum of the Weyl quantization of the indicatrix of a concave cone is

$$1 - \operatorname{Spectrum}\big(\operatorname{Op_w}(H(x)H(\xi))\big).$$

We may sum-up the situation by the following theorem.

Theorem 7.1.1. *Let Σ_θ be a convex cone in $\mathbb{R}^2$ with aperture $\theta \in [0, 2\pi]$ (cf. (5.3.3)) and let $\mathcal{A}_\theta$ be the self-adjoint bounded operator with the indicator function of Σ_θ as a Weyl symbol.*

(1) *If $\theta = 0$, we have $\mathcal{A}_\theta = 0$.*

(2) *If $\theta \in (0, \pi)$, the operator $\mathcal{A}_\theta$ is unitarily equivalent to $\operatorname{Op_w}(H(x)H(\xi))$, thus with spectrum $[\mu_2^-, \mu_2^+]$ with $\mu_2^- < 0 < 1 < \mu_2^+$, as given in Theorem 5.2.4.*

(3) *If $\theta = \pi$, Σ_π is a half-space and $\mathcal{A}_\pi$ is a proper orthogonal projection, thus with spectrum $\{0, 1\}$.*

(4) *If $\theta \in (\pi, 2\pi)$, Σ_θ is a concave cone and the operator $\mathcal{A}_\theta$ is unitarily equivalent to*

$$\operatorname{Id} - \operatorname{Op_w}(H(x)H(\xi)),$$

thus with spectrum $[1-\mu_2^+, 1-\mu_2^-]$ (see footnote[1]).

(5) *If $\theta = 2\pi$, we have $\mathcal{A}_{2\pi} = \operatorname{Id}$.*

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[1] So that we have in particular, from (2), the inequalities $1-\mu_2^+ < 0 < 1 < 1-\mu_2^-$.

Remark 7.1.2. It is only in the trivial cases $\theta \in \{0, \pi, 2\pi\}$ that A_θ is an orthogonal projection. These cases are also characterized (among cones) by the fact that the spectrum of $\mathcal{A}_\theta$ is included in $[0, 1]$.

Remark 7.1.3. It is interesting to remark that all operators $\mathcal{A}_\theta$ for $\theta \in (0, \pi)$ are unitarily equivalent and thus with constant spectrum $[\mu_2^-, \mu_2^+]$ as given in Theorem 5.2.4. Nevertheless, the sequence $(\mathcal{A}_\theta)_{0<\theta<\pi}$ is weakly converging to the orthogonal projection $\mathcal{A}_\pi$ whose spectrum is $\{0, 1\}$: indeed for $\phi \in \mathscr{S}(\mathbb{R})$, $\psi \in \mathscr{S}(\mathbb{R})$, we have

$$\langle \mathcal{A}_\theta \phi, \psi \rangle_{L^2(\mathbb{R})} = \iint_{\Sigma_\theta} \underbrace{\mathcal{W}(\phi, \psi)}_{\in \mathscr{S}(\mathbb{R}^2)}(x, \xi) dx d\xi,$$

and thus the Lebesgue dominated convergence theorem implies that

$$\lim_{\theta \to \pi_-} \langle \mathcal{A}_\theta \phi, \psi \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\pi \phi, \psi \rangle_{L^2(\mathbb{R})}. \tag{7.1.3}$$

On the other hand, for $u, v \in L^2(\mathbb{R})$ and sequences $(\phi_k)_{k \geq 1}, (\psi_k)_{k \geq 1}$ in $\mathscr{S}(\mathbb{R})$ with respective limits u, v in $L^2(\mathbb{R})$, we have

$$\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\theta (u - \phi_k), v \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})},$$

so that

$$\begin{aligned} &\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})} \\ &\quad = \langle \mathcal{A}_\theta (u - \phi_k), v \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} + \langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})}, \\ &\qquad - \langle \mathcal{A}_\pi (u - \phi_k), v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, v - \psi_k \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, \psi_k \rangle_{L^2(\mathbb{R})}, \end{aligned}$$

implying

$$\begin{aligned} &|\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}| \\ &\qquad \leq (\mu_2^+ + 1)\big(\|u - \phi_k\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} + \|v - \psi_k\|_{L^2(\mathbb{R})} \|\phi_k\|_{L^2(\mathbb{R})}\big) \\ &\qquad\quad + |\langle \mathcal{A}_\theta \phi_k, \psi_k \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi \phi_k, \psi_k \rangle_{L^2(\mathbb{R})}|, \end{aligned}$$

and thus, using (7.1.3), we get

$$\begin{aligned} &\limsup_{\theta \to 0_+} |\langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} - \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}| \\ &\qquad \leq (\mu_2^+ + 1)\big(\|u - \phi_k\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})} + \|v - \psi_k\|_{L^2(\mathbb{R})} \|\phi_k\|_{L^2(\mathbb{R})}\big). \end{aligned}$$

Taking now the infimum with respect to k of the right-hand side in the above inequality, we obtain indeed the weak convergence

$$\lim_{\theta \to 0_+} \langle \mathcal{A}_\theta u, v \rangle_{L^2(\mathbb{R})} = \langle \mathcal{A}_\pi u, v \rangle_{L^2(\mathbb{R})}.$$

Of course, we cannot have strong convergence of the bounded self-adjoint $\mathcal{A}_\theta$ towards (the bounded self-adjoint) A_π because of their respective spectra and the same lines can be written on the weak limit 0 when $\theta \to 0_+$ of $\mathcal{A}_\theta$.

7.2 Triangles

We may consider general "triangles" in the plane that we define as

$$\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} = \big\{(x,\xi) \in \mathbb{R}^2, L_j(x,\xi) \ge c_j, j \in \{1,2,3\}\big\},$$

c_j are real numbers and L_j are linear forms. To avoid degenerate situations, we shall assume that

$$\text{for } j \ne k, \quad dL_j \wedge dL_k \ne 0, \quad |\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3}| > 0 \quad \text{and} \quad \mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} \text{ is not a cone.} \tag{7.2.1}$$

Note that this includes standard triangles (convex envelope of three non-colinear points) but also sets with infinite area such as

$$\big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x + \xi \ge \lambda\big\}, \quad \text{where } \lambda \text{ is a positive parameter.} \tag{7.2.2}$$

Without loss of generality, we may assume that $L_1(x,\xi) - c_1 = x, L_2(x,\xi) - c_2 = \xi$, so that

$$\mathcal{T}_{L_1,L_2,L_3}^{c_1,c_2,c_3} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, ax + b\xi \ge \nu\big\},$$

where a, b, λ are real parameters with $a \ne 0, b \ne 0$ from the assumption (7.2.1); using the symplectic mapping $(x,\xi) \mapsto (\mu x, \xi/\mu)$ with $\mu = \sqrt{|b/a|}$, we see that the condition $ax + b\xi \ge \nu$ becomes

$$x \operatorname{sign} a + \xi \operatorname{sign} b \ge \lambda = \nu/\sqrt{|ab|}, \quad \text{i.e.} \quad \begin{cases} x + \xi & \ge \tilde{\nu}, \\ x - \xi & \ge \tilde{\nu}, \\ -x + \xi & \ge \tilde{\nu}, \\ -x - \xi & \ge \tilde{\nu}. \end{cases}$$

The first case requires $\tilde{\nu} > 0$ and the other cases $\tilde{\nu} < 0$. The only case with finite area is the fourth case

$$\mathcal{T}_{4,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x + \xi \le \lambda\big\} \text{ triangle with area } \lambda^2/2, \lambda > 0. \tag{7.2.3}$$

The second case is

$$\mathcal{T}_{2,\lambda} = \big\{(x,\xi) \in \mathbb{R}^2, x \ge 0, \xi \ge 0, x - \xi \ge -\lambda\big\}, \quad \lambda > 0, \tag{7.2.4}$$

the third case is

$$\mathcal{T}_{3,\lambda} = \{(x,\xi) \in \mathbb{R}^2, x \geq 0, \xi \geq 0, \xi - x \geq -\lambda\}, \quad \lambda > 0, \tag{7.2.5}$$

and the first case is

$$\mathcal{T}_{1,\lambda} = \{(x,\xi) \in \mathbb{R}^2, x \geq 0, \xi \geq 0, \xi + x \geq \lambda\}, \quad \lambda > 0. \tag{7.2.6}$$

Proposition 7.2.1. *Let $\mathcal{T}_{4,\lambda}$ be a triangle with finite non-zero area in the plane given by (7.2.3), where λ is a positive parameter. Then, the operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is unitarily equivalent to the operator with kernel*

$$\tilde{k}_{4,\lambda}(x,y) = \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}. \tag{7.2.7}$$

The operator $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is self-adjoint and bounded on $L^2(\mathbb{R})$ so that

$$\|\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \leq \frac{1}{2}\Big(\mu_2^+ + \sqrt{1+(\mu_2^+)^2}\Big) := \tilde{\mu}_3, \tag{7.2.8}$$

where μ_2^+ is given in (7.1.1).

Proof. The kernel $k_{4,\lambda}$ of $\operatorname{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})$ is such that

$$\begin{aligned}
k_{4,\lambda}(x,y) &= \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\int_0^{\lambda-\frac{x+y}{2}} e^{2i\pi(x-y)\xi}d\xi \\
&= \mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{(e^{2i\pi(x-y)(\lambda-\frac{x+y}{2})}-1)}{2i\pi(x-y)} \\
&= e^{i\pi(\lambda x-\frac{x^2}{2})}\mathbf{1}_{[0,\lambda]}\Big(\frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}e^{-i\pi(\lambda y-\frac{y^2}{2})},
\end{aligned}$$

proving (7.2.7). We note now that the kernel of the operator with Weyl symbol $H(\xi)$ $H(\lambda-\xi-x)$ is

$$\ell_\lambda(x,y) = e^{i\pi(\lambda x-\frac{x^2}{2})}H\Big(\lambda-\frac{x+y}{2}\Big)\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}e^{-i\pi(\lambda y-\frac{y^2}{2})},$$

and that

$$\operatorname{Op_w}\big(H(\xi)H(\lambda-\xi-x)\big)$$

is unitarily equivalent to the operator $\operatorname{Op_w}(H(x)H(\xi))$ as given by Theorem 7.1.1. We get then

$$\begin{aligned}
k_{4,\lambda}(x,y) = H(x+y)\ell_\lambda(x,y) &= H(x)\ell_\lambda(x,y)H(y) \\
&+ H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)H\Big(\lambda-\frac{x+y}{2}\Big) \\
&\times\frac{\sin(\pi(x-y)(\lambda-\frac{x+y}{2}))}{\pi(x-y)}\times e^{i\pi(\lambda x-\frac{x^2}{2})}e^{-i\pi(\lambda y-\frac{y^2}{2})},
\end{aligned}$$

and we have thus

$$\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}}) = H\mathrm{Op_w}\left(H(\xi)H(\lambda-\xi-x)\right)H + \Omega_\lambda,$$

where the kernel $\omega_\lambda(x,y)$ of the operator Ω_λ verifies

$$\begin{aligned}|\omega_\lambda(x,y)| &\le \frac{H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)}{\pi|x-y|}\\ &= \frac{H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)}{\pi(|x|+|y|)}.\end{aligned}$$

We obtain, thanks to Proposition A.5.1 (2), that

$$\iint |\omega_\lambda(x,y)||u(y)||u(x)|dydx \le \|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})}.$$

As a result, we find that

$$|\langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{4,\lambda}})u,u\rangle_{L^2(\mathbb{R})}| \le \mu_2^+\|Hu\|^2_{L^2(\mathbb{R})} + \|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})},$$

proving (7.2.8). ■

Proposition 7.2.2. *Let $\mathcal{T}_{1,\lambda}$ be a triangle with infinite area in the plane given by (7.2.6), where λ is a positive parameter. Then, the operator $\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})$ is unitarily equivalent to the operator with kernel*

$$\tilde{k}_{1,\lambda}(x,y) = \mathbf{1}_{[0,\lambda]}\left(\frac{x+y}{2}\right)\frac{\sin\big(\pi(x-y)(\lambda-\frac{x+y}{2})\big)}{\pi(x-y)}.$$

The operator $\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})$ is self-adjoint and bounded on $L^2(\mathbb{R})$ so that

$$\|\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \frac{1}{2}\left(\mu_2^+ + \sqrt{\frac{1}{4}+(\mu_2^+)^2}\right) \approx 1.066294188078, \quad (7.2.9)$$

where μ_2^+ is given in (7.1.1).

Proof. We note that the kernel of the operator $\mathrm{Op_w}(H(x+\xi-\lambda)H(\xi))$ is

$$\ell_1(x,y) = e^{2i\pi(x-y)\max(0,\lambda-\frac{x+y}{2})}\frac{1}{2}\left(\delta_0(y-x)+\frac{1}{i\pi(y-x)}\right),$$

so that

$$\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}}) = H\underbrace{\mathrm{Op_w}(H(x+\xi-\lambda)H(\xi))}_{\substack{\text{unitarily equivalent to}\\ \mathrm{Op_w}(H(x)H(\xi))}}H + \Omega_{1,\lambda}, \quad (7.2.10)$$

where the kernel $\omega_{1,\lambda}$ of the operator $\Omega_{1,\lambda}$ is equal to

$$H(x+y)\big(H(x)\check{H}(y)+\check{H}(x)H(y)\big)\ell_1(x,y),$$

and such that

$$|\omega_{1,\lambda}(x,y)| \le H(x+y)\frac{(H(x)\check{H}(y)+\check{H}(x)H(y))}{2\pi(|x|+|y|)},$$

and, thanks to Proposition A.5.1 (2), we get from (7.2.10) that

$$|\langle \mathrm{Op_w}(\mathbf{1}_{\mathcal{T}_{1,\lambda}})u,u\rangle_{L^2(\mathbb{R})}| \le \mu_2^+\|Hu\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|\check{H}u\|_{L^2(\mathbb{R})}\|Hu\|_{L^2(\mathbb{R})},$$

which gives (7.2.9). ■

We leave for the reader to check the two other cases (7.2.4), (7.2.5), which are very similar as well as the degenerate cases excluded by (7.2.1), which are in fact easier to tackle.

Theorem 7.2.3. *Let*

$$\mathscr{T} = \big\{\mathcal{T}^{c_1,c_2,c_3}_{L_1,L_2,L_3}\big\}_{\substack{c_j\in\mathbb{R},\ L_j \\ \text{linear form on } \mathbb{R}^2}}$$

be the set of triangles of $\mathbb{R}^2$. *For all* $\mathcal{T}\in\mathscr{T}$, *the operator* $\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}})$ *is bounded on* $L^2(\mathbb{R})$, *self-adjoint and we have*

$$\begin{aligned} 1.007680 \approx \mu_2^+ &= \sup_{\mathcal{C}\ \text{cone}} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{C}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \\ \le \mu_3^+ &= \sup_{\mathcal{T}\ \text{triangle}} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{T}})\|_{\mathcal{B}(L^2(\mathbb{R}))} \le \tilde{\mu}_3 \approx 1.213668. \end{aligned}$$

N.B. The L^2 boundedness is easy to prove since it is obvious for triangles with finite areas and in the case of triangles with infinite area, we may note that in the case (7.2.6) (resp., (7.2.4), (7.2.5)) they are the union of two cones (resp., one cone) with a strip $[0,1]\times\mathbb{R}_+$. What matters most in the above statement is the effective explicit bound. Our result does not give an explicit value for μ_3^+ and it is quite likely that the bound given by $\tilde{\mu}_3$ is way too large.

Proof. The second inequality is proven in Propositions 7.2.1 and 7.2.2, whereas the first inequality is a consequence of Theorem 5.3.1. ■

Remark 7.2.4. This implies that for any $u\in L^2(\mathbb{R})$ and any $\mathcal{T}\in\mathscr{T}$, we have

$$\left|\iint_{\mathcal{T}} \mathcal{W}(u,u)(x,\xi)\,dx\,d\xi\right| \le \tilde{\mu}_3\|u\|^2_{L^2(\mathbb{R})}, \quad \text{with } \tilde{\mu}_3 \approx 1.213668.$$

7.3 Convex polygons

We want to tackle now the general case of a convex polygon in the plane. We consider

$$L_1, \dots, L_N,$$

to be N linear forms of x, ξ ($L_j(x,\xi) = a_j\xi - \alpha_j x = [(x,\xi);(a_j,\alpha_j)]$) and $c_1, \dots, c_N$ some real constants. We consider the convex polygon

$$\mathcal{P} = \big\{(x,\xi) \in \mathbb{R}^2, \forall j \in \{1,\dots,N\}, L_j(x,\xi) - c_j \ge 0\big\}, \tag{7.3.1}$$

so that

$$\mathbf{1}_{\mathcal{P}}(x,\xi) = \prod_{1\le j\le N} H\big(L_j(x,\xi) - c_j\big).$$

Definition 7.3.1. Let $N \in \mathbb{N}^*$, let $L_1, \dots, L_N$ be linear forms on $\mathbb{R}^2$ and let $c_1, \dots, c_N$ be real numbers. The polygon with N sides $\mathcal{P}_{L_1,\dots,L_N}^{c_1,\dots,c_N}$ is defined by (7.3.1). We shall denote by $\mathscr{P}_N$ the set of all polygons with N sides.

N.B. Since we may take some $L_j = 0$ in (7.3.1), we see that $\mathscr{P}_N \subset \mathscr{P}_{N+1}$.

Note as above that it includes some convex subsets of the plane with infinite area such as (7.2.2).

Theorem 7.3.2. *Let $\mathscr{P}_N$ be the set of convex polygons with N sides of the plane $\mathbb{R}^2$. We define*

$$\mu_N^+ = \sup_{\mathcal{P}\in\mathscr{P}_N} \|\mathrm{Op_w}(\mathbf{1}_{\mathcal{P}})\|_{\mathcal{B}(L^2(\mathbb{R}))}.$$

Then, μ_2^+ is given by Theorem 5.2.4 and

$$\forall N \ge 3, \quad \mu_N^+ \le \sqrt{N/2}.$$

Proof. Using an affine symplectic transformation, we may assume that $L_N(x,\xi) - c_N = x$, so that

$$\mathbf{1}_{\mathcal{P}}(x,\xi) = H(x) \prod_{1\le j\le N-1} H\big(a_j\xi - \alpha_j x - c_j\big),$$

and the kernel of the operator $\mathrm{Op_w}(\mathbf{1}_{\mathcal{P}})$ is

$$k_N(x,y) = H(x+y)\int e^{2i\pi(x-y)\xi} \prod_{1\le j\le N-1} H\left(a_j\xi - \alpha_j\Big(\frac{x+y}{2}\Big) - c_j\right) d\xi.$$

As a result, we have

$$k_N(x,y) = H(x+y)k_{N-1}(x,y),$$

where k_{N-1} is the kernel of $\operatorname{Op}_w(\mathbf{1}_{\mathcal{P}_{N-1}})$, where

$$\mathcal{P}_{N-1} = \{(x,\xi) \in \mathbb{R}^2, \forall j \in \{1,\dots,N-1\}, L_j(x,\xi) - c_j \ge 0\}.$$

We may assume inductively that for any convex polygon $\mathcal{P}_k$ with $k \le N-1$ sides, there exist μ_k^+ such that

$$\operatorname{Op}_w(\mathbf{1}_{\mathcal{P}_k}) \le \mu_k^+,$$

where μ_k^+ depends only on k and not on the area of the polygon, a fact already proven for $k = 1, 2, 3$. We note that with $A_N = \operatorname{Op}_w(\mathbf{1}_{\mathcal{P}_N})$, we have with H standing for the operator of multiplication by $H(x)$,

$$HA_NH = HA_{N-1}H, \quad A_{N-1} = \operatorname{Op}_w(\mathbf{1}_{\mathcal{P}_{N-1}}),$$

since the kernel of HA_NH is

$$H(x)H(y)k_N(x,y) = H(x+y)H(x)H(y)k_{N-1}(x,y) = H(x)H(y)k_{N-1}(x,y).$$

Also, we have, with $\check{H}(x) = H(-x)$, that $\check{H}A_N\check{H} = 0$, since the kernel of that operator is

$$\check{H}(x)\check{H}(y)H(x+y)k_{N-1}(x,y) = 0.$$

We have thus

$$A_N = HA_{N-1}H + 2\operatorname{Re}\check{H}A_NH, \tag{7.3.2}$$

and the kernel of $2\operatorname{Re}\check{H}A_NH$ is

$$\omega_N(x,y) = H(x+y)\big(\check{H}(x)H(y) + \check{H}(y)H(x)\big)k_{N-1}(x,y).$$

We calculate now

$$k_{N-1}(x,y) = \int e^{2i\pi(x-y)\xi} \prod_{1\le j\le N-1} H\Big(a_j\xi - \alpha_j(\frac{x+y}{2}) - c_j\Big)d\xi.$$

We check first the j such that $a_j = 0$ (and thus $\alpha_j \ne 0$)[2]. Without loss of generality, we may assume that this happens for $1 \le j < N_0$ so that with some interval J of the real line, $\tilde{\alpha}_j = \alpha_j/a_j, \tilde{c}_j = c_j/a_j$,

$$k_{N-1}(x,y) = \mathbf{1}_J(\frac{x+y}{2}) \int e^{2i\pi(x-y)\xi} \prod_{\substack{N_0\le j\le N-1\\ a_j>0}} H\Big(\xi - \tilde{\alpha}_j(\frac{x+y}{2}) - \tilde{c}_j\Big)$$
$$\times \prod_{\substack{N_0\le j\le N-1\\ a_j<0}} \check{H}\Big(\xi - \tilde{\alpha}_j(\frac{x+y}{2}) - \tilde{c}_j\Big)d\xi.$$

[2]In this induction proof, we may assume that all the linear forms L_j, $1 \le j \le N$ are different from 0, otherwise we may use the induction hypothesis.

We note that the integration domain is

$$\psi\Big(\frac{x+y}{2}\Big) = \max_{\substack{N_0 \le j \le N-1 \\ a_j > 0}} \Big(\tilde{\alpha}_j\Big(\frac{x+y}{2}\Big) + \tilde{c}_j\Big)$$
$$\le \xi \le \min_{\substack{N_0 \le j \le N-1 \\ a_j < 0}} \tilde{\alpha}_j\Big(\frac{x+y}{2}\Big) + \tilde{c}_j = -\phi\Big(\frac{x+y}{2}\Big),$$

with ϕ, ψ convex piecewise affine functions; since $\phi + \psi$ is also a convex function, we get the – convex – constraint $(\phi + \psi)((x+y)/2) \le 0$, so that $(x+y)/2$ must belong to a subinterval $\tilde{J}$ of the interval J. As a result we get that

$$\begin{aligned}
&k_{N-1}(x, y) \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big) \frac{e^{-2i\pi(x-y)\phi(\frac{x+y}{2})} - e^{2i\pi(x-y)\psi(\frac{x+y}{2})}}{2i\pi(x-y)} \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big) e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})} \frac{e^{-i\pi(x-y)(\phi+\psi)(\frac{x+y}{2})} - e^{i\pi(x-y)(\phi+\psi)(\frac{x+y}{2})}}{2i\pi(x-y)} \\
&= \mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big) e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})} \frac{\sin(\pi(x-y)(\phi+\psi)(\frac{x+y}{2}))}{\pi(y-x)},
\end{aligned}$$

and thus the kernel of $2\operatorname{Re} \check{H} A_N H$ is

$$\omega_N(x, y) = H(x+y)\big(\check{H}(x)H(y) + \check{H}(y)H(x)\big)\mathbf{1}_{\tilde{J}}\Big(\frac{x+y}{2}\Big)$$
$$\times\, e^{-i\pi(x-y)(\phi-\psi)(\frac{x+y}{2})} \frac{\sin(\pi(x-y)(\phi+\psi)(\frac{x+y}{2}))}{\pi(y-x)},$$

so that, thanks to Proposition A.5.1 (2),

$$2\operatorname{Re}\langle \check{H} A_N H u, u\rangle \le \|Hu\| \|\check{H}u\|,$$

and with (7.3.2) we obtain, $\langle A_N u, u\rangle \le \mu_{N-1}^{+}\|Hu\|^2 + \|Hu\|\|\check{H}u\|$, and we get

$$\mu_N^{+} \le \frac{\mu_{N-1}^{+} + \sqrt{(\mu_{N-1}^{+})^2 + 1}}{2}.$$

This implies that

$$\forall N \ge 3, \quad \mu_N^{+} \le \sqrt{N/2},$$

since it is true for $N = 3$ and[3] if we assume that it is true for some $N \ge 3$, we get

$$\mu_{N+1}^{+} \le \frac{\mu_N^{+} + \sqrt{(\mu_N^{+})^2 + 1}}{2} \le \frac{1}{2}\Big(\sqrt{\frac{N}{2}} + \sqrt{\frac{N+2}{2}}\Big) \le \sqrt{\frac{N+1}{2}},$$

[3] Indeed, we have $\mu_3^{+} \le \tilde{\mu}_3 < 1.2137 < 1.2247 \approx \sqrt{3/2}$.

where the latter inequality follows from the concavity of the square-root function since we have for a concave function F,

$$\frac{1}{2}\frac{N}{2} + \frac{1}{2}\frac{N+2}{2} = \frac{N+1}{2}$$

and thus

$$\frac{1}{2}F\Big(\frac{N}{2}\Big) + \frac{1}{2}F\Big(\frac{N+2}{2}\Big) \le F\Big(\frac{N+1}{2}\Big).$$

The proof of Theorem 7.3.2 is complete. ∎

Remark 7.3.3. The above result is weak by its dependence on the number of sides, but it should be pointed out that it is independent of the area of the polygon (which could be infinite). Another general comment is concerned with convexity: although Flandrin's conjecture is not true, there is still something special about convex subsets of the phase space and it is in particular interesting that an essentially explicit calculation of the kernel of the operator $\mathrm{Op_w}(\mathbf{1}_{\mathscr{P}})$ is tractable when $\mathscr{P}$ is a polygon with N sides of $\mathbb{R}^2$. Something analogous could probably be done with convex polytopes of $\mathbb{R}^{2n}$.

7.4 Symbols supported in a half-space

Theorem 7.4.1. (1) *Let A be a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$ such that its Weyl symbol $a(x,\xi)$ is supported in $\mathbb{R}_+ \times \mathbb{R}^{2n-1}$. Then, with $\check{H}$ standing for the orthogonal projection onto*

$$\big\{u \in L^2(\mathbb{R}^n), \operatorname{supp} u \subset \mathbb{R}_- \times \mathbb{R}^{n-1}\big\},$$

we have $\check{H}A\check{H} = 0$.

(2) *Let A be as above; if A is a non-negative operator, then with $H = I - \check{H}$, we have*

$$\check{H}A = A\check{H} = 0, \quad A = HAH,$$

N.B. We have seen explicit examples of bounded self-adjoint operators such that the Weyl symbol is supported in $x \ge 0$ but for which $\check{H}AH \ne 0$: the quarter-plane operator (see Section 5.1) has the Weyl symbol $H(x)H(\xi)$, the kernel of

$$\check{H}\mathrm{Op_w}(H(x)H(\xi))H \text{ is } \check{H}(x)H(y)H(x+y)\frac{1}{2i\pi}\mathrm{pv}\frac{1}{y-x},$$

which is not the zero distribution and, according to the above result, this alone implies that

$$\mathrm{Op_w}\left(H(x)H(\xi)\right)$$

cannot be non-negative.

Proof. Let us prove first that $\check{H}A\check{H} = 0$; let $\phi, \psi \in C_c^\infty(\mathbb{R}^n)$ such that

$$\operatorname{supp}\phi \cup \operatorname{supp}\psi \subset (-\infty, 0) \times \mathbb{R}^{n-1}.$$

Since the Wigner distribution $\mathcal{W}(\phi,\psi)$ belongs to $\mathscr{S}(\mathbb{R}^{2n})$ and is given by the integral

$$\mathcal{W}(\phi,\psi)(x,\xi) = \int_{\mathbb{R}^n} \phi\Big(x+\frac{z}{2}\Big)\bar{\psi}\Big(x-\frac{z}{2}\Big)e^{-2i\pi z\cdot\xi}dz,$$

we infer right away[4] that $\operatorname{supp}\mathcal{W}(\phi,\psi) \subset (-\infty,0)\times\mathbb{R}^{2n-1}$. We know also that

$$\langle A\phi,\psi\rangle_{L^2(\mathbb{R}^n)} = \langle A\phi,\psi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(\phi,\psi)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})} = 0.$$

As a result, the $L^2(\mathbb{R}^n)$ bounded operator $\check{H}A\check{H}$ is such that, for $u, v \in L^2(\mathbb{R}^n)$, ϕ,ψ as above,

$$\begin{aligned}
\langle \check{H}A\check{H}u, v\rangle_{L^2(\mathbb{R}^n)} &= \langle \check{H}A\check{H}\check{H}u, \check{H}v\rangle_{L^2(\mathbb{R}^n)} \\
&= \langle \check{H}A\check{H}(\check{H}u-\phi), \check{H}v\rangle_{L^2(\mathbb{R}^n)} + \langle \check{H}A\check{H}\phi, \check{H}v-\psi\rangle_{L^2(\mathbb{R}^n)} \\
&\quad + \underbrace{\langle \check{H}A\check{H}\phi, \psi\rangle_{L^2(\mathbb{R}^n)}}_{\langle A\phi,\psi\rangle_{L^2(\mathbb{R}^n)}=0},
\end{aligned}$$

so that

$$\begin{aligned}
&|\langle \check{H}A\check{H}u, v\rangle_{L^2(\mathbb{R}^n)}| \\
&\le \|A\|_{\mathcal{B}(L^2(\mathbb{R}^n))}\big(\|\check{H}u-\phi\|_{L^2(\mathbb{R}^n)}\|v\|_{L^2(\mathbb{R}^n)} + \|\check{H}v-\psi\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}\big).
\end{aligned}$$

Using now that the set $\{\phi \in C_c^\infty(\mathbb{R}^n), \operatorname{supp}\phi \subset (-\infty,0)\times\mathbb{R}^{n-1}\}$ is dense[5] in

$$\big\{w \in L^2(\mathbb{R}^n), \operatorname{supp} w \subset (-\infty,0]\times\mathbb{R}^{n-1}\big\}, \tag{7.4.1}$$

---

[4]In the integrand, we must have, $x_1 + \frac{z_1}{2} \le -\varepsilon_0 < 0, x_1 - \frac{z_1}{2} \le -\varepsilon_1 < 0$ and thus $x_1 \le -(\varepsilon_0+\varepsilon_1)/2$

[5]Let χ_0 be a function satisfying (5.2.1) and let w be in the set (7.4.1). Let $(\phi_k)_{k\ge1}$ be a sequence in $C_c^\infty(\mathbb{R}^n)$ converging in $L^2(\mathbb{R}^n)$ towards w; the function defined by

$$\tilde{\phi}_k(x) = \chi_0(-kx_1)\phi_k(x),$$

belongs to $C_c^\infty(\mathbb{R}^n)$, is supported in $(-\infty,-1/k]\times\mathbb{R}^{n-1}$, and that sequence converges in $L^2(\mathbb{R}^n)$ towards w since

$$\|\tilde{\phi}_k - w\|_{L^2(\mathbb{R}^n)} \le \underbrace{\|\chi_0(-kx_1)\big(\phi_k(x)-w(x)\big)\|_{L^2(\mathbb{R}^n)}}_{\le\|\phi_k-w\|_{L^2(\mathbb{R}^n)}\to 0 \text{ when } k\to+\infty.} + \|(\chi_0(-kx_1)-1)w(x)\|_{L^2(\mathbb{R}^n)}$$

and $\|(\chi_0(-kx_1)-1)w(x)\|^2_{L^2(\mathbb{R}^n)} \le \int \mathbf{1}\{-\frac{2}{k}\le x_1\le 0\}|w(x)|^2dx$ which has also limit 0 when k goes to $+\infty$ by the Lebesgue dominated convergence theorem.

we obtain that $\langle \check{H} A \check{H} u, v\rangle_{L^2(\mathbb{R}^n)} = 0$ and the first result. Let us assume that the operator A is non-negative. We have

$$A = B^2, \quad B = B^* \text{ bounded self-adjoint.}$$

It implies with $L^2(\mathbb{R}^n)$ norms and dot-products,

$$\begin{aligned}
\langle Au, u\rangle &= \langle HAHu, u\rangle + 2\operatorname{Re}\langle \check{H} AHu, \check{H} u\rangle \\
&= \langle HBBHu, u\rangle + 2\operatorname{Re}\langle \check{H} BBHu, \check{H} u\rangle \\
&= \|BHu\|^2 + 2\operatorname{Re}\langle BHu, B\check{H} u\rangle \\
&= \|BHu + B\check{H} u\|^2 - \|B\check{H} u\|^2 \\
&= \|Bu\|^2 - \|B\check{H} u\|^2 = \langle Au, u\rangle - \|B\check{H} u\|^2,
\end{aligned}$$

and thus $B\check{H} = 0$, so that $\check{H} B = 0$ and thus $\check{H} B^2 = \check{H} A = 0 = A\check{H}$, so that $\check{H} AH = 0 = HA\check{H}$, and $A = HAH$, concluding the proof of (2). ■

Corollary 7.4.2. *Let A be a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$ such that its Weyl symbol is supported in $\mathbb{R}_+ \times \mathbb{R}^{2n-1}$ and such that* $\operatorname{Re}(\check{H} AH) \neq 0$*, then the spectrum of A intersects* $(-\infty, 0)$.

Proof. We have from (1) in the previous theorem,

$$A = (H + \check{H})A(H + \check{H}) = HAH + 2\operatorname{Re} HA\check{H},$$

and from (2), if A were non-negative, we would have $A\check{H} = 0$ and $\operatorname{Re} HA\check{H} = 0$, contradicting the assumption. ■

Remark 7.4.3. If $\mathscr{C}$ is a compact convex body of $\mathbb{R}^{2n}$, we may use the fact (see, e.g., [45]) that

$$\mathscr{C} = \bigcap_{\substack{\mathfrak{H}_j \text{ closed half-spaces} \\ \text{containing } K}} \mathfrak{H}_j.$$

Then, of course $\operatorname{Op_w}(\mathbf{1}_{\mathscr{C}})$ is a bounded self-adjoint operator on $L^2(\mathbb{R}^n)$, and if $\mathfrak{H}_j$ is defined by

$$\mathfrak{H}_j = \{(x, \xi) \in \mathbb{R}^2, L_j(x, \xi) \geq c_j\},$$

where L_j is a linear form on $\mathbb{R}^2$ and c_j a real constant, we obtain with the symplectic covariance of the Weyl calculus, setting

$$\mathscr{H}_j(x, \xi) = H\big(L_j(x, \xi) - c_j\big),$$

that for all $\mathfrak{H}_j$ closed half-spaces containing $\mathscr{C}$, we have

$$\operatorname{Op_w}(\mathbf{1}_{\mathscr{C}}) = \operatorname{Op_w}(H_j)\operatorname{Op_w}(\mathbf{1}_{\mathscr{C}})\operatorname{Op_w}(H_j) + 2\operatorname{Re}\operatorname{Op_w}(\check{H}_j)\operatorname{Op_w}(\mathbf{1}_{\mathscr{C}})\operatorname{Op_w}(H_j),$$

where $\check{H}(x, \xi) = H(-L_j(x, \xi) + c_j)$.