Chapter 8

Open questions and conjectures

In this section, we review the rather long list of conjectures formulated in the text and we try to classify their statements by rating their respective interest, relevance and difficulty. We should keep in mind that the study of $\mathrm{Op_w}(\mathbf{1}_E)$ for a subset E of the phase space is highly correlated to some particular set of special functions related to E: Hermite functions and Laguerre polynomials for ellipses, Airy functions for parabolas, homogeneous distributions for hyperbolas and so on. It is quite likely that the "shape" of E will determine the type of special functions to be studied to getting a diagonalisation of the operator $\mathrm{Op_w}(\mathbf{1}_E)$.

8.1 Anisotropic ellipsoids and paraboloids

Conjecture 8.1.1. Let E be an ellipsoid in $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure. Then, the operator $\mathrm{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ (which is obvious from (1.2.5)) and we have

$$\mathrm{Op_w}(\mathbf{1}_E) \le \mathrm{Id}\,. \tag{8.1.1}$$

A sharp version of this result was proven for $n = 1$ in the 1988 P. Flandrin's article [13], and was improved to an isotropic higher-dimensional setting in paper [39] by E. Lieb and Y. Ostrover. Without isotropy, it remains a conjecture. As described in more details in Section 3.4, it can be reformulated as a problem on Laguerre polynomials. That conjecture is a very natural one and it would be quite surprising that a counterexample to (8.1.1) could occur from an anisotropic ellipsoid[1]. We introduced in Section 4.4 a conjecture on anisotropic paraboloids directly related to Conjecture 8.1.1.

Conjecture 8.1.2. Let E be an anisotropic paraboloid in $\mathbb{R}^{2n}$ equipped with its canonical symplectic structure. Then, the operator $\mathrm{Op_w}(\mathbf{1}_E)$ is bounded on $L^2(\mathbb{R}^n)$ and we have

$$\mathrm{Op_w}(\mathbf{1}_E) \le \mathrm{Id}\,. \tag{8.1.2}$$

In terms of special functions, it is related to a property of Airy-type functions. As a contrast with ellipses, we do not expect (8.1.2) to leave any room for improvement whereas (8.1.1) can certainly be improved with its right-hand side replaced by a smaller operator as in (3.2.2).

[1] We mean by anisotropic ellipsoid a set of type (3.3.2) where $0 < a_1 < a_2 < \cdots < a_n$.

8.2 Balls for the ℓ^p norm

We have seen in Section 5.3.2 that the quantization of the indicatrix of an ℓ^p ball could have a spectrum intersecting $(1, +\infty)$ when $p \neq 2$. More generally one could raise the following question.

Question 8.2.1. Let $p \in [1, +\infty]$, $p \neq 2$ and let $\mathbb{B}_p^{2n}$ be the unit ℓ^p ball in $\mathbb{R}^{2n}$. For $\lambda > 0$, we define the operator

$$P_{n,p,\lambda} = \mathrm{Op_w}(\mathbf{1}_{\lambda \mathbb{B}_p^{2n}}).$$

Is it possible to say something on the spectrum of the operator $P_{n,p,\lambda}$, even in a two-dimensional phase space ($n = 1$)? Is there an asymptotic behaviour for the upper bound of the spectrum of $P_{n,p,\lambda}$ when λ goes to $+\infty$?

8.3 On generic pulses in $L^2(\mathbb{R}^n)$

We have seen that the set $\mathscr{G}$ defined in (6.3.2) is generic in the Baire category sense, but our explicit examples were quite simplistic.

Question 8.3.1. Let $\mathscr{G}$ be defined in (6.3.2). Does there exist $u \in \mathscr{G}$ such that the set $E_+(u)$ (defined in (6.4.3)) is connected?

8.4 On convex bodies

Conjecture 8.4.1. For $N \geq 2$, we define

$$\mu_N^+ = \sup_{\substack{\mathscr{P} \text{ convex bounded} \\ \text{polygon with } N \text{ sides}}} \mathrm{Spectrum}(\mathrm{Op_w}(\mathbf{1}_{\mathscr{P}})).$$

Then, the sequence $(\mu_N^+)_{N \geq 2}$ is increasing[2] and there exists $\alpha > 0$ such that

$$\forall N \geq 2, \quad \mu_N^+ \leq \alpha \ln N.$$

N.B. Theorem 7.3.2 is a small step in this direction.

A stronger version of Conjecture 8.4.1 would be the following conjecture.

Conjecture 8.4.2. We define

$$\mu^+ = \sup_{\substack{\mathscr{C} \text{ convex} \\ \text{bounded}}} \mathrm{Spectrum}(\mathrm{Op_w}(\mathbf{1}_{\mathscr{C}})).$$

Then, we have $\mu^+ < +\infty$.

[2] According to our Definition 7.3.1 of the set $\mathscr{P}_N$ of polygons with N sides is increasing with respect to N.

The invalid Flandrin's conjecture was $\mu^+ = 1$ and we know now that $\mu_+ \geq \mu_2^+ > 1$ as given by (7.1.2).

Question 8.4.3. There is a diagonalisation of quantization of the indicator function of ellipsoids, paraboloids, and hyperbolic regions. Is there a non-quadratic example of diagonalisation?

Note that the quarter-plane, studied in Section 5.1, is somehow a degenerate hyperbolic region, but could be seen as a first answer to the above question. In the phase space $\mathbb{R}^{2n}$, an argument of homogeneity, similar to the one used for the quarter-plane, can probably be useful for handling integrals of the Wigner distribution on cones.

Question 8.4.4. The value of μ_2^+ is known explicitly, but for μ_3^+, we have only the upperbound $\tilde{\mu}_3$ as given by Theorem 7.2.3. Is it possible to determine explicitly the value of μ_3^+, either by answering Question 8.4.3, or via another argument?

Conjecture 8.4.5. Let $\mathscr{C}$ be a proper closed convex subset of $\mathbb{R}^2$ with positive Lebesgue measure such that $\operatorname{Op}_{\mathrm{w}}(\mathbf{1}_{\mathscr{C}})$ is bounded self-adjoint on $L^2(\mathbb{R})$ (that assumption is useless if Conjecture 8.4.2 is proven) with a spectrum included in $[0, 1]$. Then, $\mathscr{C}$ is the strip $[0, 1] \times \mathbb{R}$, up to an affine symplectic map.

All the explicitly available examples are compatible with that conjecture (see also Remark 7.1.2) and the second part of Theorem 7.4.1 is also an indication in that direction. It would be nice in that instance to reach a spectral characterization of a subset modulo the affine symplectic group.