Appendix A

A.1 Fourier transform, Weyl quantization, harmonic oscillator

A.1.1 Fourier transform

For $f \in \mathscr{S}(\mathbb{R}^N)$, we define its Fourier transform by

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2i\pi x\cdot\xi} f(x)dx, \tag{A.1.1}$$

and we obtain the inversion formula

$$f(x) = \int_{\mathbb{R}^N} e^{2i\pi x\cdot\xi} \hat{f}(\xi)d\xi. \tag{A.1.2}$$

Both formulas can be extended to tempered distributions: for $T \in \mathscr{S}'(\mathbb{R}^N)$, we define the tempered distribution $\widehat{T}$ by

$$\langle \widehat{T}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)} = \langle T, \hat{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^N),\mathscr{S}(\mathbb{R}^N)}. \tag{A.1.3}$$

Note also that with this normalization, it is natural to introduce the operators D_x^α defined for $\alpha \in \mathbb{N}^N$ by

$$D_x^\alpha u = D_{x_1}^{\alpha_1}\cdots D_{x_N}^{\alpha_n} u, \quad D_{x_j} u = \frac{\partial u}{2i\pi\partial x_j}, \tag{A.1.4}$$

so that

$$\widehat{D_x^\alpha u} = \xi^\alpha \hat{u}(\xi),$$

with

$$\xi^\alpha = \xi_1^{\alpha_1}\cdots\xi_N^{\alpha_N}.$$

It follows readily from (A.1.1), (A.1.2), and (A.1.3) that for $u \in \mathscr{S}'(\mathbb{R}^n)$, the inversion formula

$$\check{\hat{\hat{u}}} = u, \tag{A.1.5}$$

holds true, where the distribution $\check{u}$ (extending (1.1.10)) is defined by

$$\langle \check{u}, \phi\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle u, \check{\phi}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)}.$$

Using (1.2.6) and denoting the Fourier transformation by $\mathcal{F}$, (A.1.5) read

$$\sigma_0\mathcal{F}^2 = \mathrm{Id}, \quad [\mathcal{F}, \sigma_0] = 0, \text{ so that } \mathcal{F}^* = \mathcal{F}^{-1} = \sigma_0\mathcal{F} = \mathcal{F}\sigma_0. \tag{A.1.6}$$

This normalization yields simple formulas for the Fourier transform of Gaussian functions: for A a real-valued symmetric positive definite $n \times n$ matrix, we define the function v_A in the Schwartz space by

$$v_A(x) = e^{-\pi\langle Ax,x\rangle},$$

and we have

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}.$$

Similarly, when B is a real-valued symmetric non-singular $n \times n$ matrix, the function w_B defined by

$$w_B(x) = e^{i\pi\langle Bx,x\rangle}$$

is in $L^\infty(\mathbb{R}^n)$ and thus a tempered distribution and we have

$$\widehat{w_B}(\xi) = |\det B|^{-1/2} e^{\frac{i\pi}{4}\operatorname{sign} B} e^{-i\pi\langle B^{-1}\xi,\xi\rangle}, \tag{A.1.7}$$

where sign B stands for the *s*ignature of B that is, with $\mathtt{E}$ the set of eigenvalues of B (which are real and non-zero),

$$\operatorname{sign} B = \underbrace{\operatorname{Card}(\mathtt{E} \cap \mathbb{R}_+)}_{\nu_+(B)} - \underbrace{\operatorname{Card}(\mathtt{E} \cap \mathbb{R}_-)}_{\nu_-(B)=\operatorname{index}(B)}.$$

The integer $\nu_-(B)$ is called the *index* of B, noted index (B); formula (A.1.7) can be written as

$$e^{-i\pi n/4}\mathcal{F}\left(e^{i\pi\langle Bx,x\rangle}\right) = i^{-\operatorname{index} B}|\det B|^{-1/2} e^{-i\pi\langle B^{-1}\xi,\xi\rangle}, \tag{A.1.8}$$

since $\nu_+ + \nu_- = n$ (as B is non-singular),

$$e^{\frac{i\pi n}{4}} e^{-\frac{i\pi\nu_-}{2}} = e^{\frac{i\pi}{4}(\nu_+ + \nu_- - 2\nu_-)} = e^{\frac{i\pi}{4}\operatorname{sign}(B)}.$$

We note also that

$$\operatorname{sign}(\det B) = (-1)^{\operatorname{index} B},$$

so that

$$\left(i^{-\operatorname{index} B}|\det B|^{-1/2}\right)^2 = (-1)^{\nu_-}|\det B|^{-1} = \operatorname{sign}(\det B)|\det B|^{-1} = (\det B)^{-1},$$

and thus the prefactor $i^{-\operatorname{index} B}|\det B|^{-1/2}$ in the right-hand side of (A.1.8) is a square-root of $1/\det B$.

With H standing for the characteristic function of $\mathbb{R}_+$, we have

$$1 = H + \check{H}, \quad \delta_0 = \widehat{H} + \check{\widehat{H}},$$

$$D\operatorname{sign} = \frac{\delta_0}{i\pi}, \quad \widehat{D\operatorname{sign}} = \frac{1}{i\pi}, \quad \xi\widehat{\operatorname{sign}} = \frac{1}{i\pi}, \quad \widehat{\operatorname{sign}} = \frac{1}{i\pi}\operatorname{pv}\frac{1}{\xi}, \quad \text{(principal value)}$$

the latter formula following from the fact that

$$\xi\left(\widehat{\operatorname{sign}} - \operatorname{pv}\frac{1}{i\pi\xi}\right) = 0,$$

which implies

$$\widehat{\operatorname{sign}} - \operatorname{pv}\frac{1}{i\pi\xi} = c\delta_0 = 0,$$

since $\widehat{\operatorname{sign}} - \frac{1}{i\pi\xi}$ is odd. We infer from that

$$\hat{H} - \check{\hat{H}} = \widehat{\operatorname{sign}} = \operatorname{pv}\frac{1}{i\pi\xi},$$

and

$$\hat{H} = \frac{\delta_0}{2} + \operatorname{pv}\frac{1}{2i\pi\xi}.$$

Lemma A.1.1. *Let T be a compactly supported distribution on $\mathbb{R}^n$ such that*

$$\forall N \in \mathbb{N}, \quad \langle\xi\rangle^N \widehat{T}(\xi) \quad \textit{is bounded}, \quad \textit{with } \langle\xi\rangle = \sqrt{1+|\xi|^2}. \tag{A.1.9}$$

Then, T is a C^∞ function.

Proof. Note that $\widehat{T}$ is an entire function, as the Fourier transform of a compactly supported distribution. Moreover, from (A.1.9) with $N = n+1$, we get that $\widehat{T}$ belongs to $L^1(\mathbb{R}^n)$ and thus T is a continuous function. Moreover, we have for any $\alpha \in \mathbb{N}^n$,

$$(D_x^\alpha T)(x) = \int e^{2i\pi x\cdot\xi} \underbrace{\xi^\alpha \widehat{T}(\xi)}_{\in L^1(\mathbb{R}^n)} d\xi,$$

so that T is a C^∞ function. ∎

Proposition A.1.2. *Let $\rho > 0$ and let f be a holomorphic function on a neighborhood of $\{z \in \mathbb{C}, |\operatorname{Im} z| \le \rho\}$ such that*

$$\forall y \in [-\rho, \rho], \quad \int |f(x+iy)|dx < +\infty, \tag{A.1.10}$$

$$\lim_{R\to+\infty} \int_{|y|\le\rho} |f(\pm R + iy)|dy = 0. \tag{A.1.11}$$

Then, we have

$$\forall \xi \in \mathbb{R}, \quad |\hat{f}(\xi)| \le Ce^{-2\pi\rho|\xi|},$$

with

$$C = \max(C_+, C_-), \quad C_\pm = \int_{\mathbb{R}} |f(x \pm i\rho)|dx.$$

Conversely, if f is a bounded measurable function such that $\hat{f}(\xi)$ is $O(e^{-2\pi r|\xi|})$ for some $r > 0$, then f is holomorphic on $\{z \in \mathbb{C}, |\operatorname{Im} z| < r\}$.

Proof. If f is holomorphic near $\{z \in \mathbb{C}, |\operatorname{Im} z| \le \rho\}$, satisfies (A.1.10) and (A.1.11), then Cauchy's formula shows that for $|y| \le \rho$,

$$\begin{aligned}\int_{\mathbb{R}} e^{-2i\pi(x+iy)\xi} f(x+iy)dx &= e^{2\pi y\xi} \lim_{R\to+\infty} \int_{-R}^{R} e^{-2i\pi x\xi} f(x+iy)dx \\ &= \lim_{R\to+\infty} \int_{[-R+iy,R+iy]} e^{-2i\pi z\xi} f(z)dz \\ &= \lim_{R\to+\infty} \int_{[-R+iy,-R]\cup[-R,R]\cup[R,R+iy]} e^{-2i\pi z\xi} f(z)dz \\ &= \hat{f}(\xi) + \lim_{R\to+\infty} \Big(\int_0^y e^{-2i\pi(R+it)\xi} f(R+it)idt \\ &\qquad - \int_0^y e^{-2i\pi(-R+it)\xi} f(-R+it)idt \Big).\end{aligned}$$

We have for $|y| \le \rho$,

$$\left| \int_0^y e^{-2i\pi(\pm R+it)\xi} f(\pm R+it)idt \right| \le \int_{|t|\le\rho} |f(\pm R+it)|dt e^{2\pi\rho|\xi|},$$

which goes to 0 when R goes to $+\infty$, thanks to (A.1.11), so that for all $y \in [-\rho, \rho]$, we have

$$\int_{\mathbb{R}} e^{-2i\pi(x+iy)\xi} f(x+iy)dx = \hat{f}(\xi),$$

which implies for $y = -\rho \operatorname{sign} \xi$ (taken as 0, if $\xi = 0$)

$$|\hat{f}(\xi)| \le \int_{\mathbb{R}} |f(x \mp i\rho)|dx\, e^{-2\pi\rho|\xi|} \underbrace{\le}_{\text{from (A.1.10)}} Ce^{-2\pi\rho|\xi|},$$

proving the first part of the proposition. Let us consider now a function f in $L^\infty(\mathbb{R})$ such that $\hat{f}(\xi)$ is $O(e^{-2\pi r|\xi|})$ for some $r > 0$, and let $\rho \in (0, r)$. We have $f(x) = \int e^{2i\pi x\xi} \hat{f}(\xi)d\xi$ and for $|y| \le \rho$, we have $\int_{\mathbb{R}} e^{2\pi|y||\xi|}|\hat{f}(\xi)|d\xi < +\infty$, so that f is holomorphic on $\{z \in \mathbb{C}, |\operatorname{Im} z| < r\}$ with

$$f(x+iy) = \int_{\mathbb{R}} e^{2i\pi(x+iy)\xi} \hat{f}(\xi)d\xi,$$

concluding the proof. ∎

A.1.2 Weyl quantization

Let $a \in \mathscr{S}'(\mathbb{R}^{2n})$. We have defined the operator $\operatorname{Op_w}(a)$, continuous from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$, in Section 1.2.1 with the formula

$$\langle \operatorname{Op_w}(a)u, \bar{v}\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle a, \mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

where the Wigner function $\mathcal{W}(u,v)$ is defined in Definition 1.1.1. We note that the sesquilinear mapping $\mathscr{S}(\mathbb{R}^n)\times\mathscr{S}(\mathbb{R}^n)\ni(u,v)\mapsto\mathcal{W}(u,v)\in\mathscr{S}(\mathbb{R}^{2n})$ is continuous so that the above bracket of duality $\langle a,\mathcal{W}(u,v)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})}$, makes sense. We note as well that a temperate distribution $a\in\mathscr{S}'(\mathbb{R}^{2n})$ gets quantized by a continuous operator $\operatorname{Op_w}(a)$ from $\mathscr{S}(\mathbb{R}^n)$ into $\mathscr{S}'(\mathbb{R}^n)$.

Lemma A.1.3. *Let a be a tempered distribution on $\mathbb{R}^{2n}$ and let b be a polynomial of degree d on $\mathbb{R}^{2n}$. Then, we have*

$$a\sharp b=\sum_{0\le k\le d}\omega_k(a,b),\quad \textit{with}$$

$$\omega_k(a,b)=\frac{1}{(4i\pi)^k}\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)(\partial_x^\alpha\partial_\xi^\beta b)(x,\xi),\tag{A.1.12}$$

$$\omega_k(b,a)=(-1)^k\omega_k(a,b).\tag{A.1.13}$$

The Weyl symbol of the commutator $[\operatorname{Op_w}(a),\operatorname{Op_w}(b)]$ *is*

$$c(a,b)=2\sum_{\substack{0\le k\le d\\ k\ \text{odd}}}\omega_k(a,b).$$

If the degree of b is smaller than 2, we have

$$c(a,b)=2\omega_1(a,b)=\frac{1}{2\pi i}\{a,b\},$$

and if a is a function of b, the commutator $[\operatorname{Op_w}(a),\operatorname{Op_w}(b)]=0$.

Remark A.1.4. In particular, if $q(x,\xi)$ is a quadratic polynomial and $a(x,\xi)=H(1-q(x,\xi))$, is the characteristic function of the set $\{(x,\xi),q(x,\xi)\le 1\}$, then we have $[\operatorname{Op_w}(a),\operatorname{Op_w}(q)]=0$.

Proof. Applying (1.2.2), (1.2.3), we obtain that this lemma follows from (A.1.13), that we check now

$$\begin{aligned}(4i\pi)^k\omega_k(a,b)&=\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\beta|}}{\alpha!\beta!}(\partial_\xi^\alpha\partial_x^\beta a)(x,\xi)(\partial_x^\alpha\partial_\xi^\beta b)(x,\xi)\\&=\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{|\alpha|}}{\alpha!\beta!}(\partial_\xi^\beta\partial_x^\alpha a)(x,\xi)(\partial_x^\beta\partial_\xi^\alpha b)(x,\xi)\\&=\sum_{|\alpha|+|\beta|=k}\frac{(-1)^{k-|\beta|}}{\alpha!\beta!}(\partial_\xi^\beta\partial_x^\alpha a)(x,\xi)(\partial_x^\beta\partial_\xi^\alpha b)(x,\xi)\\&=(-1)^k(4i\pi)^k\omega_k(b,a),\end{aligned}$$

which is the sought result. ∎

Remark A.1.5. We can note that formula (1.2.61) is non-local in the sense that for $a, b \in \mathscr{S}(\mathbb{R}^{2n})$ with disjoint supports, although all $\omega_k(a, b)$ (given by (A.1.12)) are identically 0, the function $a \sharp b$ (which belongs to $\mathscr{S}(\mathbb{R}^{2n})$) is different from 0; let us give an example. Let $\chi_0 \in C_c^\infty(\mathbb{R}; [0, 1])$ with support $[-1 + \varepsilon_0, 1 - \varepsilon_0]$ with $\varepsilon_0 \in (0, 1)$ and let us consider in $\mathbb{R}^2$,

$$a(x, \xi) = \chi_0(x) e^{-\pi\xi^2}, \quad b(x, \xi) = \chi_0(x - 2) e^{-\pi\xi^2},$$

so that a, b both belong to $\mathscr{S}(\mathbb{R}^2)$ and

$$\operatorname{supp} a = [-1 + \varepsilon_0, 1 - \varepsilon_0] \times \mathbb{R}, \quad \operatorname{supp} b = [1 + \varepsilon_0, 3 - \varepsilon_0] \times \mathbb{R},$$

so that the supports are disjoint and all $\omega_k(a, b)$ are identically vanishing. We check now

$$\begin{aligned}
&(a\sharp b)(x, \xi) \\
&= 4 \iiiint \chi_0(y) e^{-\pi\eta^2} \chi_0(z - 2) e^{-\pi\zeta^2} e^{-4i\pi(\xi-\eta)(x-z)} e^{4i\pi(x-y)(\xi-\zeta)} dy\, d\eta\, dz\, d\zeta \\
&= 4 \iint \chi_0(y) \chi_0(z - 2) e^{-4\pi(x-z)^2} e^{-4\pi(x-y)^2} e^{4i\pi\xi(z-x+x-y)} dy\, dz \\
&= 4 \Big(\int \chi_0(y) e^{-4i\pi\xi y} e^{-4\pi(x-y)^2} dy \Big) \Big(\int \chi_0(z) e^{4i\pi\xi z} e^{-4\pi(x-2-z)^2} dz \Big),
\end{aligned}$$

so that

$$(a\sharp b)(0, 0) = 4 \underbrace{\Big(\int \chi_0(y) e^{-4\pi y^2} dy \Big)}_{>0} \underbrace{\Big(\int \chi_0(z) e^{-4\pi(2+z)^2} dz \Big)}_{>0} > 0.$$

A.1.3 Some explicit computations

We may also calculate with

$$u_a(x) = (2a)^{1/4} e^{-\pi a x^2}, a > 0, \tag{A.1.14}$$

$$\begin{aligned}
\mathcal{W}(u_a, u_a)(x, \xi) &= (2a)^{1/2} \int e^{-2i\pi z\cdot\xi} e^{-\pi a|x - \frac{z}{2}|^2} e^{-\pi a|x + \frac{z}{2}|^2} dz \\
&= (2a)^{1/2} \int e^{-2i\pi z\cdot\xi} e^{-2\pi a x^2} e^{-\pi a z^2/2} dz \\
&= (2a)^{1/2} e^{-2\pi a x^2} 2^{1/2} a^{-1/2} e^{-\pi \frac{2}{a} \xi^2} \\
&= 2 e^{-2\pi(a x^2 + a^{-1}\xi^2)},
\end{aligned}$$

which is also a Gaussian function on the phase space (and positive function). The calculation of

$$\mathcal{W}(u_a', u_a')(x, \xi)$$

is interesting since we have

$$4\pi^2\langle D_x b^w D_x u_a, \bar u_a\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle b^w u'_a, \bar u'_a\rangle_{\mathscr{S}'(\mathbb{R}^n),\mathscr{S}(\mathbb{R}^n)} = \langle b, \mathcal{W}(u'_a, u'_a)\rangle_{\mathscr{S}'(\mathbb{R}^{2n}),\mathscr{S}(\mathbb{R}^{2n})},$$

and for $b(x,\xi)$ real-valued we have

$$\xi\sharp b\sharp\xi = \Big(\xi b + \frac{b'_x}{4i\pi}\Big)\sharp\xi = \xi^2 b + \frac{b'_x\xi}{4i\pi} - \frac{\partial_x}{4i\pi}\Big(\xi b + \frac{b'_x}{4i\pi}\Big) = \xi^2 b + \frac{b''_{xx}}{16\pi^2},$$

so that

$$4\pi^2\iint 2e^{-2\pi(ax^2+a^{-1}\xi^2)}\Big(\xi^2 b + \frac{b''_{xx}}{16\pi^2}\Big)dxd\xi = \langle b, \mathcal{W}(u'_a, u'_a)\rangle,$$

proving that

$$\begin{aligned}\mathcal{W}(u'_a, u'_a)(x,\xi) &= 2e^{-2\pi(ax^2+a^{-1}\xi^2)}4\pi^2\xi^2 + \frac14 2\partial_x^2\big(e^{-2\pi(ax^2+a^{-1}\xi^2)}\big)\\ &= 2e^{-2\pi(ax^2+a^{-1}\xi^2)}\Big(4\pi^2\xi^2 + \frac14((-4\pi a x)^2 - 4\pi a)\Big)\\ &= 8\pi^2 e^{-2\pi(ax^2+a^{-1}\xi^2)}a\Big(a^{-1}\xi^2 + ax^2 - \frac{1}{4\pi}\Big).\end{aligned}$$

We obtain that the function $\mathcal{W}(u'_a, u'_a)$ is negative on

$$a^{-1}\xi^2 + ax^2 < \frac{1}{4\pi},$$

which has area $1/4$. We may note as well for consistency for u_a given by (A.1.14), we have

$$u'_a = (2a)^{1/4}(-2\pi a x)e^{-\pi a x^2}, \quad \|u'_a\|^2_{L^2} = \pi a,$$

and

$$\begin{aligned}\iint \mathcal{W}(u'_a, u'_a)(x,\xi)dxd\xi &= 8\pi^2 a\iint e^{-2\pi(y^2+\eta^2)}\Big(y^2+\eta^2 - \frac{1}{4\pi}\Big)dyd\eta\\ &= \frac{8\pi^2 a}{8\pi} = \pi a = \|u'_a\|^2_{L^2}.\end{aligned}$$

For $\lambda > 0$ and $a \in \mathscr{S}'(\mathbb{R}^{2n})$, we define

$$a_\lambda(x,\xi) = a(\lambda^{-1}x, \lambda\xi),$$

and we find that

$$(a_\lambda)^w = U^*_\lambda a^w U_\lambda, \tag{A.1.15}$$
$$\text{for}\quad f \in \mathscr{S}(\mathbb{R}^n),\ (U_\lambda f)(x) = f(\lambda x)\lambda^{n/2},\quad U^*_\lambda = U_{\lambda^{-1}} = (U_\lambda)^{-1}.$$

We note that the above formula is a particular case of *Segal's formula* (see, e.g., [33, Theorem 2.1.2]).

A.1.4 The harmonic oscillator

The harmonic oscillator $\mathcal{H}_n$ in n dimensions is defined as the operator with Weyl symbol $\pi(|x|^2 + |\xi|^2)$ and thus from (A.1.15), we find that

$$\mathcal{H} = U_{\sqrt{2\pi}} \frac{1}{2}\big(|x|^2 + 4\pi^2|\xi|^2\big)^w U^*_{\sqrt{2\pi}} = U_{\sqrt{2\pi}} \frac{1}{2}(-\Delta + |x|^2) U^*_{\sqrt{2\pi}}.$$

We shall define in one dimension the Hermite function of level $k \in \mathbb{N}$, by

$$\psi_k(x) = \frac{(-1)^k}{2^k\sqrt{k!}} 2^{1/4} e^{\pi x^2} \left(\frac{d}{\sqrt{\pi}\,dx}\right)^k (e^{-2\pi x^2}), \tag{A.1.16}$$

and we find that $(\psi_k)_{k\in\mathbb{N}}$ is a Hilbertian orthonormal basis on $L^2(\mathbb{R})$. The one-dimensional harmonic oscillator can be written as

$$\mathcal{H}_1 = \sum_{k\geq 0} \left(\frac{1}{2} + k\right) \mathbb{P}_k, \tag{A.1.17}$$

where $\mathbb{P}_k$ is the orthogonal projection onto ψ_k.

In n dimensions, we consider a multi-index $(\alpha_1, \ldots, \alpha_n) = \alpha \in \mathbb{N}^n$ and we define on $\mathbb{R}^n$, using the one-dimensional (A.1.16),

$$\Psi_\alpha(x) = \prod_{1\leq j\leq n} \psi_{\alpha_j}(x_j), \quad \mathcal{E}_k = \operatorname{Vect}\{\Psi_\alpha\}_{\alpha\in\mathbb{N}^n, |\alpha|=k}, \quad |\alpha| = \sum_{1\leq j\leq n} \alpha_j. \tag{A.1.18}$$

We note that the dimension of $\mathcal{E}_{k,n}$ is

$$\binom{k+n-1}{n-1},$$

and that (A.1.17) holds with $\mathbb{P}_{k;n}$ standing for the orthogonal projection onto $\mathcal{E}_{k,n}$; the lowest eigenvalue of $\mathcal{H}_n$ is $n/2$ and the corresponding eigenspace is one-dimensional in all dimensions, although in two and more dimensions, the eigenspaces corresponding to the eigenvalue $\frac{n}{2} + k, k \geq 1$ are multi-dimensional with dimension $\binom{k+n-1}{n-1}$. The n-dimensional harmonic oscillator can be written as

$$\mathcal{H}_n = \sum_{k\geq 0} \left(\frac{n}{2} + k\right) \mathbb{P}_{k;n},$$

where $\mathbb{P}_{k;n}$ stands for the orthogonal projection onto $\mathcal{E}_{k,n}$ defined above. We have in particular

$$\mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, |\alpha|=k} \mathbb{P}_\alpha, \quad \text{where } \mathbb{P}_\alpha \text{ is the orthogonal projection onto } \Psi_\alpha. \tag{A.1.19}$$

A.1.5 On the spectrum of the anisotropic harmonic oscillator

The standard n-dimensional harmonic oscillator is the operator

$$\mathcal{H}_n = \pi \sum_{1\le j\le n} (D_j^2 + x_j^2), \quad D_j = \frac{1}{2\pi i}\partial_{x_j},$$

and its spectral decomposition is

$$\mathcal{H} = \sum_{k\ge 0}\left(\frac{n}{2}+k\right)\mathbb{P}_{k;n}, \quad \mathbb{P}_{k;n} = \sum_{\alpha\in\mathbb{N}^n, \alpha_1+\dots+\alpha_n=k} \mathbb{P}_{\alpha_1}\otimes\dots\otimes\mathbb{P}_{\alpha_n},$$

where $\mathbb{P}_{\alpha_j}$ stands for the orthogonal projection onto the one-dimensional Hermite function with level α_j. Now let us consider for $\mu = (\mu_1, \dots, \mu_n)$ with $\mu_j > 0$, the operator

$$\mathcal{H}_{(\mu)} = \pi \sum_{1\le j\le n} \mu_j (D_j^2 + x_j^2) = \pi \mathrm{Op_w}(q_\mu(x,\xi)),$$

with

$$q_\mu(x,\xi) = \sum_{1\le j\le n} \mu_j(x_j^2 + \xi_j^2).$$

With the notation $|\mu| = \sum_{1\le j\le n}\mu_j$ and $\mu\cdot\alpha = \sum_{1\le j\le n}\mu_j\alpha_j$, we have

$$\mathcal{H}_{(\mu)} = \sum_{\alpha\in\mathbb{N}^n}\left(\frac{|\mu|}{2}+\mu\cdot\alpha\right)\underbrace{(\mathbb{P}_{\alpha_1}\otimes\dots\otimes\mathbb{P}_{\alpha_n})}_{\mathbb{P}_\alpha},$$

so that the eigenspaces are the same as for $\mathcal{H}_n$ but the arithmetic properties of μ make possible that all eigenvalues $(\frac{|\mu|}{2} + \mu\cdot\alpha)$ are simple. For instance for

$$n = 2, 0 < \mu_1 < \mu_2, \quad \frac{\mu_2}{\mu_1} \notin \mathbb{Q},$$

if $\beta \in \mathbb{Z}^2$ is such that $\mu_1\beta_1 + \mu_2\beta_2 = 0$, this implies that $\beta = 0$ and thus that all the eigenvalues of $\mathcal{H}_{(\mu)}$ are simple.

Remark A.1.6. If $0 < \mu_1 \le \dots \le \mu_n$ and if for all $j \in [2,n]$ we have $\mu_j/\mu_1 \in \mathbb{N}$, we then have for $\alpha\in\mathbb{N}^n$,

$$\alpha\cdot\mu = \underbrace{\mu_1\left(\alpha_1 + \sum_{2\le j\le n}\frac{\alpha_j\mu_j}{\mu_1}\right)}_{\beta_1} = \beta\cdot\mu, \quad \beta = (\beta_1, 0, \dots, 0)\in\mathbb{N}^n.$$

Sinus cardinal. It is a classical result of Distribution Theory that the weak limit when $\lambda \to +\infty$ of the sinus cardinal $\frac{\sin(\lambda x)}{x}$ is $\pi\delta_0$, where δ_0 is the Dirac mass at 0, but we wish to extend that result to more general test functions.

Lemma A.1.7. *Let f be a function in $L^1_{loc}(\mathbb{R})$ such that*

$$\int_{|\tau|\ge 1} \frac{|f(\tau)|}{|\tau|} d\tau < +\infty \quad \text{and} \quad \exists a \in \mathbb{C} \text{ so that } \int_{|\tau|\le 1} \frac{|f(\tau)-a|}{|\tau|} d\tau < +\infty.$$

Then, we have

$$\lim_{\lambda\to+\infty} \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau} f(\tau) d\tau = a. \tag{A.1.20}$$

N.B. In particular, if f is a Hölderian function such that $f(\tau)/\tau \in L^1(\{|\tau| \ge 1\})$ we get that the left-hand side of (A.1.20) equals $f(0)$.

Proof. Let χ_0 be a function in $C_c^\infty(\mathbb{R})$ equal to 1 near the origin and let us define $\chi_1 = 1 - \chi_0$. We have

$$\int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau} f(\tau) d\tau = \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi} \underbrace{\frac{\big(f(\tau)-a\big)}{\tau}\chi_0(\tau)}_{\in L^1(\mathbb{R})} d\tau + a \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\chi_0(\tau) d\tau$$
$$+ \int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi} \underbrace{f(\tau)\tau^{-1}\chi_1(\tau)}_{\in L^1(\mathbb{R})} d\tau,$$

so that the limit when $\lambda \to +\infty$ of the first and the third integral is zero, thanks to the Riemann–Lebesgue lemma. We note also that

$$\frac{\sin(\lambda\tau)}{\pi\tau} = \widehat{\mathbf{1}_{[-\frac{\lambda}{2\pi},\frac{\lambda}{2\pi}]}}(\tau),$$

and applying Plancherel's formula to the second integral yields

$$\int_{\mathbb{R}} \frac{\sin(\lambda\tau)}{\pi\tau}\chi_0(\tau) d\tau = \int_{|t|\le\lambda/(2\pi)} \widehat{\chi_0}(t) dt,$$

whose limit when $\lambda \to +\infty$ is $\int_{\mathbb{R}} \widehat{\chi_0}(t)dt = \chi_0(0) = 1$, thanks to the Lebesgue dominated convergence theorem, completing the proof of the lemma. ■

A.2 Further properties of the metaplectic group

A.2.1 Another set of generators for the metaplectic group

Definition A.2.1. Let P, L, Q be $n \times n$ real matrices such that $P = P^*$, $Q = Q^*$ and $\det L \ne 0$. We define the operator $\mathcal{M}_{P,L,Q}$ by the formula

$$(\mathcal{M}_{P,L,Q}u)(x) = e^{-i\pi n/4}(\det L)^{1/2} \int_{\mathbb{R}^n} e^{i\pi\{\langle Px,x\rangle - 2\langle Lx,y\rangle + \langle Qy,y\rangle\}} u(y) dy.$$

N.B. In that definition, $(\det L)^{1/2}$ stands for a choice of a square-root of the real number $\det L$, that is $\pm\sqrt{\det L}$ if $\det L > 0$ and $\pm i\sqrt{-\det L}$ if $\det L < 0$.

With $m(L) \in \mathbb{Z}/4\mathbb{Z}$ defined by (1.2.34) we shall also define

$$\big(\mathcal{M}^{\{m(L)\}}_{P,L,Q}u\big)(x) = e^{-\frac{i\pi n}{4}} e^{\frac{i\pi m(L)}{2}} |\det L|^{1/2} \int_{\mathbb{R}^n} e^{i\pi\{\langle Px,x\rangle - 2\langle Lx,y\rangle + \langle Qy,y\rangle\}} u(y)dy.$$

Proposition A.2.2. *The operator $\mathcal{M}_{P,L,Q}$ given in Definition A.2.1 is an automorphism of $\mathscr{S}(\mathbb{R}^n)$ and of $\mathscr{S}'(\mathbb{R}^n)$ which is a unitary operator on $L^2(\mathbb{R}^n)$ belonging to the metaplectic group (cf. Definition 1.2.13). Moreover, the metaplectic group is generated by the set*

$$\{\mathcal{M}_{P,L,Q}\}_{\substack{P=P^*,Q=Q^*\\ \det L\neq 0}}.$$

Proof. Using the notation (1.2.28) and (1.2.37), we see that[1]

$$M^{\{m(B)\}}_{A,B,C} = \mathcal{M}^{\{m(B)+n\}}_{A,-B,C}\mathcal{F}e^{-i\pi n/4}, \quad \mathcal{M}^{\{m(L)\}}_{P,L,Q} = M^{\{m(L)-n\}}_{P,-L,Q}\big(\mathcal{F}e^{-i\pi n/4}\big)^{-1}, \tag{A.2.1}$$

and (1.2.44) imply that the set $\{\mathcal{M}_{P,L,Q}\}$ is included in $\mathrm{Mp}(n)$ (second formula in (A.2.1)) whereas the fact that

$$\mathcal{F}e^{-i\pi n/4} = \mathcal{M}^{\{0\}}_{0,I_n,0},$$

the first formula in (A.2.1) and Definition 1.2.13 imply that $\mathrm{Mp}(n)$ is generated by the set $\{\mathcal{M}_{P,L,Q}\}$, proving the proposition. ∎

Remark A.2.3. From (A.2.1), we deduce, noting $m(I_n) \in \{0,2\}, m(-I_n) \in \{n, n+2\}$,

$$-\mathrm{Id}_{L^2(\mathbb{R}^n)} = M^{\{2\}}_{0,I_n,0} = \mathcal{M}^{\{n+2\}}_{0,-I_n,0}\mathcal{M}^{\{0\}}_{0,I_n,0},$$

so that

$$\mathcal{M}^{\{m(L)+2\}}_{P,L,Q} = -\mathcal{M}^{\{m(L)\}}_{P,L,Q} = \mathcal{M}^{\{n+2\}}_{0,-I_n,0}\mathcal{M}^{\{0\}}_{0,I_n,0}\mathcal{M}^{\{m(L)\}}_{P,L,Q}.$$

---

[1] We note that $m(B)+n \in \{m(-B), m(-B)+2\}$ modulo 4: indeed, we have modulo 4

$$\begin{cases} \text{for } n \text{ even}, & \underbrace{\{0,2\}}_{\det B>0} +n = \underbrace{\{0,2\}}_{\det(-B)>0}, \quad \underbrace{\{1,3\}}_{\det B<0} +n = \underbrace{\{1,3\}}_{\det(-B)<0}, \\ \text{for } n \text{ odd}, & \underbrace{\{0,2\}}_{\det B>0} +n = \underbrace{\{1,3\}}_{\det(-B)<0}, \quad \underbrace{\{1,3\}}_{\det B<0} +n = \underbrace{\{0,2\}}_{\det(-B)>0}. \end{cases}$$

We have also $m(L)-n \in \{m(-L), m(-L)+2\}$ since we know already (from the above in that footnote) that $m(L)-n \in \{m(-L), m(-L)+2\} - 2n$, which gives $m(L)-n \in \{m(-L), m(-L)+2\}$ for n even; for $n = 2l+1$ odd we get the same result since

$$m(L)-n \in \{m(-L), m(-L)+2\} - 4l - 2 = \{m(-L)-2, m(-L)\} = \{m(-L)+2, m(-L)\}.$$

Lemma A.2.4. *With the homomorphism* Ψ *defined in* (1.2.46) *and defining*

$$\Lambda_{P,L,Q} = \Psi(\mathcal{M}_{P,L,Q}),$$

we find that

$$\Lambda_{P,L,Q} = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^* & PL^{-1} \end{pmatrix}.$$

Proof. Indeed, from the second formula in (A.2.1), (1.2.38), (1.2.27), and (1.2.47) we get that

$$\Lambda_{P,L,Q} = \Xi_{P,-L,Q}\,\Xi^{-2}_{-I_n,2^{1/2}I_n,-I_n} = \begin{pmatrix} -L^{-1} & L^{-1}Q \\ -PL^{-1} & -L^* + PL^{-1}Q \end{pmatrix}\begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

providing the sought result. ■

Lemma A.2.5. *Let* P_j, L_j, Q_j, $j = 1, 2$ *be as in Definition* A.2.1 *and let us assume that*

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} = e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}, \quad \phi \in \mathbb{R}. \tag{A.2.2}$$

Then, we have

$$P_1 + Q_2 = Q_1 + P_2 = 0, \quad L_2 = -L_1^*, \quad e^{i\phi} \in \{\pm 1\}. \tag{A.2.3}$$

Proof. The assumption (A.2.2) implies that both sides of the equality belong to $\mathrm{Mp}(n)$ and

$$\Lambda_{P_1,L_1,Q_1}\Lambda_{P_2,L_2,Q_2} = \Psi\big(e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}\big) = I_{2n},$$

where the last equality follows from the fact that $e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}$ commutes with every operator $\mathrm{Op_w}(L_Y)$ given in Lemma 1.2.17. We have thus

$$\begin{pmatrix} L_1^{-1}Q_1 & L_1^{-1} \\ P_1L_1^{-1}Q_1 - L_1^* & P_1L_1^{-1} \end{pmatrix}\begin{pmatrix} L_2^{-1}Q_2 & L_2^{-1} \\ P_2L_2^{-1}Q_2 - L_2^* & P_2L_2^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix},$$

so that

first line × second column: $L_1^{-1}Q_1L_2^{-1} + L_1^{-1}P_2L_2^{-1} = 0 \Longrightarrow Q_1 + P_2 = 0,$

second line × first column: $(P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1}Q_2 + P_1L_1^{-1}(P_2L_2^{-1}Q_2 - L_2^*) = 0,$

second line × second column: $(P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1} + P_1L_1^{-1}P_2L_2^{-1} = I_n,$

which gives

$$(P_1L_1^{-1}Q_1 - L_1^*)L_2^{-1} + P_1L_1^{-1}\underbrace{P_2}_{-Q_1}L_2^{-1} = I_n \Rightarrow -L_1^*L_2^{-1} = I_n \Rightarrow L_2 = -L_1^*,$$

$$P_1L_1^{-1}Q_1L_2^{-1}Q_2\underbrace{-L_1^*L_2^{-1}}_{I_n}Q_2 + P_1L_1^{-1}\underbrace{P_2}_{-Q_1}L_2^{-1}Q_2 - P_1\underbrace{L_1^{-1}L_2^*}_{-I_n} = 0 \Rightarrow P_1 + Q_2 = 0,$$

providing the sought formulas in (A.2.3), except for the last one. Let κ_j be the kernel of $\mathcal{M}_{P_j,L_j,Q_j}$ and let $\kappa = \kappa_1 \circ \kappa_2$ be the kernel of the composition (in the left-hand side of (A.2.2)). We have consequently

$$\begin{aligned}
&\kappa(x,y)\\
&= (\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}\int e^{i\pi\{P_1x^2-2L_1x\cdot z+Q_1z^2+P_2z^2-2L_2z\cdot y+Q_2y^2\}}dz\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\int e^{-2i\pi\{z\cdot(L_1x+L_2^*y)\}}dz\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\delta_0(L_1x+L_2^*y)\\
&= (\det L_1)^{1/2}(\det(-L_1^*))^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2-P_1y^2\}}\delta_0(x-y)|\det L_1|^{-1},
\end{aligned}$$

entailing

$$\begin{aligned}
e^{i\phi}\delta_0(x-y)\underbrace{=}_{(A.2.2)}\kappa(x,y) &= e^{i\frac{\pi}{2}(m(L_1)+m(L_1^*)+n)}\delta_0(x-y)e^{-i\pi n/2}\\
&= e^{i\pi m(L_1)}\delta_0(x-y),
\end{aligned}$$

proving that $e^{i\phi} = e^{i\pi m(L_1)} \in \{\pm 1\}$. The proof of the lemma is complete. ∎

Claim A.2.6. Let P, L, Q be as in Definition A.2.1. Then, we have

$$(\mathcal{M}_{P,L,Q}^{\{m(L)\}})^{-1} = \mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}}, \tag{A.2.4}$$

and moreover $n - m(L) \in \{m(-L^*), m(-L^*)+2\}$ modulo 4.

Proof of Claim A.2.6. Indeed, calculating the kernel κ of $\mathcal{M}_{P,L,Q}^{\{m(L)\}}\mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}}$, we get

$$\begin{aligned}
\kappa(x,y) &= e^{\frac{i\pi}{2}(m(L)+n-m(L)-n)}|\det L|\int e^{i\pi\{Px^2-2Lx\cdot z+Qz^2-Qz^2+2L^*z\cdot y-Py^2\}}dz\\
&= |\det L|e^{i\pi\{Px^2-Py^2\}}\delta_0(Lx-Ly) = \delta_0(x-y),
\end{aligned}$$

so that

$$\mathcal{M}_{P,L,Q}^{\{m(L)\}}\mathcal{M}_{-Q,-L^*,-P}^{\{n-m(L)\}} = \mathrm{Id}_{L^2(\mathbb{R}^n)}$$

and since $\mathcal{M}_{P,L,Q}$ is unitary, this proves (A.2.4). The last assertion is equivalent to $m(L) \in \{n - m(-L^*), n - m(-L^*) - 2\}$. Since the latter set is equal to $\{-m(L), -m(L)-2\}$ and the mapping

$$\mathbb{Z}/4\mathbb{Z} \ni x \mapsto -x \in \mathbb{Z}/4\mathbb{Z},$$

leaves invariant the sets $\{0,2\}, \{1,3\}$, we obtain the sought result, concluding the proof of the claim. ∎

Proposition A.2.7. *Let P_j, L_j, Q_j, $j = 1, 2$ be as in Definition A.2.1 and let us assume that*

$$\det(Q_1 + P_2) \neq 0.$$

Then, there exist P, L, Q, as in Definition A.2.1 such that

$$\mathcal{M}^{\{m(L_1)\}}_{P_1,L_1,Q_1}\mathcal{M}^{\{m(L_2)\}}_{P_2,L_2,Q_2} = \mathcal{M}^{\{m(L_1)+m(L_2)-\text{index}\,(Q_1+P_2)\}}_{P,L,Q}.$$

More precisely, we have

$$\begin{aligned}&P = P_1 - L_1^*(Q_1 + P_2)^{-1}L_1, \quad Q = Q_2 - L_2(Q_1 + P_2)^{-1}L_2^*,\\ &L = L_2(Q_1 + P_2)^{-1}L_1.\end{aligned}$$

Moreover, we have

$$m(L_1) + m(L_2) - \text{index}\,(Q_1 + P_2) \in \{m(L), m(L) + 2\} \mod 4.$$

Proof. The kernel κ of $\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}$ is

$$\begin{aligned}&\kappa(x, y)\\ &= (\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}\int e^{i\pi\{P_1x^2-2L_1x\cdot z+Q_1z^2+P_2z^2-2L_2z\cdot y+Q_2y^2\}}dz\\ &= (\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2+Q_2y^2\}}\\ &\quad\times\int e^{-2i\pi(L_1x+L_2^*y)\cdot z}e^{i\pi(Q_1+P_2)z^2}dz\\ &= (\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}e^{i\pi\{P_1x^2+Q_2y^2\}}e^{-i\pi(Q_1+P_2)^{-1}(L_1x+L_2^*y)^2}\\ &\quad\times|\det(Q_1 + P_2)|^{-1/2}e^{i\frac{\pi}{4}\,\text{sign}(Q_1+P_2)},\end{aligned}$$

according to formula (A.1.7) (see also (A.1.8)), noting that the matrix $Q_1 + P_2$ is real symmetric and non-singular. As a result, we have

$$\begin{aligned}&\kappa(x, y)\\ &= e^{i\pi\{(P_1-L_1^*(Q_1+P_2)^{-1}L_1)x^2+(Q_2-L_2(Q_1+P_2)^{-1}L_2^*y^2)\}}e^{-2i\pi\{L_2(Q_1+P_2)^{-1}L_1x\cdot y\}}\\ &\quad\times(\det L_1)^{1/2}(\det L_2)^{1/2}e^{-i\pi n/2}|\det(Q_1 + P_2)|^{-1/2}e^{i\frac{\pi}{4}\,\text{sign}(Q_1+P_2)}.\end{aligned}$$

We note that, with E_{12} standing for the eigenvalues of $Q_1 + P_2$,

$$\nu_+ = \text{Card}(\mathrm{E}_{12} \cap \mathbb{R}_+), \quad \nu_- = \text{Card}(\mathrm{E}_{12} \cap \mathbb{R}_-) = \text{index}\,(Q_1 + P_2),$$

implying that the kernel κ is given by

$$\kappa(x, y) = e^{i\frac{\pi}{2}\left(m(L_1)+m(L_2)-n+\frac{1}{2}(\nu_+-\nu_-)\right)}|\det L|^{1/2}e^{i\pi\{Px^2-2Lx\cdot y+Qy^2\}}, \quad \text{(A.2.5)}$$

with

$$P = P_1 - L_1^*(Q_1 + P_2)^{-1}L_1, \quad Q = Q_2 - L_2(Q_1 + P_2)^{-1}L_2^*, \quad \text{(A.2.6)}$$
$$L = L_2(Q_1 + P_2)^{-1}L_1.$$

Checking the unit factor in front of the right-hand side of (A.2.5), we note that $\nu_+ + \nu_- = n$ since $Q_1 + P_2$ is non-singular and we get

$$e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-n+\frac{1}{2}(\nu_+-\nu_-))} = e^{-\frac{i\pi n}{4}} e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\frac{n}{2}+\frac{1}{2}(\nu_+-\nu_-))}$$
$$= e^{-\frac{i\pi n}{4}} e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\nu_-)}.$$

We have also, since index $(Q_1 + P_2) =$ index $(Q_1 + P_2)^{-1}$,

$$\left(e^{i\frac{\pi}{2}(m(L_1)+m(L_2)-\nu_-)}\right)^2 = \operatorname{sign}(\det L_1)\operatorname{sign}(\det L_2)(-1)^{\nu_-}$$
$$= \operatorname{sign}(\det L_1)\operatorname{sign}(\det L_2)\operatorname{sign}(\det(Q_1 + P_2)^{-1})$$
$$= \operatorname{sign}(\det L),$$

entailing that

$$\kappa(x, y) = e^{-\frac{i\pi n}{4}} (\det L)^{1/2} e^{i\pi\{Px^2 - 2Lx\cdot y + Qy^2\}},$$

concluding the proof of the proposition. ∎

Lemma A.2.8. *Let P_j, L_j, Q_j, $j = 1, 2, 3$ be as in Definition A.2.1. Then, there exist (P', L', Q'), (P'', L'', Q'') as in Definition A.2.1 such that*

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{P',L',Q'}\mathcal{M}_{P'',L'',Q''}. \quad \text{(A.2.7)}$$

Proof. If $\det(Q_1 + P_2) \neq 0$, Lemma A.2.7 implies that $\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2} = \mathcal{M}_{P',L',Q'}$ so that (A.2.7) is satisfied with $(P'', L'', Q'') = (P_3, L_3, Q_3)$. We may thus assume in the sequel that $\det(Q_1 + P_2) = 0$. Then, the kernel of $Q_1 + P_2$ is of dimension $r \in [\![1, n]\!]$; let us define J_r as the orthogonal projection onto $\ker(Q_1 + P_2)$.

Claim A.2.9. The matrix $J_r + (Q_1 + P_2)^2$ is positive definite (thus invertible).

Proof. Indeed, if $J_r x + (Q_1 + P_2)^2 x = 0$, we obtain by taking the dot-product with x that

$$\|J_r x\|^2 + \|(Q_1 + P_2)x\|^2 = 0 \Longrightarrow x \in \ker(Q_1 + P_2), J_r x = 0 \Longrightarrow x = 0.$$

This matrix is also non-negative, proving the claim. ∎

Let us define the real $n \times n$ symmetric matrix

$$P = \mu L_2\big[J_r + (Q_1 + P_2)^2\big]^{-1} L_2^* - Q_2, \quad \text{(A.2.8)}$$

where μ is a positive parameter to be chosen later; we note that $P + Q_2$ is invertible. Also, we have

$$L_2^*\big(Q_2 + P\big)^{-1} L_2 - (Q_1 + P_2) = \mu^{-1}\big[J_r + (Q_1 + P_2)^2\big] - (Q_1 + P_2),$$

which is invertible if μ (is different from 0 and) does not meet the spectrum of $Q_1 + P_2$ (see footnote[2]). We have also

$$\begin{aligned} P - P_3 &= \mu L_2\big[J_r + (Q_1 + P_2)^2\big]^{-1} L_2^* - (Q_2 + P_3) \\ &= L_2\big\{\mu\big[J_r + (Q_1 + P_2)^2\big]^{-1} - L_2^{-1}(Q_2 + P_3)L_2^{*-1}\big\} L_2^*, \end{aligned}$$

which is invertible for μ large enough[3]. Eventually, defining

$$\lambda_0 = \max(\text{Spectrum}\,|Q_2 + P_1|),$$

the condition

$$\mu > \max\big\{\lambda_0, \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|, \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|\lambda_0^2\big\},$$

implies that, with P given by (A.2.8), we obtain that the matrices

$$P + Q_2, \quad Q_1 + P_2 - L_2^*\big(Q_2 + P\big)^{-1} L_2, \quad P - P_3 \text{ are invertible.} \tag{A.2.9}$$

Using now Lemma A.2.7 and the first property in (A.2.9), we get that we can find $\tilde{P}, \tilde{L}, \tilde{Q}$ as in Definition A.2.1 such that

$$\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0} = \mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}},$$

with (thanks to (A.2.6)),

$$\tilde{P} = P_2 - L_2^*(Q_2 + P)^{-1} L_2.$$

We check now

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0} = \mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}},$$

---

[2]The symmetric matrices $Q_1 + P_2$ and J_r can be diagonalised simultaneously so that the invertibility of

$$\mu^{-1}\big[J_r + (Q_1 + P_2)^2\big] - (Q_1 + P_2)$$

is equivalent to $\mu \neq 0$, $\mu^{-1}\lambda_j^2 \neq \lambda_j$, i.e., $\mu \neq \lambda_j$, where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$.

[3]Indeed, the eigenvalues of $[J_r + (Q_1 + P_2)^2]^{-1}$ are 1 and λ_j^{-2} where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$. To secure the invertibility of $P - P_3$, it is thus enough to have

$$\min(\mu, \mu\lambda_j^{-2}) > \|L_2^{-1}(Q_2 + P_3)L_2^{*-1}\|,$$

where the λ_j are the non-zero eigenvalues of $Q_1 + P_2$.

and we note that

$$Q_1 + \tilde{P} = Q_1 + P_2 - L_2^*(Q_2 + P)^{-1}L_2 \quad \text{is invertible,}$$

thanks to the second property in (A.2.9) so that, from Lemma A.2.7, we can find P', L', Q' as in Definition A.2.1 such that

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{\tilde{P},\tilde{L},\tilde{Q}} = \mathcal{M}_{P',L',Q'},$$

and this yields

$$\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0} = \mathcal{M}_{P',L',Q'}. \tag{A.2.10}$$

Finally, we check

$$\underbrace{\mathcal{M}_{P,I_n,0}^{-1}}_{\substack{=\mathcal{M}_{0,-I_n,-P}\\ \text{cf. Claim A.2.6}}}\ \mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{0,-I_n,-P}\mathcal{M}_{P_3,L_3,Q_3},$$

and since $-P + P_3$ is invertible (thanks to the third property in (A.2.9)), we obtain, using once again Lemma A.2.7, that we can find P'', L'', Q'' as in Definition A.2.1 such that

$$\mathcal{M}_{P,I_n,0}^{-1}\mathcal{M}_{P_3,L_3,Q_3} = \mathcal{M}_{P'',L'',Q''}. \tag{A.2.11}$$

Gathering the information above, we find that

$$\begin{aligned}&\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P_3,L_3,Q_3}\\ &\quad = \underbrace{\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\mathcal{M}_{P,I_n,0}}_{\mathcal{M}_{P',L',Q'},\ (\text{A.2.10})}\underbrace{\mathcal{M}_{P,I_n,0}^{-1}\mathcal{M}_{P_3,L_3,Q_3}}_{\mathcal{M}_{P'',L'',Q''},\ (\text{A.2.11})},\end{aligned}$$

which ends the proof of the lemma. ■

Proposition A.2.10. *The metaplectic group* Mp(n) *is equal to the set*

$$\{\mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\}_{\substack{P_j=P_j^*,Q_j=Q_j^*\\ \det L_j\neq 0}}.$$

In other words, every metaplectic operator of Mp(n) *is the product of two operators of type* $\mathcal{M}_{P,L,Q}$ *as given by Definition* A.2.1.

Proof. From Proposition A.2.2, the metaplectic group is generated by the $\mathcal{M}_{P,L,Q}$ and since the inverse of $\mathcal{M}_{P,L,Q}$ is $\mathcal{M}_{-Q,-L^*,-P}$, thanks to Claim A.2.6, it is enough to check the products

$$\mathcal{M}_{P_1,L_1,Q_1}\cdots\mathcal{M}_{P_N,L_N,Q_N}$$

for $N \geq 3$. Lemma A.2.8 is tackling the case $N = 3$ and a trivial recurrence on N provides the result of the proposition. ■

Theorem A.2.11. *Let M be an element of* $\mathrm{Mp}(n)$ *such that* $M = e^{i\phi}\,\mathrm{Id}_{L^2(\mathbb{R}^n)}, \phi \in \mathbb{R}$. *Then, $e^{i\phi}$ belongs to the set $\{-1, 1\}$. In other words, the intersection of the metaplectic group with the unit circle (identified to the unitary operators in $L^2(\mathbb{R}^n)$ defined by the mappings $v \mapsto zv$ where $z \in \mathbb{S}^1 \subset \mathbb{C}$) is reduced to the set $\{-1, 1\}$.*

Proof. Using Proposition A.2.10, the result follows from Lemma A.2.5. ∎

We may go back to the description given by Proposition 1.2.11 and Definition 1.2.13.

Proposition A.2.12. *The metaplectic group* $\mathrm{Mp}(n)$ *is equal to the set*

$$\{M_{A_1,B_1,C_1} M_{A_2,B_2,C_2}\}_{\substack{A_j = A_j^*, C_j = C_j^*,\\ \det B_j \neq 0}},$$

where the operators $M_{A,B,C}$ are defined in Proposition 1.2.11.

Proof. Let M be in $\mathrm{Mp}(n)$. We have

$$\begin{aligned}
M &= (M_{A_1,B_1,C_1})^{\pm 1} \cdots (M_{A_N,B_N,C_N})^{\pm 1} \\
&\underbrace{=}_{(A.2.1)} \left(\mathcal{M}_{A_1,-B_1,C_1} e^{-i\pi n/4} \mathcal{F}\right)^{\pm 1} \cdots \left(\mathcal{M}_{A_N,-B_N,C_N} e^{-i\pi n/4} \mathcal{F}\right)^{\pm 1} \\
&= \left(\mathcal{M}_{A_1,-B_1,C_1} \mathcal{M}_{0,I_n,0}\right)^{\pm 1} \cdots \left(\mathcal{M}_{A_N,-B_N,C_N} \mathcal{M}_{0,I_n,0}\right)^{\pm 1},
\end{aligned}$$

and since from Claim A.2.6, we have

$$\mathcal{M}_{A,B,C}^{-1} = \mathcal{M}_{-C,-B^*,-A},$$

we find that M is in fact a product of $2N$ terms of type $\mathcal{M}_{P,L,Q}$, and thanks to Proposition A.2.10, we get

$$\begin{aligned}
M &= \mathcal{M}_{P_1,L_1,Q_1} \mathcal{M}_{P_2,L_2,Q_2} = \underbrace{\mathcal{M}_{P_1,L_1,Q_1} e^{-i\pi n/4} \mathcal{F}}_{M_{P_1,-L_1,Q_1}} \underbrace{\left(e^{-i\pi n/4} \mathcal{F}\right)^{-1} \mathcal{M}_{P_2,L_2,Q_2}}_{\left(\mathcal{M}_{-Q_2,-L_2^*,-P_2} e^{-i\pi n/4} \mathcal{F}\right)^{-1}} \\
&= M_{P_1,-L_1,Q_1} \left(M_{-Q_2,-L_2^*,-P_2}\right)^{-1} \\
&= M_{P_1,-L_1,0} M_{0,I_n,Q_1} \left(M_{-Q_2,-L_2^*,0} M_{0,I_n,-P_2}\right)^{-1} \\
&= M_{P_1,-L_1,0} M_{0,I_n,Q_1} M_{0,I_n,P_2} \left(M_{-Q_2,-L_2^*,0}\right)^{-1} \\
&= M_{P_1,-L_1,0} M_{0,I_n,Q_1+P_2} \left(M_{-Q_2,-L_2^*,0}\right)^{-1} \qquad \text{(cf. formula (1.2.33))} \\
&= M_{P_1,-L_1,Q_1+P_2} M_{A'',B'',0} \quad \text{(cf. Lemma A.2.14 below in the next subsection)},
\end{aligned}$$

proving the proposition. ∎

A.2.2 On some subgroups of the metaplectic group

We have seen in (1.2.24), (1.2.22) some equivalent conditions for a matrix

$$\Xi = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \quad \text{where } P, Q, R, S \text{ are } n \times n \text{ real matrices,} \tag{A.2.12}$$

to be symplectic. We note here that when $\Xi \in \mathrm{Sp}(n, \mathbb{R})$, we have

$$\Xi^{-1} = \begin{pmatrix} S^* & -Q^* \\ -R^* & P^* \end{pmatrix}, \tag{A.2.13}$$

as it is easily checked from (1.2.24), (1.2.22). When $\det P \neq 0$, we proved that $\Xi = \Xi_{A,B,C}$ as defined in (1.2.19). Also from (A.2.13), we get that if $\det S \neq 0$ we have

$$\Xi^{-1} = \Xi_{A,B,C},$$

so that

$$\Xi = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A & I_n \end{pmatrix}.$$

Some other properties of the same type are available when $\det Q$ or $\det R$ are different from 0. Indeed, we have for $\Xi \in \mathrm{Sp}(n, \mathbb{R})$ and σ given by (1.2.15),

$$\Xi\sigma = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \sigma = \begin{pmatrix} -Q & P \\ -S & R \end{pmatrix} \underbrace{=}_{\text{if } \det Q \neq 0} \Xi_{A,B,C}, \tag{A.2.14}$$

so that

$$\Xi = -\Xi_{A,B,C}\sigma = \begin{pmatrix} I_n & 0 \\ A & I_n \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B^* \end{pmatrix} \begin{pmatrix} I_n & -C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

If we have $\det R \neq 0$, using the two first equalities in (A.2.14), we get that $(\Xi\sigma)^{-1} = \Xi_{A,B,C}$, which gives

$$\Xi = \begin{pmatrix} I_n & C \\ 0 & I_n \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{*-1} \end{pmatrix} \begin{pmatrix} I_n & 0 \\ -A & I_n \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

However, it is indeed possible when $n \geq 2$ to have a symplectic matrix in $\mathrm{Sp}(n, \mathbb{R})$ in the form (A.2.12) such that all blocks are singular, as shown in the following remark.

Remark A.2.13. The 4×4 matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

belongs to $\mathrm{Sp}(2, \mathbb{R})$ although all the block 2×2 matrices P, Q, R, S, are singular (with rank 1).

Lemma A.2.14. *With $M_{A,B,C}$ defined in Proposition 1.2.11, the sets*

$$\mathcal{L} = \{M_{A,B,0}\}_{\substack{A=A^* \\ \det B \neq 0}}, \quad \mathcal{R} = \{M_{0,B,C}\}_{\substack{C=C^* \\ \det B \neq 0}}, \tag{A.2.15}$$

are subgroups of the metaplectic group (cf. Definition 1.2.13).

Proof. Indeed, $\mathcal{L}$ contains the identity of $L^2(\mathbb{R}^n)$ and we have for $v \in L^2(\mathbb{R}^n)$,

$$\begin{aligned} M_{A_1,B_1,0} M_{A_2,B_2,0}^{-1} v &= M_{A_1,B_1,0}\big\{M_{0,B_2^{-1},0}\{e^{-i\pi A_2 x^2} v(x)\}\big\} \\ &= M_{A_1,B_1,0}\big\{e^{-i\pi B_2^{*-1} A_2 B_2^{-1} x^2} v(B_2^{-1}x)\big\}(\det B_2)^{-1/2} \\ &= e^{i\pi A_1 x^2} e^{-i\pi B_1^* B_2^{*-1} A_2 B_2^{-1} B_1 x^2} v(B_2^{-1} B_1 x)(\det B_1)^{1/2}(\det B_2)^{-1/2} \\ &= e^{i\pi (A_1 - B_1^* B_2^{*-1} A_2 B_2^{-1} B_1) x^2} v(B_2^{-1} B_1 x)(\det B_1)^{1/2}(\det B_2)^{-1/2} \\ &= (M_{A_1 - B_1^* B_2^{*-1} A_2 B_2^{-1} B_1, B_2^{-1} B_1, 0} v)(x), \end{aligned}$$

so that $M_{A_1,B_1,0} M_{A_2,B_2,0}^{-1}$ belongs to the set $\mathcal{L}$ in (A.2.15), proving that $\mathcal{L}$ is indeed a subgroup of the metaplectic group. We note also that the bijective mapping

$$\mathcal{L} \ni M \mapsto F^* M F \in \mathcal{R}, \tag{A.2.16}$$

(F stands for the Fourier transformation) sends $\mathcal{L}$ onto $\mathcal{R}$ since we have

$$\begin{aligned} F^* M_{A,B,0} F &= F^* M_{A,I_n,0} F F^* M_{0,B,0} F = M_{0,I_n,A} M_{0,B^{*-1},0} \\ &= M_{0,B^{*-1},B^{*-1}AB^{-1}}. \end{aligned} \tag{A.2.17}$$

Moreover, the mapping (A.2.16) is obviously one-to-one and is also onto since, given $B_1 \in \mathrm{Gl}(n, \mathbb{R})$ and C_1 a symmetric $n \times n$ matrix, we see from (A.2.17) that

$$F^* M_{B_1^{-1} C_1 B_1^{*-1}, B_1^{*-1}, 0} F = M_{0,B_1,C_1}.$$

The mapping (A.2.16) also extends to a group isomorphism of $\mathrm{Mp}(n)$, proving the lemma. ∎

Remark A.2.15. We may note that

$$\begin{aligned} (M_{A_1,B_1,0} M_{A_2,B_2,0} v)(x) &= e^{i\pi A_1 x^2}(M_{A_2,B_2,0} v)(B_1 x)(\det B_1)^{1/2} \\ &= e^{i\pi (A_1 + B_1^* A_2 B_1) x^2} v(B_2 B_1 x)(\det B_1)^{1/2}(\det B_2)^{1/2} \\ &= (M_{A_1 + B_1^* A_2 B_1, B_2 B_1, 0} v)(x), \end{aligned}$$

so that the internal binary operation $\star$ can be defined on the set $\{(A, B)\}_{\substack{A=A^* \\ \det B \neq 0}}$ as

$$(A_1, B_1) \star (A_2, B_2) = (A_1 + B_1^* A_2 B_1, B_2 B_1),$$

for which the identity is $(0, I_n)$ and the inverse

$$(A, B)^{-1} = (-B^{*-1} A B^{-1}, B^{-1}).$$

Remark A.2.16. A consequence of Lemma A.2.14 is, with Ψ defined in (1.2.46), that

$$\{\Psi(M_{A,B,0})\}_{\substack{A=A^*\\ \det B\neq 0}} = \{\Xi_{A,B,0}\}_{\substack{A=A^*\\ \det B\neq 0}}, \quad \{\Psi(M_{0,B,C})\}_{\substack{C=C^*\\ \det B\neq 0}} = \{\Xi_{0,B,C}\}_{\substack{C=C^*\\ \det B\neq 0}},$$

are subgroups of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$.

Proposition A.2.17. *The metaplectic group* $\mathrm{Mp}(n)$ *is equal to the set*

$$\left\{M_{A_1,B_1,C_1} M_{A_2,B_2,C_2}\right\}_{\substack{A_j=A_j^*,C_j=C_j^*\\ \det B_j\neq 0}}.$$

In other words, every metaplectic operator of $\mathrm{Mp}(n)$ *is the product of two operators of type* $M_{A,B,C}$ *as given by Proposition* 1.2.11.

Proof. Let $M \in \mathrm{Mp}(n)$; using Proposition A.2.10, we may assume that

$$\begin{aligned}
M &= \mathcal{M}_{P_1,L_1,Q_1}\mathcal{M}_{P_2,L_2,Q_2}\\
&= \mathcal{M}_{P_1,L_1,Q_1}\mathcal{F}e^{-i\pi n/4}\left(\mathcal{F}e^{-i\pi n/4}\right)^{-1}\mathcal{M}_{P_2,L_2,Q_2}\\
\text{(A.2.1)} &= M_{P_1,-L_1,Q_1}\left(\mathcal{M}^{-1}_{P_2,L_2,Q_2}\mathcal{F}e^{-i\pi n/4}\right)^{-1}\\
\text{(Claim A.2.6)} &= M_{P_1,-L_1,Q_1}\left(\mathcal{M}_{-Q_2,-L_2^*,-P_2}\mathcal{F}e^{-i\pi n/4}\right)^{-1}\\
\text{(A.2.1), (1.2.33)} &= M_{P_1,-L_1,Q_1}M^{-1}_{-Q_2,L_2^*,-P_2}\\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1}\left(M_{-Q_2,L_2^*,0}M_{0,I_n,-P_2}\right)^{-1}\\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1}M_{0,I_n,P_2}M^{-1}_{-Q_2,L_2^*,0}\\
&= M_{P_1,-L_1,0}M_{0,I_n,Q_1+P_2}M^{-1}_{-Q_2,L_2^*,0}\\
&= M_{P_1,-L_1,Q_1+P_2}M^{-1}_{-Q_2,L_2^*,0}\\
\text{(using Lemma A.2.14)} &= M_{P_1,-L_1,Q_1+P_2}M_{A',B',0},
\end{aligned}$$

proving the sought result. ∎

Remark A.2.18. We have used two different sets of generators of the metaplectic group. First the set $\mathcal{G}_1 = \{M^{\{m(B)\}}_{A,B,C}\}$ given by (1.2.35) which is somewhat natural, also allowing us to recover the operator $e^{-i\pi n/4}\mathcal{F}$ where the phase factor appears via formula (1.2.38). The Identity appears clearly as $M^{\{0\}}_{0,I_n,0}$, but the inverse of $M^{\{m(B)\}}_{A,B,C}$ cannot always be expressed within $\mathcal{G}_1$.

Also, we have the set $\mathcal{G}_2 = \left\{\mathcal{M}^{\{m(B)\}}_{A,B,C}\right\}$ given in Definition A.2.1, which incorporates a phase prefactor $e^{-i\pi n/4}$, looking a priori rather arbitrary but of course necessary for the sequel (this prefactor is also suggested by (1.2.38)); here to express the identity, we need to write it as $\mathcal{M}^{\{0\}}_{0,I_n,0}\mathcal{M}^{\{n\}}_{0,-I_n,0}$, but the inverse of $\mathcal{M}^{\{m(B)\}}_{A,B,C}$ is easily obtained by Claim A.2.6 within $\mathcal{G}_2$. Certainly the description given by $\mathcal{G}_2$ is much better, in particular because the calculations leading to Lemma A.2.5 and Proposition A.2.7 are rather easy as well as the proof of Lemma A.2.8; a statement analogous to Proposition A.2.10 for $\mathscr{G}_1$ is true (cf. Proposition A.2.12), but its proof is quite indirect and relies heavily on the results for $\mathscr{G}_2$.

A.3 Mehler's formula

We provide here a couple of statements related to the so-called Mehler's formula, appearing as particular cases of L. Hörmander's study in [22] (see also the more recent K. Pravda-Starov' article [42]). In the general framework, we consider a complex-valued quadratic form Q on the phase space $\mathbb{R}^{2n}$ such that $\operatorname{Re} Q \le 0$: we want to quantize the Gaussian function (here X stands for (x, ξ)) $\mathbf{a}(X) = e^{\langle QX, X\rangle}$, and to relate the operator with Weyl symbol $\mathbf{a}$ to the operator

$$\exp\left\{\operatorname{Op}_{\mathrm{w}}(\langle QX, X\rangle)\right\}.$$

Lemma A.3.1. *For* $\operatorname{Re} t \ge 0$, $t \notin i\pi(2\mathbb{Z}+1)$, *we have in* n *dimensions,*

$$\left(\cosh(t/2)\right)^n \exp -t\pi \operatorname{Op}_{\mathrm{w}}(|x|^2 + |\xi|^2) = \operatorname{Op}_{\mathrm{w}}\left(e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)}\right).$$

In particular, for $t = -2is$, $s \in \mathbb{R}$, $s \notin \frac{\pi}{2}(1+2\mathbb{Z})$, we have in n dimensions

$$(\cos s)^n \exp\left(2i\pi s \operatorname{Op}_{\mathrm{w}}(|x|^2 + |\xi|^2)\right) = \operatorname{Op}_{\mathrm{w}}\left(e^{2i\pi \tan s(|x|^2+|\xi|^2)}\right). \tag{A.3.1}$$

Lemma A.3.2. *For any* $z \in \mathbb{C}, \operatorname{Re} z \ge 0$, *we have in* n *dimensions*

$$\operatorname{Op}_{\mathrm{w}}\left(\exp -\left(2z\pi\left(|\xi|^2 + |x|^2\right)\right)\right) = \frac{1}{(1+z)^n} \sum_{k\ge 0} \left(\frac{1-z}{1+z}\right)^k \mathbb{P}_{k;n}, \tag{A.3.2}$$

where $\mathbb{P}_{k;n}$ *is defined in Section A.1.4 and the equality holds between* $L^2(\mathbb{R}^n)$*-bounded operators.*

We provide first a proof of a particular case of the results of [22].

Lemma A.3.3. *For* $\operatorname{Re} t \ge 0$, $t \notin i\pi(2\mathbb{Z}+1)$, *we have in* n *dimensions,*

$$(\cosh(t/2))^n \exp -t\pi \operatorname{Op}_{\mathrm{w}}(|x|^2 + |\xi|^2) = \operatorname{Op}_{\mathrm{w}}\left(e^{-2\tanh(\frac{t}{2})\pi(x^2+\xi^2)}\right). \tag{A.3.3}$$

Proof. By tensorisation, it is enough to prove that formula for $n = 1$, which we assume from now on. We define

$$L = \xi + ix, \quad \bar{L} = \xi - ix, \quad M(t) = \beta(t)\operatorname{Op}_{\mathrm{w}}(e^{-\alpha(t)\pi L\bar{L}}),$$

where α, β are smooth functions of t to be chosen below. Assuming $\beta(0) = 1, \alpha(0) = 0$, we find that $M(0) = \mathrm{Id}$ and

$$\dot{M} + \pi \mathrm{Op_w}(|L|^2)M = \mathrm{Op_w}\big(\dot{\beta} e^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2 e^{-\alpha\pi|L|^2} + \pi(|L|^2)\sharp\beta e^{-\alpha\pi|L|^2}\big).$$

We have from (1.2.3), since $\partial_x\partial_\xi|L|^2 = 0$,

$$\begin{aligned}
|L|^2\sharp e^{-\alpha\pi|L|^2} &= |L|^2 e^{-\alpha\pi|L|^2} + \frac{1}{4i\pi}\overbrace{\{|L|^2, e^{-\alpha\pi|L|^2}\}}^{=0} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2}\big(\partial_\xi^2(|L|^2)\partial_x^2 e^{-\alpha\pi|L|^2} + \partial_x^2(|L|^2)\partial_\xi^2 e^{-\alpha\pi|L|^2}\big) \\
&= |L|^2 e^{-\alpha\pi|L|^2} \\
&\quad + \frac{1}{(4i\pi)^2}\frac{1}{2} e^{-\alpha\pi|L|^2}\big(2((-2\alpha\pi x)^2 - 2\alpha\pi) + 2((-2\alpha\pi\xi)^2 - 2\alpha\pi)\big) \\
&= |L|^2 e^{-\alpha\pi|L|^2}\left(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi}{4\pi^2} e^{-\alpha\pi|L|^2},
\end{aligned}$$

so that

$$\begin{aligned}
&\dot{M} + \pi\mathrm{Op_w}(|L|^2)M \\
&\quad = \mathrm{Op_w}\Bigg(\dot{\beta} e^{-\alpha\pi|L|^2} - \beta\dot{\alpha}\pi|L|^2 e^{-\alpha\pi|L|^2} \\
&\qquad\qquad + \pi\beta|L|^2 e^{-\alpha\pi|L|^2}\left(1 - \frac{4\alpha^2\pi^2}{16\pi^2}\right) + \frac{\alpha\pi\beta}{4\pi} e^{-\alpha\pi|L|^2}\Bigg) \\
&\quad = \mathrm{Op_w}\left(e^{-\alpha\pi|L|^2}\left\{|L|^2\left(-\pi\dot{\alpha}\beta + \pi\beta\left(1 - \frac{\alpha^2}{4}\right)\right) + \dot{\beta} + \frac{\alpha\beta}{4}\right\}\right).
\end{aligned}$$

We solve now

$$\dot{\alpha} = 1 - \frac{\alpha^2}{4}, \quad \alpha(0) = 0 \Longleftrightarrow \alpha(t) = 2\tanh(t/2),$$

and

$$4\dot{\beta} + \alpha\beta = 0, \quad \beta(0) = 1 \Longleftrightarrow \beta(t) = \frac{1}{\cosh(t/2)}.$$

We obtain that

$$\dot{M} + \pi\mathrm{Op_w}(|L|^2)M = 0, \ M(0) = \mathrm{Id},$$

and this implies

$$\beta(t)\mathrm{Op_w}(e^{-\alpha(t)\pi L\bar{L}}) = M(t) = \exp -t\pi(|L|^2)^w,$$

which proves (A.3.3). ∎

In particular, for $t = -2is, s \in \mathbb{R}, s \notin \frac{\pi}{2}(1+2\mathbb{Z})$, we have in n dimensions

$$(\cos s)^n \exp\big(2i\pi s \mathrm{Op_w}(|x|^2 + |\xi|^2)\big) = \mathrm{Op_w}\left(e^{2i\pi \tan s(|x|^2+|\xi|^2)}\right). \tag{A.3.4}$$

Lemma A.3.4. *For any $z \in \mathbb{C}$, $\operatorname{Re} z \geq 0$, we have in n dimensions*

$$\mathrm{Op_w}\left(\exp -\big(2z\pi(|\xi|^2 + |x|^2)\big)\right) = \frac{1}{(1+z)^n}\sum_{k\geq 0}\left(\frac{1-z}{1+z}\right)^k \mathbb{P}_{k;n},$$

where $\mathbb{P}_{k;n}$ is defined in Section A.1.4 and the equality holds between $L^2(\mathbb{R}^n)$-bounded operators.

Proof. Starting from (A.3.4), we get for $\tau \in \mathbb{R}$, in n dimensions,

$$(\cos(\arctan \tau))^n \exp\big(2i\pi \arctan \tau \mathrm{Op_w}(|x|^2 + |\xi|^2)\big) = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

so that using the spectral decomposition of the (n-dimensional) harmonic oscillator and (A.8.1), we get

$$(1+\tau^2)^{-n/2}\sum_{k\geq 0} e^{2i(\arctan\tau)(k+\frac{n}{2})}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

which implies

$$(1+\tau^2)^{-n/2}\sum_{k\geq 0}\frac{(1+i\tau)^{2k+n}}{(1+\tau^2)^{k+\frac{n}{2}}}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

entailing

$$\sum_{k\geq 0}\frac{(1+i\tau)^k}{(1-i\tau)^{k+n}}\mathbb{P}_{k;n} = \mathrm{Op_w}\left(e^{2i\pi\tau(|x|^2+|\xi|^2)}\right),$$

proving the lemma by analytic continuation (we may refer the reader as well to [50, pages 204–205] and note that for any $z \in \mathbb{C}, \operatorname{Re} z \geq 0$, we have $|\frac{1-z}{1+z}| \leq 1$). ∎

A.4 Laguerre polynomials

A.4.1 Classical Laguerre polynomials

The Laguerre polynomials $\{L_k\}_{k\in\mathbb{N}}$ are defined by

$$L_k(x) = \sum_{0\leq l\leq k}\frac{(-1)^l}{l!}\binom{k}{l}x^l = e^x\frac{1}{k!}\left(\frac{d}{dx}\right)^k\{x^k e^{-x}\} = \left(\frac{d}{dx}-1\right)^k\left\{\frac{x^k}{k!}\right\}, \tag{A.4.1}$$

and we have

$$
\begin{aligned}
L_0 &= 1,\\
L_1 &= -X + 1,\\
L_2 &= \frac{1}{2}(X^2 - 4X + 2),\\
L_3 &= \frac{1}{6}(-X^3 + 9X^2 - 18X + 6),\\
L_4 &= \frac{1}{24}(X^4 - 16X^3 + 72X^2 - 96X + 24),\\
L_5 &= \frac{1}{120}(-X^5 + 25X^4 - 200X^3 + 600X^2 - 600X + 120),\\
L_6 &= \frac{1}{720}(X^6 - 36X^5 + 450X^4 - 2400X^3 + 5400X^2 - 4320X + 720),\\
L_7 &= \frac{-X^7 + 49X^6 - 882X^5 + 7350X^4 - 29400X^3 + 52920X^2 - 35280X + 5040}{5040}.
\end{aligned}
$$

We get also easily from the above definition that

$$L'_{k+1} = L'_k - L_k,$$

since with $T = d/dX - 1$

$$L'_k - L_k = TL_k = T^{k+1}\Big(\frac{X^k}{k!}\Big) = T^{k+1}\Big(\frac{d}{dX}\frac{X^{k+1}}{(k+1)!}\Big) = \frac{d}{dX}L_{k+1}.$$

Formula (6.8) and Theorem 12 in the R. Askey and G. Gasper's article [2] provide the inequalities

$$\forall k \in \mathbb{N}, \forall x \geq 0, \quad \sum_{0 \leq l \leq k} (-1)^l L_l(x) \geq 0. \tag{A.4.2}$$

This result follows as well from formula (73) in the 1940 paper [12] by E. Feldheim. Let us calculate the Fourier transform of the Laguerre polynomials, we have

$$L_k(x) = \Big(\frac{d}{dx} - 1\Big)^k \Big\{\frac{x^k}{k!}\Big\},$$

so that

$$\widehat{L_k}(\xi) = (2i\pi\xi - 1)^k \Big(\frac{-1}{2i\pi}\Big)^k \frac{\delta_0^{(k)}}{k!} = \frac{(-1)^k}{k!}\Big(\xi - \frac{1}{2i\pi}\Big)^k \delta_0^{(k)}(\xi).$$

As a result, defining for $k \in \mathbb{N}, t \in \mathbb{R}$,

$$M_k(t) = (-1)^k H(t) e^{-t} L_k(2t), \quad H = \mathbf{1}_{\mathbb{R}_+}, \tag{A.4.3}$$

we find, using the homogeneity of degree $-k-1$ of $\delta_0^{(k)}$,

$$\begin{aligned}
\widehat{M_k}(\tau) &= \frac{1}{2}\frac{(-1)^k}{k!}\left(\frac{\tau}{2}-\frac{1}{2i\pi}\right)^k \delta_0^{(k)}\left(\frac{\tau}{2}\right) * \frac{(-1)^k}{1+2i\pi\tau} \\
&= (-1)^k (\frac{d}{d\sigma})^k \left\{\frac{(\sigma-\frac{1}{i\pi})^k/k!}{1+2i\pi(\tau-\sigma)}\right\}_{|\sigma=0} \\
\widehat{M_k}(\tau) &= \sum_l (-1)^k \binom{k}{l} \frac{(\sigma-\frac{1}{i\pi})^{k-l}}{(k-l)!} \frac{(k-l)!(2i\pi)^{k-l}}{\big(1+2i\pi(\tau-\sigma)\big)^{1+k-l}}_{|\sigma=0} \\
&= \sum_l (-1)^k \binom{k}{l} \frac{(-2)^{k-l}}{\big(1+2i\pi\tau\big)^{1+k-l}} \\
&= \frac{(-1)^k}{(1+2i\pi\tau)} \sum_l \binom{k}{l} \frac{(-2)^{k-l}}{\big(1+2i\pi\tau\big)^{k-l}} \\
&= \frac{(-1)^k}{(1+2i\pi\tau)} \left(1-\frac{2}{\big(1+2i\pi\tau\big)}\right)^k \\
&= \frac{(-1)^k}{(1+2i\pi\tau)} \left(\frac{-1+2i\pi\tau}{1+2i\pi\tau}\right)^k = \frac{1}{(1+2i\pi\tau)}\left(\frac{1-2i\pi\tau}{1+2i\pi\tau}\right)^k
\end{aligned}$$

so that

$$\widehat{M_k}(\tau) = \frac{(1-2i\pi\tau)^k}{(1+2i\pi\tau)^{k+1}} = \frac{(1-2i\pi\tau)^{2k+1}}{(1+4\pi^2\tau^2)^{k+1}}. \tag{A.4.4}$$

A.4.2 Generalized Laguerre polynomials

Let α be a complex number and let k be a non-negative integer such that $\alpha + k \notin (-\mathbb{N}^*)$. We define the generalized Laguerre polynomial L_k^α by

$$L_k^\alpha(x) = x^{-\alpha} e^x \left(\frac{d}{dx}\right)^k \left\{e^{-x}\frac{x^{k+\alpha}}{k!}\right\} = x^{-\alpha}\left(\frac{d}{dx}-1\right)^k \left\{\frac{x^{k+\alpha}}{k!}\right\}. \tag{A.4.5}$$

We note that L_k^α is indeed a polynomial with degree k with the formula

$$\begin{aligned}
L_k^\alpha(x) &= \sum_{k_1+k_2=k} \frac{1}{k!}\binom{k}{k_1}(-1)^{k_2}\Gamma(k+\alpha+1)\frac{x^{k-k_1}}{\Gamma(k+\alpha+1-k_1)} \\
&= \sum_{0\le k_1\le k} \frac{(-1)^{k_2}}{k_1!(k-k_1)!}\Gamma(k+\alpha+1)\frac{x^{k-k_1}}{\Gamma(k+\alpha+1-k_1)} \\
&= \sum_{0\le l\le k} \binom{k+\alpha}{k-l}\frac{(-1)^l x^l}{l!}.
\end{aligned} \tag{A.4.6}$$

N.B. We recall that the function $1/\Gamma$ is an entire function with simple zeroes at $-\mathbb{N}$. As a result to make sense for the binomial coefficient

$$\binom{k+\alpha}{k-l} = \frac{\Gamma(k+\alpha+1)}{(k-l)!\Gamma(l+\alpha+1)},$$

we need to make sure that $k+\alpha+1 \notin -\mathbb{N}$, i.e., $\alpha \notin -\mathbb{N}^* - k$.

Lemma A.4.1. *Let $\alpha \in \mathbb{C}\backslash(-\mathbb{N}^*)$ and let k be a non-negative integer. For $\alpha = 0$, we have $L_k^\alpha = L_k$, where L_k is the classical Laguerre polynomial defined in* (A.4.1). *Moreover, we have for $l \le k$,*

$$\left(\frac{d}{dX}\right)^l L_k^\alpha = (-1)^l L_{k-l}^{\alpha+l}. \tag{A.4.7}$$

Proof. Indeed, we have from (A.4.6)

$$\begin{aligned}\left(\frac{d}{dX}\right)^l L_k^\alpha &= (-1)^l \sum_{l\le m\le k} \binom{k+\alpha}{k-m} \frac{(-1)^{m-l} X^{m-l}}{(m-l)!} \\ &= (-1)^l \sum_{0\le r\le k-l} \binom{k-l+\alpha+l}{k-r-l} \frac{(-1)^r X^r}{r!} = (-1)^l L_{k-l}^{\alpha+l},\end{aligned}$$

proving the sought formula. ∎

A.5 Singular integrals

Proposition A.5.1. (1) *The (Hardy) operator with distribution kernel*

$$\frac{H(x)H(y)}{\pi(x+y)}$$

is self-adjoint bounded on $L^2(\mathbb{R})$ with spectrum $[0, 1]$ *and thus norm 1.*

(2) *The (modified Hardy) operators with respective distribution kernels*

$$H(x-y)\frac{H(x)H(y)}{\pi(x+y)}, \quad H(y-x)\frac{H(x)H(y)}{\pi(x+y)},$$

are bounded on $L^2(\mathbb{R})$ with norm $1/2$.

Proof. Let us prove (1): for $\phi \in L^2(\mathbb{R}_+)$, we define for $t \in \mathbb{R}$, $\tilde{\phi}(t) = \phi(e^t)e^{t/2}$, and we have to check the kernel

$$\frac{e^{t/2}e^{s/2}}{\pi(e^t+e^s)} = \frac{1}{\pi(e^{(t-s)/2}+e^{-(t-s)/2})} = \frac{1}{2\pi}\operatorname{sech}\left(\frac{t-s}{2}\right),$$

which is a convolution kernel. Using now the classical formula

$$\int e^{-2i\pi x\xi} \operatorname{sech} x dx = \pi \operatorname{sech}(\pi^2 \xi),$$

we get that

$$\frac{1}{2\pi} \int \operatorname{sech}\left(\frac{t}{2}\right) e^{-2i\pi t\tau} dt = \operatorname{sech}(\pi^2 2\tau),$$

a smooth function whose range is $(0, 1]$, proving the first part of the proposition. To obtain (2), we observe with the notations $\phi(t) = u(e^t)e^{t/2}$, $\psi(s) = v(e^s)e^{s/2}$ that we have to check

$$\begin{aligned} &\iint H(s-t) \frac{e^{t/2}e^{s/2}}{\pi(e^t + e^s)} \phi(t)\bar{\psi}(s) dt ds \\ &= \iint \frac{H(s-t)}{\pi(e^{(t-s)/2} + e^{-(t-s)/2})} \phi(t)\bar{\psi}(s) dt ds = \langle R * \phi, \psi \rangle_{L^2(\mathbb{R})}, \end{aligned}$$

with

$$R(t) = \frac{H(t)}{2\pi \cosh(t/2)}, \quad \widehat{R}(\tau) = \frac{1}{2\pi} \int_0^{+\infty} \operatorname{sech}(t/2) e^{-2i\pi t\tau} dt,$$

so that[4]

$$|\widehat{R}(\tau)| \le \widehat{R}(0) = \frac{1}{2\pi} \int_0^{+\infty} \operatorname{sech}(t/2) dt = \frac{1}{2},$$

yielding the sought result. ∎

A.6 On some auxiliary functions

A.6.1 A preliminary quadrature

Lemma A.6.1. *We have*

$$\int_0^{\pi/2} (\csc s - \operatorname{csch} s) ds = \int_{\pi/2}^{+\infty} \operatorname{csch} s ds = \operatorname{Log}\left(\coth \frac{\pi}{4}\right),$$

with $\csc s = 1/\sin s, \operatorname{csch} s = 1/\sinh s$.

Proof. Note that the function $[0, \pi/2] \ni s \mapsto \frac{\sinh s - \sin s}{\sinh s \sin s}$, is continuous. Moreover, we have

$$\int \frac{ds}{\sin s} = \frac{1}{2} \operatorname{Log}\left(\frac{1 - \cos s}{1 + \cos s}\right) \quad \text{and} \quad \int \frac{ds}{\sinh s} = \frac{1}{2} \operatorname{Log}\left(\frac{\cosh s - 1}{\cosh s + 1}\right),$$

---

[4]We recall that $\frac{d}{ds} \arctan(\sinh s) = \operatorname{sech} s$.

so that

$$\int_{\varepsilon}^{\pi/2} (\csc s - \operatorname{csch} s) ds$$

$$= \frac{1}{2}\Big[\operatorname{Log}\Big(\frac{1-\cos s}{1+\cos s}\Big)\Big]_{\varepsilon}^{\pi/2} - \Big[\frac{1}{2}\operatorname{Log}\Big(\frac{\cosh s - 1}{\cosh s + 1}\Big)\Big]_{\varepsilon}^{\pi/2}$$

$$= \frac{1}{2}\operatorname{Log}\underbrace{\Big(\Big(\frac{1+\cos\varepsilon}{1-\cos\varepsilon}\Big)\Big(\frac{\cosh\varepsilon - 1}{\cosh\varepsilon + 1}\Big)\Big)}_{=\frac{(2+O(\varepsilon^2))(\frac{\varepsilon^2}{2}+O(\varepsilon^4))}{(\frac{\varepsilon^2}{2}+O(\varepsilon^4))(2+O(\varepsilon^2))}\to 1 \text{ for } \varepsilon\to 0} + \frac{1}{2}\operatorname{Log}\Big(\frac{\cosh\frac{\pi}{2}+1}{\cosh\frac{\pi}{2}-1}\Big),$$

so that we obtain

$$\int_0^{\pi/2} (\csc s - \operatorname{csch} s) ds = \frac{1}{2}\operatorname{Log}\Big(\frac{e^{\pi/2}+e^{-\pi/2}+2}{e^{\pi/2}+e^{-\pi/2}-2}\Big) = \operatorname{Log}\frac{\cosh(\pi/4)}{\sinh(\pi/4)},$$

which is the first result. Also, we have $\int_{\pi/2}^{+\infty} \operatorname{csch} s ds = \frac{1}{2}\operatorname{Log}(\frac{\cosh(\pi/2)+1}{\cosh(\pi/2)-1})$, yielding the second result. ■

A.6.2 Study of the function ρ_σ

We study in this section the real-valued Schwartz function ρ_σ given in (5.2.10). Using the notations

$$\omega = 2\pi\tau, \quad \kappa = 2\pi\sigma, \quad \nu = \sqrt{\kappa/\omega}, \tag{A.6.1}$$

we have

$$\rho_\sigma(\tau) = \int_{\mathbb{R}} \frac{s}{\sinh s} e^{2i\omega(s-\nu^2\tanh s)} ds = \int_{\mathbb{R}} \frac{s}{\sinh s}\cos\big(2\omega(s-\nu^2\tanh s)\big) ds.$$

Defining the holomorphic function F by

$$F(z) = \frac{z}{\sinh z} e^{2i\omega(z-\nu^2\tanh z)}, \tag{A.6.2}$$

we see that F has simple poles at $i\pi\mathbb{Z}^*$ and essential singularities at $i\pi(\frac{1}{2}+\mathbb{Z})$. We already know that the function ρ_σ belongs to the Schwartz space, but we want to prove a more precise exponential decay. We start with the calculation of

$$\int_{\mathbb{R}+i\frac{\pi}{4}} F(z)dz = \int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{\sinh(t+i\frac{\pi}{4})} e^{2i\omega(t+i\frac{\pi}{4}-\nu^2\tanh(t+i\frac{\pi}{4}))} dt$$

$$= e^{-\pi\omega/2} 2\sqrt{2}\int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{(1+i)e^t-(1-i)e^{-t}} e^{2i\omega t} e^{-2i\omega\nu^2\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}} dt$$

$$= e^{-\pi\omega/2}\sqrt{2}\int_{\mathbb{R}} \frac{t+i\frac{\pi}{4}}{\sinh t + i\cosh t} e^{2i\omega t} e^{-2i\omega\nu^2\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}} dt.$$

We have

$$\operatorname{Im}\left(\frac{e^t(1+i)-e^{-t}(1-i)}{e^t(1+i)+e^{-t}(1-i)}\right)=\operatorname{Im}\left(\frac{\sinh t+i\cosh t}{\cosh t+i\sinh t}\right)=\frac{1}{\cosh^2 t+\sinh^2 t},$$

so that

$$\begin{aligned}\left|\int_{\mathbb{R}+i\frac{\pi}{4}}F(z)dz\right| &\le e^{-\frac{\pi\omega}{2}}\sqrt{2}\int_{\mathbb{R}}\frac{\sqrt{t^2+(\frac{\pi}{4})^2}}{\sqrt{\sinh^2 t+\cosh^2 t}}e^{\frac{2\omega\nu^2}{\sinh^2 t+\cosh^2 t}}dt\\ &= e^{-\frac{\pi\omega}{2}}\sqrt{2}e^{2\kappa}\int_{\mathbb{R}}\frac{\sqrt{t^2+(\frac{\pi}{4})^2}}{\sqrt{\sinh^2 t+\cosh^2 t}}dt \le 6e^{-\frac{\pi\omega}{2}}e^{2\kappa}.\end{aligned} \tag{A.6.3}$$

Claim A.6.2. We have

$$\lim_{R\to+\infty}\oint_{[R,R+i\pi/4]}F(z)dz=\lim_{R\to+\infty}\oint_{[-R,-R+i\pi/4]}F(z)dz=0.$$

Proof of Claim A.6.2. We note first that

$$\oint_{[-R,-R+i\pi/4]}F(z)dz=-\overline{\oint_{[R,R+i\pi/4]}F(z)dz},$$

so that it is enough to prove one equality. Indeed, for $R>0$, we have

$$\oint_{[R,R+i\pi/4]}F(z)dz=\int_0^{\pi/4}\frac{R+it}{\sinh(R+it)}e^{2i\omega(R+it-\nu^2\tanh(R+it))}i\,dt,$$

so that

$$\begin{aligned}&\left|\oint_{[R,R+i\pi/4]}F(z)dz\right|\\ &\quad\le\int_0^{\pi/4}\frac{2\sqrt{R^2+t^2}}{|e^{R+it}||1-e^{-2R-2it}|}e^{-2\omega t}e^{2\kappa\operatorname{Im}(\tanh(R+it))}dt\\ &\quad\le e^{-R}\frac{\sqrt{4R^2+\pi^2/4}}{1-e^{-2R}}\int_0^{\pi/4}e^{2\kappa\left|\frac{1-e^{-2R-2it}}{1+e^{-2R-2it}}\right|}dt\\ &\quad\le e^{-R}\frac{\sqrt{4R^2+\pi^2/4}}{1-e^{-2R}}\frac{\pi}{4}e^{\frac{4\kappa}{(1-e^{-2R})}},\end{aligned}$$

proving the claim. ■

Lemma A.6.3. *We have for $\tau>0,\sigma\ge 0$, ρ_σ given in (5.2.10),*

$$|\rho_\sigma(\tau)|\le 6e^{-\pi^2\tau}e^{4\pi\sigma}. \tag{A.6.4}$$

Proof. We have, with the notations (A.6.1), F given in (A.6.2) and $\gamma_R = [-R, -R + i\frac{\pi}{4}] \cup [-R + i\frac{\pi}{4}, R + i\frac{\pi}{4}] \cup [R + i\frac{\pi}{4}, R]$,

$$\rho_\sigma(\tau) = \lim_{R\to+\infty} \int_{[-R,R]} F(s)ds = \lim_{R\to+\infty} \left(\oint_{\gamma_R} F(z)dz \right) \underbrace{=}_{\text{Claim (A.6.2)}} \oint_{\mathbb{R}+\frac{i\pi}{4}} F(z)dz,$$

so that (A.6.3) implies the lemma. ∎

A.6.3 On the function ψ_ν

Let $\nu \in (0, 1)$ be given. We study first the function ϕ_ν defined on $[0, \pi/2)$ by

$$\phi_\nu(s) = s - \nu^2 \tan s, \quad \text{so that} \quad \phi_\nu'(s) = 1 - \nu^2(1 + \tan^2 s) = \frac{\cos^2 s - \nu^2}{\cos^2 s},$$

so that

s	0		s_ν		t_ν		$\frac{\pi}{2}$
$\phi_\nu'(s)$	$1-\nu^2$	$+$	0	$-$		$-$	
$\phi_\nu(s)$	0	$\nearrow$	$\phi_\nu(s_\nu)$	$\searrow$	0	$\searrow$	$-\infty$

(A.6.5)

We have

$$\begin{cases} s_\nu = \arccos \nu = \frac{\pi}{2} - \nu + O(\nu^3), \\ \phi_\nu(s_\nu) = \arccos \nu - \nu\sqrt{1-\nu^2} = \frac{\pi}{2} - 2\nu + O(\nu^3), \end{cases} \quad \text{for } \nu \to 0. \tag{A.6.6}$$

The function ϕ_ν is concave on $(0, \pi/2)$ since we have there

$$\phi_\nu''(s) = -\nu^2(-2)(\cos s)^{-3}(-\sin s) = -\nu^2 2(\cos s)^{-3} \sin s \le 0.$$

We have defined in (5.2.45)

$$\psi_\nu(\omega) = \frac{e^{-\pi\omega}}{2\pi} \int_0^{\pi/2} \frac{e^{2\omega\phi_\nu(s)} - 1}{\sin s} ds. \tag{A.6.7}$$

Let us start with an elementary lemma.

Lemma A.6.4. *Let $\lambda > 0$ be given. Defining*

$$J(\lambda) = e^{-\lambda} \int_0^\lambda \frac{e^\sigma - 1}{\sigma} d\sigma,$$

we have

$$J(\lambda) = \lambda^{-1} + O(\lambda^{-2}), \quad \lambda \to +\infty, \tag{A.6.8}$$

$$\forall \lambda > 0, \quad J(\lambda) \ge \lambda^{-1} - \lambda^{-2}. \tag{A.6.9}$$

Proof. Indeed, we have for $\lambda > 0$,

$$\begin{aligned}\lambda J(\lambda) &= \lambda e^{-\lambda}\sum_{k\ge 1}\int_0^\lambda \frac{\sigma^{k-1}}{k!}d\sigma = \lambda e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^k}{k!k} = e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^{k+1}}{(k+1)!}\frac{k+1}{k}\\ &= e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^{k+1}}{(k+1)!} + e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^{k+1}}{(k+1)!}\frac{1}{k}\\ &= e^{-\lambda}(e^{\lambda}-1-\lambda) + \lambda^{-1}\underbrace{\left(e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^{k+2}}{(k+1)!}\frac{1}{k}\right)}_{R(\lambda)},\end{aligned} \tag{A.6.10}$$

with

$$0 \le R(\lambda) \le e^{-\lambda}\sum_{k\ge 1}\frac{\lambda^{k+2}}{(k+2)!}\frac{k+2}{k} \le e^{-\lambda}\left(e^{\lambda}-1-\lambda-\frac{\lambda^2}{2}\right)\times 3 = O(1), \tag{A.6.11}$$

so that

$$\lambda J(\lambda) = e^{-\lambda}(e^{\lambda}-1-\lambda)+\lambda^{-1}O(1) = 1+\lambda^{-1}O(1)-(1+\lambda)e^{-\lambda} = 1+\lambda^{-1}O(1),$$

proving (A.6.8). Note also that (A.6.10), (A.6.11) imply, since $R(\lambda) \ge 0$,

$$\lambda J(\lambda) \ge 1 - e^{-\lambda}(1+\lambda),$$

so that $J(\lambda) \ge \lambda^{-1} - e^{-\lambda}(1+\lambda^{-1})$, and thus[5] the sought result (A.6.9). ∎

Remark A.6.5. Considering now the function φ_0 defined by

$$\varphi_0(\omega) = \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega s}-1}{\sin s}ds,$$

we find that, for $\omega \ge 0$, using Lemma A.6.4,

$$\varphi_0(\omega) \ge \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi/2}\frac{e^{2\omega s}-1}{s}ds = \frac{e^{-\pi\omega}}{2\pi}\int_0^{\pi\omega}\frac{e^{\sigma}-1}{\sigma}d\sigma = \frac{1}{2\pi}J(\pi\omega),$$

so that

$$\varphi_0(\omega) \ge \frac{1}{2\pi^2\omega} - \frac{1}{2\pi^3\omega^2}.$$

It is our goal now to prove a minoration of the same flavour for the function (A.6.7) defined above.

[5]We leave for the reader to check that for $\lambda > 0$, $e^{-\lambda}(1+\lambda^{-1}) \le \lambda^{-2}$, which boils down to study $q(\lambda) = e^{-\lambda}(\lambda^2+\lambda)$ reaching its maximum for $\lambda \in \mathbb{R}_+$, at $\lambda_0 = (1+\sqrt{5})/2$ with $q(\lambda_0) \approx 0.84 < 1$.

Assuming $\nu \in (0, 1/2)$, we have $\frac{\pi}{3} < s_\nu < t_\nu < \frac{\pi}{2}$ (s_ν, t_ν are defined in (A.6.5), ψ_ν in (A.6.7)),

$$\begin{aligned}
2\pi e^{\pi\omega}\psi_\nu(\omega) &= \int_0^{t_\nu} \frac{e^{2\omega\phi_\nu(s)}-1}{\sin s} ds + \int_{t_\nu}^{\pi/2} \frac{e^{2\omega\phi_\nu(s)}-1}{\sin s} ds \\
&\geq \underbrace{\int_0^{t_\nu} \frac{e^{2\omega\phi_\nu(s)}-1}{\sin s} ds}_{\text{on } (0,t_\nu),\ \phi_\nu(s)\geq 0} - \int_{t_\nu}^{\pi/2} \frac{ds}{\sin s} \\
&\geq \int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)}-1}{\sin s} ds - \int_{\pi/3}^{\pi/2} \frac{ds}{\sin s} \\
&= \underbrace{\int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)}-1}{\sin s} ds}_{\substack{\text{on } (0,s_\nu) \\ \phi_\nu(s)>0 \text{ and } \phi_\nu'(s)>0}} - \frac{\ln 3}{2}. \qquad \text{(A.6.12)}
\end{aligned}$$

Claim A.6.6. For $s \in (0, \pi/2)$, we have $\phi_\nu(s) \geq \phi_\nu'(s)\sin s$. Moreover, for $s \in (0, s_\nu)$, we have $\frac{1}{\sin s} \geq \frac{\phi_\nu'(s)}{\phi_\nu(s)}$.

Proof of Claim A.6.6. Indeed, we have

$$\begin{aligned}
\phi_\nu(s) - \phi_\nu'(s)\sin s &= s - \nu^2\tan s - \sin s + \nu^2(1+\tan^2 s)\sin s \\
&= \nu^2(\sin s + \sin s\tan^2 s - \tan s) + s - \sin s \\
&= \nu^2\left(\frac{\sin s}{\cos^2 s} - \frac{\sin s}{\cos s}\right) + s - \sin s \\
&= \frac{\nu^2 \sin s}{\cos^2 s}(1-\cos s) + s - \sin s \geq 0, \quad \text{for } s \in (0, \pi/2).
\end{aligned}$$

The last part of the claim follows from the first part and the fact that $\sin s$ and $\phi_\nu(s)$ are both positive on $(0, s_\nu)$. ■

Going back now to (A.6.12), we obtain that for $\nu \in (0, 1/2)$ and $\omega > 0$, we have

$$\begin{aligned}
2\pi e^{\pi\omega}\psi_\nu(\omega) &\geq \int_0^{s_\nu} \frac{e^{2\omega\phi_\nu(s)}-1}{\phi_\nu(s)}\phi_\nu'(s) ds - \frac{\ln 3}{2} \\
&= \int_0^{2\omega\phi_\nu(s_\nu)} \frac{e^\sigma - 1}{\sigma} d\sigma - \frac{\ln 3}{2} = e^{2\omega\phi_\nu(s_\nu)} J(2\omega\phi_\nu(s_\nu)) - \frac{\ln 3}{2},
\end{aligned}$$

so that, using (A.6.9), we get

$$\psi_\nu(\omega) \geq \frac{1}{2\pi}e^{-\pi\omega}e^{2\omega\phi_\nu(s_\nu)}\left(\frac{1}{2\omega\phi_\nu(s_\nu)} - \frac{1}{(2\omega\phi_\nu(s_\nu))^2}\right) - \frac{\ln 3}{2}\frac{1}{2\pi}e^{-\pi\omega},$$

and since $\phi_\nu(s_\nu) = \frac{\pi}{2} - \varepsilon_\nu$, with $\varepsilon_\nu \in (0, \pi/2)$, we find also that ε_ν is a concave function[6] of $\nu \in (0, 1)$ and

$$\frac{\pi\nu}{2} \le \varepsilon_\nu \le 2\nu$$

so that

$$2\phi_\nu(s_\nu) = \pi - 2\varepsilon_\nu \in [\pi - 4\nu, \pi - \pi\nu],$$

so that for $\nu \in (0, 1/2]$, we have[7] (assuming $\omega > 0$),

$$\begin{aligned}\psi_\nu(\omega) &\ge \frac{1}{2\pi} e^{-\pi\omega} e^{\omega(\pi - 2\varepsilon_\nu)} \left(\frac{1}{\omega(\pi - 2\varepsilon_\nu)} - \frac{1}{(\omega(\pi - 2\varepsilon_\nu))^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega}, \\ &\ge \frac{1}{2\pi} e^{-4\nu\omega} \left(\frac{1}{\omega\pi} - \frac{1}{\omega^2(\pi - 2)^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega},\end{aligned}$$

We recall the notations (A.6.1), so that $\nu = \sqrt{\kappa/\omega}$, i.e., $\nu\omega = \sqrt{\kappa\omega}$ and we get

$$\forall \omega > 0, \quad \psi_\nu(\omega) \ge \frac{1}{2\pi} e^{-4\sqrt{\kappa\omega}} \left(\frac{1}{\pi\omega} - \frac{1}{\omega^2} \right) - \frac{\ln 3}{2} \frac{1}{2\pi} e^{-\pi\omega}, \quad \nu = \sqrt{\kappa/\omega}. \tag{A.6.13}$$

A.6.4 An explicit expression for a_{11}

According to (5.2.22), we have

$$a_{11}(\tau, \sigma) = \frac{1}{2} + \frac{1}{2\pi} \int_0^{+\infty} \frac{\sin(2\pi t\tau - 4\pi\sigma \tanh(t/2))}{\sinh(t/2)} dt. \tag{A.6.14}$$

We have used in Section 5.2 the equivalent expression $a_{11}(\tau, \sigma) = \frac{1}{2} + \widehat{T_\sigma}(\tau)$, where T_σ is defined in (5.2.9) and we were able to prove the estimate in Lemma 5.2.2. It turns out that (A.6.4) is not optimal, and it is interesting to give an "explicit" expression for a_{11} as displayed in [55]. Using the notations (A.6.1), we can write (A.6.14) as

$$\begin{aligned}a_{11}(\tau, \sigma) &= \frac{1}{2} + \frac{1}{4\pi} \int_{\mathbb{R}} \operatorname{Im} \frac{\exp i(\omega t - 2\kappa \tanh(t/2))}{\sinh(t/2)} dt \\ &= \frac{1}{2} + \operatorname{Im} \lim_{R \to +\infty} \frac{1}{2\pi} \int_{[-R,R]} \frac{\exp 2i(\omega s - \kappa \tanh s)}{\sinh s} ds.\end{aligned} \tag{A.6.15}$$

[6] We have from (A.6.6),

$$\varepsilon_\nu = \frac{\pi}{2} - \arccos\nu + \nu\sqrt{1 - \nu^2}, \quad \frac{d\varepsilon_\nu}{d\nu} = 2\sqrt{1 - \nu^2}, \quad \frac{d^2\varepsilon_\nu}{d\nu^2} = -2\nu/\sqrt{1 - \nu^2} < 0,$$

so that the concavity gives $\frac{\pi}{2}\nu \le \varepsilon_\nu \le 2\nu$.

[7] We know that $\omega(\pi - 2\varepsilon_\nu) \ge \omega(\pi - 4\nu) \ge \omega(\pi - 2)$ so that to ensure $\omega(\pi - 2\varepsilon_\nu) \ge 4$, it suffices to assume $\omega \ge 4/(\pi - 2)$.

Defining the holomorphic function G by

$$G(z) = \frac{\exp 2i(\omega z - \kappa \tanh z)}{2\pi \sinh z}, \tag{A.6.16}$$

we see that G has simple poles at $i\pi\mathbb{Z}$ and essential singularities at $i\pi(\frac{1}{2} + \mathbb{Z})$. For $R \in \mathbb{R}_+ \backslash \frac{\pi}{2}\mathbb{Z}$, $\varepsilon \in (0, \pi/2)$, we have

$$\begin{aligned} &\oint_{[-R,-\varepsilon]\cup[\varepsilon,R]} G(z)dz + \oint_{\substack{\gamma_\varepsilon^- \\ \gamma_\varepsilon^-(\theta)=\varepsilon e^{i\theta} \\ -\pi\le t\le 0}} G(z)dz + \oint_{\substack{\gamma_R^+ \\ \gamma_R^+(\theta)=Re^{i\theta} \\ 0\le t\le\pi}} G(z)dz \\ &\quad = 2i\pi \sum_{\substack{k\in\mathbb{N} \\ k\pi<2R}} \mathtt{Res}(G, ik\pi/2). \end{aligned} \tag{A.6.17}$$

Claim A.6.7. We have

$$\lim_{\varepsilon\to 0} \oint_{\gamma_\varepsilon^-} G(z)dz = \frac{i}{2}.$$

Proof. Indeed, we have

$$\begin{aligned} &\int_{-\pi}^{0} \frac{\exp 2i(\omega\varepsilon e^{i\theta} - \kappa \tanh(\varepsilon e^{i\theta}))}{2\pi \sinh(\varepsilon e^{i\theta})} i\varepsilon e^{i\theta} d\theta \\ &\quad = \frac{i}{2\pi} \int_{-\pi}^{0} \frac{e^{2i\omega\varepsilon e^{i\theta}} \varepsilon e^{i\theta}}{\sinh(\varepsilon e^{i\theta})} \exp(-2i\kappa \tanh(\varepsilon e^{i\theta}))d\theta, \end{aligned}$$

and since the function $z \mapsto \frac{ze^{2i\omega z}}{\sinh z} e^{-2i\kappa \tanh z}$ is holomorphic near 0 with value 1 at 0, we get the result of the claim. ∎

Lemma A.6.8. *We have*

$$\lim_{\mathbb{N}\ni m\to+\infty} \operatorname{Im}\left(\oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz\right) = 0.$$

Proof. Indeed, we have with $R = \frac{\pi}{4} + m\frac{\pi}{2}$,

$$\begin{aligned} &\operatorname{Im} \int_0^{\pi} \frac{\exp 2i(\omega Re^{i\theta} - \kappa \tanh(Re^{i\theta}))}{2\pi \sinh(Re^{i\theta})} iRe^{i\theta} d\theta \\ &= \frac{R}{\pi} \operatorname{Re} \int_0^{\pi} \frac{e^{2i\omega R\cos\theta} e^{-2R\omega\sin\theta} e^{i\theta}}{1 - e^{-2Re^{i\theta}}} e^{-Re^{i\theta}} \exp(-2i\kappa \tanh(Re^{i\theta}))d\theta \\ &= \frac{2R}{\pi} \int_0^{\pi/2} \operatorname{Re}\left\{\frac{e^{2i\omega R\cos\theta} e^{-2R\omega\sin\theta} e^{i\theta}}{1 - e^{-2Re^{i\theta}}} e^{-Re^{i\theta}} \exp(-2i\kappa \tanh(Re^{i\theta}))\right\} d\theta, \end{aligned}$$

so that

$$\operatorname{Im}\left(\oint_{\gamma^{+}_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz\right)$$
$$= \frac{2R}{\pi}\int_0^{\pi/2} e^{-R\cos\theta}e^{-2R\omega\sin\theta}$$
$$\times \operatorname{Re}\left\{\frac{e^{2i\omega R\cos\theta}e^{i\theta}}{1-e^{-2Re^{i\theta}}}e^{-iR\sin\theta}\exp\left(-2i\kappa\tanh(Re^{i\theta})\right)\right\}d\theta. \quad \text{(A.6.18)}$$

We have also

$$\tanh(Re^{i\theta}) = \frac{1-e^{-2Re^{i\theta}}}{1+e^{-2Re^{i\theta}}}. \quad \text{(A.6.19)}$$

Claim A.6.9. Defining for $m \in \mathbb{N}, \theta \in [0,\pi]$, $g_m(\theta) = 1 - e^{-(\frac{\pi}{2}+m\pi)e^{i\theta}}$, we find that

$$\inf_{\substack{\theta\in[0,\pi]\\ m\in\mathbb{N}}} |g_m(\theta)| = \beta_0 > 0, \qquad \inf_{\substack{\theta\in[0,\pi]\\ m\in\mathbb{N}}} |2-g_m(\theta)| = \beta_1 > 0.$$

Proof of Claim A.6.9. If it were not the case, we could find sequences $\theta_l \in [0,\pi], m_l \in \mathbb{N}$ such that

$$\lim_{l\to+\infty} e^{-(\frac{\pi}{2}+m_l\pi)e^{i\theta_l}} = 1. \quad \text{(A.6.20)}$$

Taking the logarithm of the modulus of both sides, we would get

$$\lim_{l\to+\infty}\left(\frac{\pi}{2}+m_l\pi\right)\cos\theta_l = 0,$$

i.e.,

$$\cos\theta_l = \frac{\varepsilon_l}{\frac{\pi}{2}+m_l\pi}, \qquad \lim_{l\to+\infty}\varepsilon_l = 0.$$

Going back to (A.6.20), we find then

$$\lim_{l\to+\infty} e^{-i(\frac{\pi}{2}+m_l\pi)\sin\theta_l} = 1,$$

i.e., since $\sin\theta_l \geq 0$,

$$\lim_{l\to+\infty}\exp -i\left\{\left(\frac{\pi}{2}+m_l\pi\right)\left(1-\frac{\varepsilon_l^2}{(\frac{\pi}{2}+m_l\pi)^2}\right)^{1/2}\right\} = 1,$$

implying $\lim_{l\to+\infty} e^{-i(\frac{\pi}{2}+m_l\pi)} = 1$, which is not possible since

$$e^{-i(\frac{\pi}{2}+m_l\pi)} = -i(-1)^{m_l} \in \{\pm i\},$$

proving the first inequality of the claim. The second inequality follows from the same *reductio ad absurdum*, starting with

$$\lim_{l\to+\infty} e^{-(\frac{\pi}{2}+m_l\pi)e^{i\theta_l}} = -1,$$

ending-up with an impossibility since $-1 \notin \{\pm i\}$. ∎

As a consequence of Claim A.6.9 and (A.6.19), we obtain for $R = \frac{\pi}{4} + m\frac{\pi}{2}$, $\theta \in (0, \pi)$,

$$|\tanh(Re^{i\theta})| \leq \frac{2}{\beta_1}.$$

Formula (A.6.18) gives then

$$\left|\operatorname{Im}\left(\oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} G(z)dz\right)\right| \leq \frac{2R}{\pi}\int_0^{\pi/2} e^{-R\cos\theta} e^{-2R\omega\sin\theta}\frac{1}{\beta_0}\exp\left(4\kappa/\beta_1\right)d\theta,$$

where for $\omega > 0$, the right-hand side goes to zero when R goes to $+\infty$, completing the proof of Lemma A.6.8. ∎

Lemma A.6.10. *With G defined in (A.6.16), we have*

$$2\pi\sum_{k\in\mathbb{N}} \mathtt{Res}(G, ik\pi/2) = \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})}\mathtt{Res}\left(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z}, 0\right). \tag{A.6.21}$$

Proof. We have $\mathtt{Res}(G, ik\pi/2) = \mathtt{Res}(G_k, 0)$ and with $k = 2l$,

$$G_k(z) = \frac{\exp 2i(\omega(z+\frac{ik\pi}{2}) - \kappa\tanh(z+\frac{ik\pi}{2}))}{2\pi\sinh(z+\frac{ik\pi}{2})} = \frac{e^{-2l\pi\omega}e^{2i\omega z}e^{-2i\kappa\tanh z}}{2\pi(-1)^l\sinh z},$$

so that

$$\mathtt{Res}(G_{2l}, 0) = \frac{(-1)^l e^{-2l\pi\omega}}{2\pi},$$

whereas for $k = 2l+1$, we have

$$\begin{aligned} G_{2l+1}(z) &= \frac{\exp 2i(\omega(z+il\pi+\frac{i\pi}{2}) - \kappa\tanh(z+il\pi+\frac{i\pi}{2}))}{2\pi\sinh(z+il\pi+\frac{i\pi}{2})} \\ &= \frac{e^{-(2l+1)\pi\omega}e^{2i\omega z}e^{-2i\kappa\coth z}}{2\pi(-1)^l i\cosh z}, \end{aligned}$$

so that

$$\mathtt{Res}(G_{2l+1}, 0) = \frac{(-1)^l e^{-(2l+1)\pi\omega}}{2\pi i}\mathtt{Res}\left(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z}, 0\right),$$

yielding

$$
\begin{aligned}
&2\pi \sum_{k\in\mathbb{N}} \mathtt{Res}(G, ik\pi/2)\\
&= \sum_{l\in\mathbb{N}} (-1)^l e^{-2l\pi\omega} + \sum_{l\in\mathbb{N}} \frac{(-1)^l e^{-(2l+1)\pi\omega}}{i} \mathtt{Res}\Big(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\Big),\\
&= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathtt{Res}\Big(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\Big),
\end{aligned}
$$

concluding the proof of the lemma. ∎

Proposition A.6.11. *Using the notations* (A.6.1), *with a_{11} defined in* (A.6.14) *(see also* (A.6.15)*), we have for $\tau > 0, \sigma \geq 0$,*

$$
a_{11}(\tau,\sigma) = \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{1+e^{-2\pi\omega}} \operatorname{Im}\Big\{\mathtt{Res}\Big(\frac{e^{2i(\omega z - \kappa \coth z)}}{\cosh z}, 0\Big)\Big\}. \tag{A.6.22}
$$

Proof. Taking the imaginary part of both sides in (A.6.17), and letting $R \to +\infty$, $\varepsilon \to 0_+$, we get, using (A.6.21), (A.6.15), Claim A.6.7,

$$
a_{11} - \frac{1}{2} + \operatorname{Im}\frac{i}{2} = \operatorname{Im} i\Big(\frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})}\mathtt{Res}\Big(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0\Big)\Big),
$$

which is (A.6.22). ∎

Remark A.6.12. In particular, when $\sigma = 0$, we find for $\tau > 0$

$$
1 - a_{11}(\tau, 0) = \frac{e^{-4\pi^2\tau}}{1+e^{-4\pi^2\tau}},
$$

and since (5.2.24) implies that

$$
\begin{aligned}
2\pi \operatorname{Re} a_{12}(\tau,0) &= \int_0^{+\infty} \frac{\sin(4\pi t\tau)}{\cosh t} dt = \operatorname{Im}\langle e^{i4\pi\tau t} H(t), \operatorname{sech} t\rangle_{\mathscr{S}'(\mathbb{R}_t),\mathscr{S}(\mathbb{R}_t)}\\
&= \operatorname{Im} \frac{1}{4i\pi\tau}\Big\langle \frac{d}{dt}\{e^{i4\pi\tau t}\} H(t), \operatorname{sech} t\Big\rangle\\
&= \operatorname{Im} \frac{1}{4i\pi\tau}\Big(\Big\langle \frac{d}{dt}\{e^{i4\pi\tau t} H(t)\}, \operatorname{sech} t\Big\rangle - \langle \delta_0, \operatorname{sech}\rangle\Big)\\
&= \frac{1}{4\pi\tau} - \operatorname{Im}\frac{1}{4i\pi\tau}\langle e^{i4\pi\tau t} H(t), \operatorname{sech}'(t)\rangle\\
&= \frac{1}{4\pi\tau} + O(\tau^{-3}), \quad \tau \to +\infty,
\end{aligned}
$$

we readily find that

$$
\operatorname{Re} a_{12}(\tau, 0) \gg 1 - a_{11}(\tau, 0), \quad \tau \to +\infty,
$$

providing another proof of Theorem 5.2.4 in the case $\sigma = 0$.

Remark A.6.13. Equation (5.2.41) gives also $\operatorname{Im} a_{12}(\tau,\sigma) = \frac{e^{-2\pi^2\tau}}{2} a_{11}(\tau,\sigma)$, where (5.2.22) gives, using the notations (A.6.1),

$$\begin{aligned}\operatorname{Im} a_{12}(\tau,\sigma) &= \frac{1}{4\pi}\int_0^{+\infty} \frac{\cos(t\omega - 2\kappa\coth(t/2))}{\cosh(t/2)} dt \\ &= \frac{1}{2\pi}\int_0^{+\infty} \frac{\cos(2(t\omega - \kappa\coth t))}{\cosh t} dt = \frac{1}{4\pi}\int_{\mathbb{R}} \frac{\cos(2(t\omega - \kappa\coth t))}{\cosh t} dt.\end{aligned}$$

With G given by (A.6.16), we note that

$$\tilde{G}(z) = \frac{ie^{\pi\omega}}{2} G\left(z + \frac{i\pi}{2}\right) = \frac{\exp 2i(\omega z - \kappa \coth z)}{4\pi \cosh z},$$

a holomorphic function with simple poles at $i\pi(\frac{1}{2} + \mathbb{Z})$ and essential singularities at $i\pi\mathbb{Z}$. Following now for $\tilde{G}$ the track of G in Claim A.6.7, Lemmas A.6.8, A.6.10, and Proposition A.6.11, we get

$$\operatorname{Im} a_{12}(\tau,\sigma) = \lim_{\substack{m\to+\infty \\ \varepsilon\to 0_+}} \operatorname{Re} \oint_{[-R_m,-\varepsilon]\cup[\varepsilon,R_m]} \tilde{G}(z)dz, \quad R_m = \frac{\pi}{4} + m\frac{\pi}{2}, \tag{A.6.23}$$

and we have also

$$\begin{aligned}&\oint_{[-R_m,-\varepsilon]\cup[\varepsilon,R_m]} \tilde{G}(z)dz - \oint_{\substack{\gamma_\varepsilon^+ \\ \gamma_\varepsilon^+(\theta)=\varepsilon e^{i\theta} \\ 0\le t\le\pi}} \tilde{G}(z)dz + \oint_{\substack{\gamma_{R_m}^+ \\ \gamma_{R_m}^+(\theta)=R_m e^{i\theta} \\ 0\le t\le\pi}} \tilde{G}(z)dz \\ &= 2i\pi \sum_{1\le k\le m} \mathtt{Res}(\tilde{G}, ik\pi/2) = -\pi e^{\pi\omega} \sum_{1\le k\le m} \mathtt{Res}\left(G\left(\zeta + \frac{ik\pi}{2} + \frac{i\pi}{2}\right), 0\right) \\ &= -\pi e^{\pi\omega} \sum_{2\le l\le m+1} \mathtt{Res}\left(G\left(\zeta + \frac{il\pi}{2}\right), 0\right).\end{aligned} \tag{A.6.24}$$

Claim A.6.14. We have $\lim_{\varepsilon\to 0} \oint_{\gamma_\varepsilon^+} \tilde{G}(z)dz = 0$.

Proof. Indeed, we have $-2i\kappa \coth \varepsilon e^{i\theta} = -2i\kappa \frac{1+e^{-2\varepsilon e^{i\theta}}}{1-e^{-2\varepsilon e^{i\theta}}}$ and for $\theta \in (0,\pi)$,

$$\begin{aligned}\operatorname{Im}\left(\frac{1+e^{-2\varepsilon e^{i\theta}}}{1-e^{-2\varepsilon e^{i\theta}}}\right) &= \operatorname{Im} \frac{(1+e^{-2\varepsilon e^{i\theta}})(1-e^{-2\varepsilon e^{-i\theta}})}{|1-e^{-2\varepsilon e^{i\theta}}|^2} = \operatorname{Im} \frac{e^{-2\varepsilon e^{i\theta}} - e^{-2\varepsilon e^{-i\theta}}}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= e^{-2\varepsilon\cos\theta} \operatorname{Im} \frac{e^{-2\varepsilon i \sin\theta} - e^{2\varepsilon i\sin\theta}}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= e^{-2\varepsilon\cos\theta} \operatorname{Im} \frac{-2i\sin(2\varepsilon\sin\theta)}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \\ &= -2e^{-2\varepsilon\cos\theta} \frac{\sin(2\varepsilon\sin\theta)}{|1-e^{-2\varepsilon e^{i\theta}}|^2} \le 0, \quad \text{if } \varepsilon \le \pi/4,\end{aligned}$$

so that $|e^{-2i\kappa \coth \varepsilon e^{i\theta}}| \leq 1$, implying

$$4\pi \left| \oint_{\gamma_\varepsilon^+} \tilde{G}(z) dz \right| \leq \int_0^\pi \frac{|e^{i\omega\varepsilon e^{i\theta}}|}{|\cosh \varepsilon e^{i\theta}|} \varepsilon |i e^{i\theta}| d\theta = \varepsilon \int_0^\pi \frac{e^{-\omega\varepsilon \sin\theta}}{|\cosh \varepsilon e^{i\theta}|} d\theta,$$

which goes to zero when $\varepsilon \to 0_+$, concluding the proof of Claim A.6.14. ■

Claim A.6.15. We have $\lim_{\mathbb{N}\ni m\to+\infty} \oint_{\gamma^+_{\frac{\pi}{4}+m\frac{\pi}{2}}} \tilde{G}(z) dz = 0.$

Proof. Indeed, we have, using Claim A.6.9,

$$|\coth(R_m e^{i\theta})| = \left| \frac{1+e^{-2R_m e^{i\theta}}}{1-e^{-2R_m e^{i\theta}}} \right| \leq \begin{cases} \frac{1+e^{-2R_m \cos\theta}}{\beta_0} \leq \frac{2}{\beta_0}, & \text{for } \theta \in [0, \pi/2], \\ \left| \frac{1+e^{2R_m e^{i\theta}}}{1-e^{2R_m e^{i\theta}}} \right| \leq \frac{2}{\beta_0}, & \text{for } \theta \in [\frac{\pi}{2}, \pi], \end{cases}$$

so that

$$\begin{aligned} &|\tilde{G}(R_m e^{i\theta}) i R_m e^{i\theta}| \\ &\leq R_m e^{4\kappa/\beta_0} e^{-2\omega R_m \sin\theta} \begin{cases} \left| \frac{2e^{-R_m e^{i\theta}}}{1+e^{-2R_m e^{i\theta}}} \right| \leq \frac{2e^{-R_m \cos\theta}}{\beta_1} & \text{for } \theta \in [0, \frac{\pi}{2}], \\ \left| \frac{2e^{R_m e^{i\theta}}}{1+e^{2R_m e^{i\theta}}} \right| \leq \frac{2e^{R_m \cos\theta}}{\beta_1} & \text{for } \theta \in [\frac{\pi}{2}, \pi], \end{cases} \\ &\leq \frac{2R_m}{\beta_1} e^{4\kappa/\beta_0} e^{-2\omega R_m \sin\theta - R_m |\cos\theta|}, \end{aligned}$$

which goes to 0 when m goes to $+\infty$, proving the claim. ■

Using (A.6.21), we calculate now

$$\begin{aligned} &2\pi \sum_{l\geq 2} \mathtt{Res}(G(\zeta + \frac{il\pi}{2}), 0) \\ &= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &\quad - 2\pi \big(\mathtt{Res}(G, i\pi/2) + \mathtt{Res}(G, 0) \big) \\ &= \frac{1}{1+e^{-2\pi\omega}} + \frac{e^{-\pi\omega}}{i(1+e^{-2\pi\omega})} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &\quad + i e^{-\pi\omega} \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) - 1 \\ &= -\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} - i \left(\frac{e^{-\pi\omega}}{1+e^{-2\pi\omega}} - e^{-\pi\omega} \right) \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right) \\ &= -\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} + i e^{-\pi\omega} \left(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} \right) \mathtt{Res}\left(\frac{e^{2i\omega z - 2i\kappa \coth z}}{\cosh z}, 0 \right), \end{aligned}$$

so that from (A.6.23), (A.6.24), Claims A.6.14 and A.6.15, we obtain

$$\begin{aligned}
&\operatorname{Im} a_{12}(\tau,\sigma)\\
&= -\pi e^{\pi\omega}\frac{1}{2\pi}\Bigg(-\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} - e^{-\pi\omega}\Big(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}}\Big)\\
&\qquad\qquad \times \operatorname{Im}\Big\{\mathtt{Res}\Big(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z},0\Big)\Big\}\Bigg)\\
&= e^{\pi\omega}\frac{1}{2}\Bigg(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}} + e^{-\pi\omega}\Big(\frac{e^{-2\pi\omega}}{1+e^{-2\pi\omega}}\Big)\operatorname{Im}\Big\{\mathtt{Res}\Big(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z},0\Big)\Big\}\Bigg),
\end{aligned}$$

so that

$$\operatorname{Im} a_{12}(\tau,\sigma) = \frac{e^{-\pi\omega}}{2(1+e^{-2\pi\omega})} + \frac{e^{-2\pi\omega}}{2(1+e^{-2\pi\omega})}\operatorname{Im}\Big\{\mathtt{Res}\Big(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z},0\Big)\Big\}, \tag{A.6.25}$$

recovering (A.6.22) from (5.2.41).

N.B. We note that

$$\mathtt{Res}\Big(\frac{e^{2i\omega z-2i\kappa\coth z}}{\cosh z},0\Big) = \frac{1}{2}\mathtt{Res}\Big(\frac{e^{i(\omega z-2\kappa\coth(z/2))}}{\cosh(z/2)},0\Big),$$

so that (A.6.25) corroborates formula (A14) in [55]; however, we were not able to understand formulas (A10), (A11), and (20) in [55].

A.7 Airy function

A.7.1 Standard results on the Airy function

We collect in this section a couple of classical results on the Airy function (see, e.g., Definition 7.6.8 in Section 7.6 of [23] or the references [51], [49], [29]). For all the statements of this section whose proofs are not included, we refer the reader to Chapter 9 of [35].

Definition A.7.1. The Airy function $\mathtt{Ai}$ is defined as the inverse Fourier transform of $\xi \mapsto e^{i(2\pi\xi)^3/3}$.

Proposition A.7.2. *For any $h > 0$ and all $x \in \mathbb{C}$, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi}\int e^{\frac{i}{3}(\xi+ih)^3}e^{ix(\xi+ih)}d\xi = e^{-xh}e^{\frac{h^3}{3}}\frac{1}{2\pi}\int e^{-h\xi^2}e^{i(\frac{\xi^3}{3}-\xi h^2)}e^{ix\xi}d\xi.$$

We note that the function $\mathbb{R} \ni \xi \mapsto e^{\frac{i}{3}(\xi+ih)^3}$ belongs to the Schwartz space for any $h > 0$ since

$$\frac{i}{3}(\xi+ih)^3 = -h\xi^2 + \frac{h^3}{3} + i\Big(\frac{\xi^3}{3} - \xi h^2\Big),$$

so that $e^{\frac{i}{3}(\xi+ih)^3} = e^{-h\xi^2}e^{i(\frac{\xi^3}{3}-\xi h^2)}e^{h^3/3}$.

Theorem A.7.3. *The Airy function* $\mathtt{Ai}$ *is an entire function on* $\mathbb{C}$*, real-valued on the real line, which is the unique solution of the initial value problem for the Airy equation*

$$\mathtt{Ai}''(x) - x\,\mathtt{Ai}(x) = 0, \quad \mathtt{Ai}(0) = \frac{3^{-1/6}\Gamma(1/3)}{2\pi}, \quad \mathtt{Ai}'(0) = -\frac{3^{1/6}\Gamma(2/3)}{2\pi}. \tag{A.7.1}$$

We have also, for any $x \in \mathbb{C}$,

$$\mathtt{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty} e^{-\xi^3/3} e^{-x\xi/2} \cos\left(\frac{x\xi\sqrt{3}}{2} + \frac{\pi}{6}\right) d\xi,$$

and the power series expansion of the Airy function is

$$\mathtt{Ai}(x) = \frac{1}{\pi 3^{2/3}} \sum_{k\geq 0} \frac{(3^{1/3}x)^k}{k!} \Gamma\left(\frac{k+1}{3}\right) \sin\left(2(k+1)\frac{\pi}{3}\right).$$

Lemma A.7.4. *For* $x \in \mathbb{C}\backslash\mathbb{R}_-$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi} e^{-\frac{2}{3}x^{3/2}} \int_{\mathbb{R}} e^{-x^{1/2}\xi^2} e^{i\xi^3/3} d\xi. \tag{A.7.2}$$

Proof. Using Proposition A.7.2, we get (A.7.2) for $x > 0$ (choosing $h = x^{1/2}$), and then we may use an analytic continuation argument. ∎

Theorem A.7.5. *For all* $M \in \mathbb{N}$*, for all* $x \in \mathbb{C}\backslash\mathbb{R}_-$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{2\pi} e^{-\frac{2x^{3/2}}{3}} x^{-1/4} \left\{ \sum_{0\leq l\leq M} \frac{(-1)^l}{3^{2l}(2l)!} \Gamma\left(3l + \frac{1}{2}\right) x^{-3l/2} + R_M(x) \right\},$$

with $$|R_M(x)| \leq \frac{\Gamma\left(3M+3+\frac{1}{2}\right)}{3^{2M+2}(2M+2)!} |x|^{-(\frac{3(M+1)}{2})} \left(\cos\left(\frac{\arg x}{2}\right)\right)^{-3(M+1)-\frac{1}{2}}. \tag{A.7.3}$$

For $x < 0$*, we have*

$$\mathtt{Ai}(x) = \frac{1}{|x|^{1/4}\sqrt{\pi}} \left(\sin\left(\frac{\pi}{4} + \frac{2}{3}|x|^{3/2}\right) + O(|x|^{-3/2}) \right), \tag{A.7.4}$$

$$\mathtt{Ai}'(x) = -\frac{|x|^{1/4}}{\sqrt{\pi}} \left(\cos\left(\frac{\pi}{4} + \frac{2}{3}|x|^{3/2}\right) + O(|x|^{-3/2}) \right). \tag{A.7.5}$$

Lemma A.7.6. *With* $j = e^{2i\pi/3}$ *we have for all* $x \in \mathbb{C}$,

$$\mathtt{Ai}(x) + j\,\mathtt{Ai}(jx) + j^2\,\mathtt{Ai}(j^2 x) = 0.$$

In particular, for $r \geq 0$*, we have*

$$\mathtt{Ai}(-r) = 2\,\mathrm{Re}\left(e^{\frac{i\pi}{3}}\,\mathtt{Ai}(re^{\frac{i\pi}{3}})\right). \tag{A.7.6}$$

Lemma A.7.7. *The zeroes of the Airy function are simple and located on* $(-\infty, 0)$. *We shall use the notation*

$$\mathtt{Ai}^{-1}(\{0\}) = \{\eta_k\}_{k\geq 0}, \quad \eta_{k+1} < \eta_k < 0, \quad \lim_{k\to+\infty} \eta_k = -\infty.$$

The largest zero of Ai *is* $\eta_0 \approx -2.338107410$ *and* $\mathtt{Ai}(\eta)$ *is positive for* $\eta > \eta_0$. *We have also for all* $k \geq 0$,

$$\mathtt{Ai}(\eta_{2k+1}) = 0,\ \mathtt{Ai}'(\eta_{2k+1}) < 0,\ \ \mathtt{Ai}(\eta_{2k}) = 0,\ \mathtt{Ai}'(\eta_{2k}) > 0,$$

$$\mathtt{Ai}(\eta) < 0 \ \textit{for}\ \eta \in (\eta_{2k+1}, \eta_{2k}),\ \mathtt{Ai}(\eta) > 0\ \textit{for}\ \eta \in (\eta_{2k+2}, \eta_{2k+1}), \quad \text{(A.7.7)}$$

$$\mathtt{Ai}''(\eta) > 0\ \textit{for}\ \eta \in (\eta_{2k+1}, \eta_{2k}),\ \mathtt{Ai}''(\eta) < 0\ \textit{for}\ \eta \in (\eta_{2k+2}, \eta_{2k+1}). \text{(A.7.8)}$$

N.B. The simplicity of the zeroes of the Airy function holds true for any non-zero solution of the Airy differential equation $y'' = xy$. The solutions of this ODE are analytic functions and if a is a double zero, we have $y(a) = y'(a) = 0$ and thus from the Airy equation, we get $y''(a) = 0$; we may then prove by induction on $k \geq 1$ that $y^{(l)}(a) = 0$ for $0 \leq l \leq k+1$: it is proven for $k = 1$, and if true for some $k \geq 1$, we get

$$y^{(k+2)}(x) = (xy(x))^{(k)} \Longrightarrow y^{(k+2)}(a) = 0,$$

proving the final step in the induction; as a consequence, the function has a zero of infinite order, which is impossible for a non-zero analytic function. Assertion (A.7.8) follows from the Airy differential equation (A.7.1), from (A.7.7) and $\eta_{2k} < 0$.

Remark A.7.8. For $M = 0$, $|\arg x| \leq \pi/3$, we have

$$|R_0(x)| \leq \frac{\Gamma\left(3+\frac{1}{2}\right)}{3^2(2)!}|x|^{-\frac{3}{2}}\left(\frac{\sqrt{3}}{2}\right)^{-\frac{7}{2}} = |x|^{-\frac{3}{2}}\sqrt{\pi}\frac{5}{3^{11/4}\sqrt{2}} \leq |x|^{-\frac{3}{2}} \times 0.305455,$$

so that

$$|R_0(x)| \leq 0.305455|x|^{-3/2} \quad \text{if } |\arg x| \leq \pi/3,$$
and for $|x| \geq 12$, $|\arg x| \leq \pi/3$ we have $|R_0(x)| \leq 0.007349$.

We get then for $\lambda > 0$, using (A.7.6)

$$\begin{aligned}\mathtt{Ai}(-\lambda) &= \frac{1}{\pi}\operatorname{Re}\Big(e^{i\pi/3}\lambda^{-1/4}e^{-i\frac{2}{3}\lambda^{3/2}}\big(\sqrt{\pi}e^{-i\pi/12} + R_0(\lambda e^{i\pi/3})\big)\Big)\\ &= \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\cos\left(\frac{\pi}{4} - \frac{2}{3}\lambda^{3/2}\right) + \frac{1}{\pi}\operatorname{Re}\{\lambda^{-1/4}R_0(re^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}\lambda^{3/2}}\}\\ &= \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\frac{1}{\sqrt{\pi}}\operatorname{Re}\left\{R_0(\lambda e^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}\lambda^{3/2}}\right\}\right),\end{aligned}$$

so that

$$\text{for } \lambda > 0, \quad \mathtt{Ai}(-\lambda) = \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + \tilde{R}_0(\lambda)\right), \tag{A.7.9}$$

$$\text{with} \quad |\tilde{R}_0(\lambda)| \le \lambda^{-3/2} \times 0.172335,$$

$$\text{and for } \lambda \ge 12, \quad |\tilde{R}_0(\lambda)| \le 0.004146. \tag{A.7.10}$$

Remark A.7.9. For $M = 1$, $|\arg x| \le \pi/3$, we have

$$|R_1(x)| \le \frac{\Gamma\left(6+\frac{1}{2}\right)}{3^4(4)!}|x|^{-3}\left(\frac{\sqrt{3}}{2}\right)^{-6-\frac{1}{2}}$$
$$= |x|^{-3}\sqrt{\pi}\frac{11!}{2^{21/2} \times 3^{37/4} \times 5} \le |x|^{-3} \times 0.377203,$$

and

$$\text{for } |x| \ge 12, \quad |R_1(x)| \le 0.000219,$$

so that

$$\mathtt{Ai}(-r) = \frac{1}{\sqrt{\pi}}r^{-1/4}\left(\sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}\sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right)r^{-3/2}\right.$$
$$\left. + \frac{1}{\sqrt{\pi}}\operatorname{Re}\left\{R_1(re^{i\pi/3})e^{i\pi/4}e^{-i\frac{2}{3}r^{3/2}}\right\}\right)$$
$$= \frac{1}{\sqrt{\pi}}r^{-1/4}\left(\sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}\sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right)r^{-3/2}\right.$$
$$\left. + \frac{1}{\sqrt{\pi}}\tilde{R}_1(r)\right),$$

so that

$$\text{for } r > 0, \quad |\tilde{R}_1(r)| \le r^{-3} \times 0.377203, \tag{A.7.11}$$
$$\text{for } r \ge 12, \quad |\tilde{R}_1(r)| \le 0.000219.$$

We find for $\lambda > 0$,

$$G(-\lambda) = \int_\lambda^{+\infty} \frac{1}{r^{1/4}\sqrt{\pi}}\left(\sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{18\sqrt{\pi}}r^{-3/2}\sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right)\right.$$
$$\left. + \frac{1}{\sqrt{\pi}}\tilde{R}_1(r)\right)dr, \tag{A.7.12}$$

and we have

$$\int_\lambda^{+\infty} \frac{1}{r^{3/4}\sqrt{\pi}}r^{1/2}\sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right)dr$$
$$= \cos\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)\frac{1}{\lambda^{3/4}\sqrt{\pi}} - \frac{3}{4}\int_\lambda^{+\infty} \frac{1}{r^{7/4}\sqrt{\pi}}\cos\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right)dr,$$

as well as

$$
\begin{aligned}
&-\frac{3}{4}\int_{\lambda}^{+\infty}\frac{1}{r^{7/4}\sqrt{\pi}}\cos\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&\quad=-\frac{3}{4}\int_{\lambda}^{+\infty}\frac{1}{r^{9/4}\sqrt{\pi}}r^{1/2}\cos\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&\quad=\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr,
\end{aligned}
$$

so that

$$
\begin{aligned}
\int_{\lambda}^{+\infty}\frac{1}{r^{1/4}\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr&=\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\frac{1}{\lambda^{3/4}\sqrt{\pi}}\\
&+\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr.
\end{aligned}
\tag{A.7.13}
$$

We have also

$$
\begin{aligned}
&\int_{\lambda}^{+\infty}\frac{1}{r^{1/4}}\frac{\Gamma(7/2)}{18\pi}r^{-3/2}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)dr\\
&\quad=\frac{\Gamma(7/2)}{18\pi}\int_{\lambda}^{+\infty}r^{-7/4}\sin\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)dr\\
&\quad=-\frac{\Gamma(7/2)}{18\pi}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-9/4}\\
&\qquad+\frac{\Gamma(7/2)}{18\pi}\frac{9}{4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr,
\end{aligned}
\tag{A.7.14}
$$

so that (A.7.13), (A.7.14), and (A.7.12) entail

$$
\begin{aligned}
G(-\lambda)=\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\frac{1}{\lambda^{3/4}\sqrt{\pi}}&+\frac{3}{4\sqrt{\pi}}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-9/4}\\
&-\frac{3}{4\sqrt{\pi}}\frac{9}{4}\int_{\lambda}^{+\infty}r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr\\
&-\frac{\Gamma(7/2)}{18\pi}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-9/4}\\
&+\frac{\Gamma(7/2)}{18\pi}\frac{9}{4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr\\
&+\frac{1}{\pi}\int_{\lambda}^{+\infty}r^{-1/4}\tilde{R}_1(r).
\end{aligned}
$$

We get then

$$G(-\lambda) = \frac{\lambda^{-3/4}}{\sqrt{\pi}}\Bigg(\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\frac{3}{4}\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)\lambda^{-6/4}$$
$$-\frac{3}{4}\times\frac{9}{4}\lambda^{3/4}\int_{\lambda}^{+\infty} r^{-13/4}\sin\left(\frac{\pi}{4}+\frac{2}{3}r^{3/2}\right)dr$$
$$-\frac{\Gamma(7/2)}{18\sqrt{\pi}}\cos\left(\frac{2}{3}\lambda^{3/2}-\frac{\pi}{4}\right)\lambda^{-6/4}$$
$$+\frac{\Gamma(7/2)}{18\sqrt{\pi}}\frac{9}{4}\lambda^{3/4}\int_{\lambda}^{+\infty}\cos\left(\frac{2}{3}r^{3/2}-\frac{\pi}{4}\right)r^{-13/4}dr$$
$$+\frac{\lambda^{3/4}}{\sqrt{\pi}}\int_{\lambda}^{+\infty} r^{-1/4}\tilde{R}_1(r)\Bigg),$$

so that

$$G(-\lambda) = \frac{\lambda^{-3/4}}{\sqrt{\pi}}\left(\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\lambda^{-3/2}S_1(\lambda)\right), \tag{A.7.15}$$

with

$$|S_1(\lambda)| \le \frac{3}{4}+\frac{3}{4}+\frac{\Gamma(7/2)}{18\sqrt{\pi}}+\frac{\Gamma(7/2)}{18\sqrt{\pi}}+\frac{4}{9\sqrt{\pi}}\times 0.377203 \le 1.80293,$$

where we have used (A.7.11) for the bound of the last term above. As a consequence, if $\lambda \ge 12$, we get that

$$|\lambda^{-3/2}S_1(\lambda)| \le 0.0433716. \tag{A.7.16}$$

This is allowing us to extend the proof of Lemma A.7.15 to all values. Note that the first 10 values (and more) are accessible numerically.

Since we have

$$\eta_9 = -12.82877675 < -12,$$

formulas (A.7.9), (A.7.10), (A.7.15), and (A.7.16) imply the following result.

Lemma A.7.10. *With* Ai *and* G *defined above, we have for* $-\lambda \le \eta_9$

$$\mathtt{Ai}(-\lambda) = \frac{1}{\sqrt{\pi}}\lambda^{-1/4}\left(\sin\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\tilde{R}_0(\lambda)\right),$$
$$|\tilde{R}_0(\lambda)| \le \lambda^{-3/2}\times 0.172335 \le 0.004146, \tag{A.7.17}$$
$$G(-\lambda) = \frac{\lambda^{-3/4}}{\sqrt{\pi}}\left(\cos\left(\frac{\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+\tilde{S}_1(\lambda)\right),$$
$$|\tilde{S}_1(\lambda)| \le \lambda^{-3/2}\times 1.80293 \le 0.0433716.$$

A.7.2 More on the Airy function

Proposition A.7.11. *We have*

$$\int_0^{+\infty} \mathtt{Ai}(x)dx = \frac{1}{3}. \tag{A.7.18}$$

Proof. According to Theorem A.7.5, the Airy function Ai is rapidly decreasing on the positive half-line and thus belongs to $L^1(\mathbb{R}_+)$, so that the integral in (A.7.18) makes sense. Also, we have from Theorem A.7.5 and the Lebesgue dominated convergence theorem that,

$$\int_0^{+\infty} \mathtt{Ai}(x)dx = \lim_{h\to 0_+} \int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3}, \tag{A.7.19}$$

and we shall now calculate the right-hand side of (A.7.19). We have for $h > 0$,

$$\begin{aligned}\int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3} &= \int_0^{+\infty} \frac{1}{2\pi} \int e^{-h\xi^2} e^{i(\frac{\xi^3}{3}-\xi h^2)} e^{ix\xi} d\xi dx \\ &= \int_0^{+\infty} \widehat{\psi_h}(-x)dx,\end{aligned}$$

with

$$\psi_h(\xi) = e^{-h(2\pi\xi)^2} e^{i(\frac{(2\pi\xi)^3}{3}-(2\pi\xi)h^2)}, \tag{A.7.20}$$

so that

$$\begin{aligned}\int_0^{+\infty} \mathtt{Ai}(x)e^{xh}dxe^{-h^3/3} &= \left\langle \frac{\delta_0}{2} - \frac{1}{2\pi i}\mathrm{pv}\frac{1}{\xi}, \psi_h \right\rangle_{\mathscr{S}',\mathscr{S}} \\ &= \frac{1}{2} - \frac{1}{2\pi i}\left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h(2\pi\xi)^2} e^{i(\frac{(2\pi\xi)^3}{3}-(2\pi\xi)h^2)} \right\rangle \\ &= \frac{1}{2} - \frac{1}{2\pi}\left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h\xi^2} \sin\left(\frac{\xi^3}{3} - \xi h^2\right) \right\rangle.\end{aligned}$$

We note at this point that, according to (4.2.5), the right-hand side of the above equality is for $h = 0$ equal to

$$\frac{1}{2} - \frac{1}{2\pi}\frac{\pi}{3} = \frac{1}{3},$$

so that, with (A.7.19), we are left to proving that

$$\lim_{h\to 0_+} \left\langle \mathrm{pv}\frac{1}{\xi}, e^{-h\xi^2} \sin\left(\frac{\xi^3}{3} - \xi h^2\right) \right\rangle = \frac{\pi}{3}. \tag{A.7.21}$$

We have

$$\int \frac{\sin(\frac{\xi^3}{3} - \xi h^2)}{\xi} e^{-h\xi^2} d\xi = \frac{\pi}{3} + \int \frac{\sin(\frac{\xi^3}{3} - \xi h^2)e^{-h\xi^2} - \sin(\frac{\xi^3}{3})}{\xi} d\xi$$

$$= \frac{\pi}{3} + \underbrace{\int \frac{\sin(\frac{\xi^3}{3})}{\xi}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi}_{I_1(h)} - \underbrace{\int \frac{\sin(\xi h^2)}{\xi} \cos\left(\frac{\xi^3}{3}\right)e^{-h\xi^2} d\xi}_{I_2(h)}.$$

We have

$$I_{1,1}(h) = \int_1^{+\infty} \frac{\xi^2 \sin(\frac{\xi^3}{3})}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi$$
$$= \int_1^{+\infty} \frac{\frac{d}{d\xi}(\cos(\frac{\xi^3}{3}))}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi,$$

and a simple integration by parts[8] shows that $\lim_{h\to 0} I_{1,1}(h) = 0$; we have also trivially that

$$0 = \lim_{h\to 0} \int_0^1 \frac{\xi^2 \sin(\frac{\xi^3}{3})}{\xi^3}\big(\cos(\xi h^2)e^{-h\xi^2} - 1\big)d\xi.$$

On the other hand, we have

$$|I_2(h)| \le \int h^2 e^{-h\xi^2} d\xi = O(h^{3/2}),$$

which completes the proof of (A.7.21) as well as the proof of Proposition A.7.11. ∎

Lemma A.7.12. *We have*

$$\lim_{R\to+\infty} \int_{-R}^{0} \mathtt{Ai}(x)dx = \frac{2}{3}. \tag{A.7.22}$$

Proof. Using (A.7.4), we find for $R \ge 1$,

$$\int_{-R}^{0} \mathtt{Ai}(x)dx = \int_0^R \mathtt{Ai}(-r)dr = \int_0^1 \mathtt{Ai}(-r)dr$$
$$+ \int_1^R \left(\frac{1}{r^{1/4}\sqrt{\pi}} \sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + O(r^{-7/4})\right)dr,$$

proving that the limit in the left-hand side of (A.7.22) is existing.

[8] The boundary term is easy to handle and for the derivative falling on ξ^{-3}, we use that $|\cos(\xi h^2)e^{-h\xi^2} - 1| \le 2$; if the derivative falls on the other term we get

$$\int_1^{+\infty} \frac{\cos(\frac{\xi^3}{3})}{\xi^3}\big(2h\xi\cos(\xi h^2)e^{-h\xi^2} + e^{-h\xi^2}\sin(\xi h^2)h^2\big)d\xi,$$

which goes trivially to 0 with h.

Claim A.7.13. We have

$$\lim_{h\to 0_+}\int_{-\infty}^{0} \mathtt{Ai}(x)e^{xh}dx = \int_{-\infty}^{0} \mathtt{Ai}(x)dx.$$

Proof of Claim A.7.13. We have

$$\int_{-\infty}^{0} \mathtt{Ai}(x)e^{xh}dx = \int_{-\infty}^{-1} \mathtt{Ai}(x)e^{xh}dx + \underbrace{\int_{-1}^{0} \mathtt{Ai}(x)e^{xh}dx}_{\text{with limit } \int_{-1}^{0} \mathtt{Ai}(x)dx},$$

and using (A.7.4), we have only to check

$$\begin{aligned}
&\int_{-\infty}^{-1} |x|^{-1/4}e^{xh+i\frac{2}{3}|x|^{3/2}}dx \\
&\quad= \int_{1}^{+\infty} t^{-1/4}e^{-th+i\frac{2}{3}t^{3/2}}dt \\
&\quad= -\int_{1}^{+\infty} \frac{d}{dt}\{e^{-th+i\frac{2}{3}t^{3/2}}\}(h-it^{1/2})^{-1}t^{-1/4}dt \\
&\quad= e^{-h+i\frac{2}{3}}(h-i)^{-1} \\
&\qquad+\int_{1}^{+\infty} e^{-th+i\frac{2}{3}t^{3/2}}\left((h-it^{1/2})^{-2}\frac{i}{2}t^{-3/4}-(h-it^{1/2})^{-1}\frac{1}{4}t^{-5/4}\right)dt,
\end{aligned}$$

and since the absolute value of the integrand in the last integral is bounded above by $\frac{3}{4}t^{-7/4}$, we get the result of the claim. ∎

With (A.7.19), (A.7.20), this gives

$$\begin{aligned}
\int_{-\infty}^{+\infty} \mathtt{Ai}(x)dx &= \lim_{h\to 0_+}\int_{-\infty}^{+\infty} \mathtt{Ai}(x)e^{xh}dx e^{-h^3/3} \\
&= \lim_{h\to 0_+}\left(\int_{\mathbb{R}} \widehat{\psi_h}(-\xi)d\xi = \psi_h(0)\right) = 1,
\end{aligned}$$

and Proposition A.7.11 provides the result of the lemma. ∎

A.7.3 Asymptotic expansion for the function G defined in (4.2.4)

Lemma A.7.14. *With G defined in* (4.2.4)*, we have*

$$G(-\lambda) = \lambda^{-3/4}\pi^{-1/2}\sin\left(\frac{3\pi}{4}+\frac{2}{3}\lambda^{3/2}\right)+O(\lambda^{-9/4}), \quad \lambda\to+\infty.$$

Proof. Property (A.7.22) and (A.7.4) give for $\eta = -\lambda < 0$,

$$G(\eta) = \frac{2}{3} + \int_0^{\eta} \mathtt{Ai}(\xi)d\xi = \int_{-\infty}^{\eta} \mathtt{Ai}(\xi)d\xi = \int_{\lambda}^{+\infty} \mathtt{Ai}(-r)dr$$

$$= \int_{\lambda}^{+\infty} 2\,\mathrm{Re}\big(e^{\frac{i\pi}{3}}\,\mathtt{Ai}(e^{\frac{i\pi}{3}}r)\big)dr$$

(we have used (A.7.6); we use now (A.7.3) for $M = 1, x \in e^{i\pi/3}\mathbb{R}_+$)

$$= \int_{\lambda}^{+\infty} \left(\frac{1}{r^{1/4}\sqrt{\pi}} \sin\left(\frac{\pi}{4} + \frac{2}{3}r^{3/2}\right) + \frac{\Gamma(7/2)}{3^2 2\pi} r^{-7/4} \sin\left(\frac{2}{3}r^{3/2} - \frac{\pi}{4}\right) + O(r^{-13/4})\right)dr$$

$$= (2/3)^{1/2}\pi^{-1/2} \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \sin\left(\frac{\pi}{4} + s\right)ds + \frac{(2/3)^{3/2}\Gamma(7/2)}{3^2 2\pi} \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-3/2} \sin\left(s - \frac{\pi}{4}\right)ds + O(\lambda^{-9/4}).$$

We integrate by parts in the first integral with

$$\int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \sin\left(\frac{\pi}{4} + s\right)ds$$

$$= -\int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} s^{-1/2} \frac{d}{ds}\left\{\cos\left(\frac{\pi}{4} + s\right)\right\}ds$$

$$= \left(\frac{2}{3}\lambda^{3/2}\right)^{-1/2} \cos\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + \int_{\frac{2}{3}\lambda^{3/2}}^{+\infty} (-1/2)s^{-3/2}\cos(\pi/4 + s)ds.$$

We have to deal with two integrals of type

$$\int_{\lambda^{3/2}}^{+\infty} s^{-3/2} \frac{d}{i\,ds} e^{is} ds$$

$$= i(\lambda^{3/2})^{(-3/2)} e^{i\lambda^{3/2}} - \frac{1}{i}\int_{\lambda^{3/2}}^{+\infty} (-3/2)s^{-5/2} e^{is} ds = O(\lambda^{-9/4}).$$

Eventually we find

$$G(-\lambda) = \lambda^{-3/4}\pi^{-1/2} \cos\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + O(\lambda^{-9/4}).$$ ■

With $(\eta_k)_{k\geq 0}$ standing for the decreasing sequence of the zeroes of the Airy function (cf. Lemma A.7.7), we have the following table of variation for the function G.

η	$-\infty$	$\cdots$	η_{2k+2}		η_{2k+1}		η_{2k}	$\cdots$	η_1		η_0		$+\infty$
$G''(\eta) = \mathtt{Ai}'(\eta)$	0	$\cdots$	$+$		$-$		$+$	$\cdots$	$-$		$+$		0
$G'(\eta) = \mathtt{Ai}(\eta)$	0	$\cdots$	0	$+$	0	$-$	0	$\cdots$	0	$-$	0	$+$	0
$G(\eta)$	0	$\cdots$	$G(\eta_{2k+2})$	$\nearrow$	$G(\eta_{2k+1})$	$\searrow$	$G(\eta_{2k})$	$\cdots$	$G(\eta_1)$	$\searrow$	$G(\eta_0)$	$\nearrow$	1

	$\boldsymbol{\eta}$	$\mathbf{G}(\boldsymbol{\eta})$
η_4	−7.944133589	−0.1187912133
η_3	−6.786708100	0.1333996865
η_2	−5.520559828	−0.1550343634
η_1	−4.087949444	0.1917571397
η_0	−2.338107410	−0.2743520591
η_9	−12.82877675	0.08315615192
η_8	−11.93601556	−0.08775971160
η_7	−11.00852430	0.09322050200
η_6	−10.04017434	−0.09984115980
η_5	−9.022650854	0.1080976882

Lemma A.7.15. *The zeroes of the function G on the real line are simple and make a decreasing sequence of negative numbers $(\xi_l)_{l\leq 0}$ such that*

$$\cdots \eta_{2k+2} < \xi_{2k+2} < \eta_{2k+1} < \xi_{2k+1} < \eta_{2k} < \xi_{2k} \cdots, \quad \xi_0 \approx -1.38418. \tag{A.7.23}$$

The largest ten zeroes of G are given by the following:

$$\begin{aligned} &\xi_0 = -1.38418, \quad \xi_1 = -3.33004, \quad \xi_2 = -4.86074, \quad \xi_3 = -6.18885, \\ &\xi_4 = -7.39024, \quad \xi_5 = -8.5022, \quad \xi_6 = -10.5366, \quad \xi_7 = -11.4826, \\ &\xi_8 = -12.3913, \quad \xi_9 = -13.2679. \end{aligned}$$

For all $k \in \mathbb{N}$, we have

$$G(\eta_{2k}) < 0 < G(\eta_{2k+1}), \tag{A.7.24}$$

and $G(\eta_{2k})$ (resp., $G(\eta_{2k+1})$) is a local minimum (resp., maximum) of G near η_{2k} (resp., η_{2k+1}). Moreover, $G(\eta_0)$ is an absolute minimum of the function G on the real line.

N.B. We claim also that

$$|G(\eta_{2k})| > G(\eta_{2k+1}) > |G(\eta_{2k+2})|, \tag{A.7.25}$$

but shall not provide a complete proof for that statement, which is anyway not needed is our Section 4.3.

Proof. In the first place, we know that $G(\eta_0)<0$ and G strictly increases on $[\eta_0,+\infty)$ so that $\xi_0 \approx -1.38418$ is defined as the unique zero of G on $(\eta_0, 0)$ since $G(0) = 2/3$. We may note that we found in particular that $\forall \eta > \eta_0$, $1 > G(\eta) > G(\eta_0)$. Also, the first ten zeroes of G are simple and satisfy (A.7.23), (A.7.24), and (A.7.25). Moreover, using Lemma A.7.10, we obtain that for $\lambda \geq 12$,

$$G(-\lambda) = 0 \Longrightarrow \left|\cos\left(\frac{3\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)\right| \leq 0.0433716,$$
$$\mathtt{Ai}(-\lambda) = 0 \Longrightarrow \left|\sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right)\right| \leq 0.004146,$$

As a result, if $-\lambda$ is a double zero of G we must have both inequalities above, which is impossible. As a result all zeroes of G are simple[9] and located on $(-\infty, 0)$. Let us consider the interval $[\eta_{2k+1}, \eta_{2k}]$, we have

$$\mathtt{Ai}(\eta_{2k+1}) = \mathtt{Ai}(\eta_{2k}) = 0, \quad \mathtt{Ai}'(\eta_{2k+1}) < 0 < \mathtt{Ai}'(\eta_{2k}), \quad \mathtt{Ai}'' > 0 \text{ on } (\eta_{2k+1}, \eta_{2k}).$$

As a result, we obtain that G has a local minimum at η_{2k} and a local maximum at η_{2k+1}. Moreover, we find from (A.7.17) in Lemma A.7.10 and $k \geq 5$ that

$$\max\left(\left|\sin\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right)\right|, \left|\sin\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right)\right|\right) \leq 0.004146$$

which implies that

$$\min\left(\left|\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right)\right|, \left|\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right)\right|\right) \geq 0.99999.$$

We know that $\mathtt{Ai}'(\eta_{2k}) > 0$, which implies, thanks[10] to (A.7.5)

$$\cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k}|^{3/2}\right) \leq -0.99999, \quad \cos\left(\frac{\pi}{4} + \frac{2}{3}|\eta_{2k+1}|^{3/2}\right) \geq 0.99999,$$

[9]It is not hard to obtain an asymptotic version of this, namely the same result for λ large enough. However, asymptotic methods provide asymptotic results and to get a result at a finite distance, we had to use the numerical results of Lemma A.7.10, grounded on a numerical estimate of the constants appearing in Theorem A.7.5.

[10]Here this is proven if k is large enough from (A.7.5), and we leave to the reader the proof of a numerical estimate analogous to Lemma A.7.10 for the derivative of the Airy function. A

and Lemma A.7.10 implies that $G(\eta_{2k}) < 0 < G(\eta_{2k+1})$, which is (A.7.24). Since the function G is strictly monotone decreasing on the interval $[\eta_{2k+1}, \eta_{2k}]$, it has a unique simple zero ξ_{2k+1} on the interior of this interval. Analogously, we can prove that on the interval $[\eta_{2k+2}, \eta_{2k+1}]$, it has a unique simple zero ξ_{2k+2} on the interior of this interval, proving that the sequence of zeroes of the function G is decreasing strictly with

$$\eta_{2k+2} < \xi_{2k+2} < \eta_{2k+1} < \xi_{2k+1} < \eta_{2k} < \xi_{2k}, \quad k \geq 0.$$

We shall prove a weaker statement than (A.7.25): we know that $|G(\eta_l)| < |G(\eta_0)|$ for $1 \leq l \leq 9$ from the numerical values obtained above. Moreover, if $\lambda \geq 12$ we find

$$|G(-\lambda)| \leq \lambda^{-3/4}\pi^{-1/2}(1 + 0.0433716) \leq 0.0913016 < |G(\eta_0)| = 0.2743520591,$$

proving indeed that $G(\eta_0)$ is the absolute minimum of the function G on the real line, since the desired estimate is proven for $\eta > \eta_0$ and for $\eta < \eta_0$, either $G(\eta) \geq 0$, or $-0.0913016 \leq G(\eta) < 0$ if $\eta \leq -12$. As said above, the values less than 12 are treated directly by a numerical calculation. The proof of the lemma is complete. ∎

A.8 Miscellaneous formulas

A.8.1 Some elementary formulas

We define for $\tau \in \mathbb{R}$,

$$\arctan \tau = \int_0^\tau \frac{dt}{1+t^2},$$

and we note that $\arctan \tau \in (-\pi/2, \pi/2)$,

$$\forall \tau \in \mathbb{R}, \quad \tan(\arctan \tau) = \tau, \quad \forall \theta \in (-\pi/2, \pi/2), \quad \arctan(\tan\theta) = \theta.$$

Moreover, we have for $\tau \in \mathbb{R}$,

$$e^{i \arctan \tau} = \frac{1}{\sqrt{1+\tau^2}}(1 + i\tau), \tag{A.8.1}$$

since for $\theta \in (-\pi/2, \pi/2)$, $\tau = \tan\theta$, we have

direct estimate is possible, using (A.7.2) and the identity (to be differentiated) for $\lambda > 0$,

$$\mathtt{Ai}(-\lambda) = \frac{\lambda^{-1/4}}{\sqrt{\pi}}\left\{\sin\left(\frac{\pi}{4} + \frac{2}{3}\lambda^{3/2}\right) + a_0(\lambda)\lambda^{-3/2}\right\},$$

$$a_0(\lambda) = \frac{\lambda^{3/2}}{\pi} e^{i(\frac{\pi}{3} - \frac{2}{3}\lambda^{3/2})} \int_{\mathbb{R}} e^{-\xi^2\lambda^{1/2}e^{i\pi/6}}\big(\cos(\xi^3/3) - 1\big)d\xi.$$

$$1+\tau^2=\frac{1}{\cos^2\theta}$$

and thus

$$\cos\theta>0\Longrightarrow\cos\theta=\frac{1}{\sqrt{1+\tau^2}}\Longrightarrow-\sin\theta=-\frac{1}{2}(1+\tau^2)^{-3/2}2\tau(1+\tau^2),$$

so that

$$e^{i\theta}=\frac{1}{\sqrt{1+\tau^2}}(1+i\tau).$$

Let $a\in\mathbb{R}_+$ be given. The Fourier transform of $\mathbf{1}_{[-a,a]}$ is

$$\int_{-a}^{a}e^{-2i\pi x\xi}dx=2\int_0^a\cos(2\pi x\xi)dx=\frac{2}{2\pi\xi}[\sin(2\pi x\xi)]_{x=0}^{x=a}=\frac{\sin(2\pi a\xi)}{\pi\xi}.$$

A.8.2 Taking the derivative of F_k on $\mathbb{R}_+$

We have, using a parity argument,

$$F_k(a)=\int_{\mathbb{R}}\frac{\sin a\tau}{\pi\tau}\frac{(1+i\tau)^{2k+1}}{(1+\tau^2)^{k+1}}d\tau=\sum_{0\le 2l\le 2k}\int_{\mathbb{R}}\frac{\sin a\tau}{\pi\tau}\frac{\binom{2k+1}{2l}(-1)^l\tau^{2l}}{(1+\tau^2)^{k+1}}d\tau.$$

We see also that $1+2k+2-2l=2k+3-2l\ge 3$ so that we can take the derivative of F_k and get

$$F_k'(a)=\sum_{0\le 2l\le 2k}\int_{\mathbb{R}}\frac{\cos a\tau}{\pi}\frac{\binom{2k+1}{2l}(-1)^l\tau^{2l}}{(1+\tau^2)^{k+1}}d\tau=\frac{1}{\pi}\int_{\mathbb{R}}(\cos a\tau)\operatorname{Re}\left(\frac{(1+i\tau)^k}{(1-i\tau)^{k+1}}\right)d\tau,$$

with absolutely converging integrals. For $a>0$, we have

$$F_k'(a)=\frac{1}{\pi}\int_{\mathbb{R}}(\cos a\tau)\frac{(1+i\tau)^k}{(1-i\tau)^{k+1}}d\tau,\tag{A.8.2}$$

since

$$\lim_{\lambda\to+\infty}\int_{-\lambda}^{\lambda}\frac{\tau^j\cos(a\tau)}{(1+\tau^2)^{k+1}}d\tau\quad\text{makes sense for } j\le 2k+1 \text{ (and vanishes for } j \text{ odd).}$$

A.8.3 A proof of the weak limit

We have for $u\in\mathscr{S}(\mathbb{R}^n)$, according to (1.2.1),

$$\langle(\mathbf{1}\{2\pi(x^2+\xi^2)\le a\})^w u,u\rangle=\iint_{2\pi(x^2+\xi^2)\le a}\mathcal{W}(u,u)(x,\xi)dxd\xi,$$

so that implies

$$\sum_{k\ge 0} F_k(a)\langle \mathbb{P}_k u, u\rangle_{L^2(\mathbb{R}^n)} = \iint_{2\pi(x^2+\xi^2)\le a} \mathcal{W}(u,u)(x,\xi)dxd\xi.$$

Choosing now $u = u_k$ as a normalized eigenfunction of the harmonic oscillator with eigenvalue $k+\frac{1}{2}$, we obtain

$$F_k(a) = \iint_{2\pi(x^2+\xi^2)\le a} \mathcal{W}(u_k,u_k)(x,\xi)dxd\xi.$$

Since the function $(x,\xi)\mapsto \mathcal{W}(u_k,u_k)(x,\xi)$ belongs to the Schwartz class of $\mathbb{R}^{2n}$, we find that

$$\lim_{a\to+\infty} F_k(a) = \iint_{\mathbb{R}^{2n}} \mathcal{W}(u_k,u_k)(x,\xi)dxd\xi = \|u_k\|^2_{L^2(\mathbb{R}^n)} = 1,$$

which is the sought formula.

A.8.4 A different normalization for the Wigner function

The paper [39] is using a different normalization for the Wigner distribution in n dimensions with

$$\widetilde{\mathcal{W}}(u,v)(x,\xi) = (2\pi)^{-n}\int_{\mathbb{R}^n} u\Big(x+\frac{z}{2}\Big)\bar{v}\Big(x-\frac{z}{2}\Big)e^{-iz\cdot\xi}dz.$$

The relationship with definition (1.1.4) is $\widetilde{\mathcal{W}}(u,v)(x,\xi) = \mathcal{W}(u,v)(x,\frac{\xi}{2\pi})(2\pi)^{-n}$. As a result, we find that

$$\mathcal{E}_{lo}(\mathbb{B}^{2n}(R)) = \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{|x|^2+|\xi|^2\le R^2} \widetilde{\mathcal{W}}(u,u)(x,\xi)dxd\xi,$$

is equal to

$$\begin{aligned}\sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} &\iint_{|x|^2+4\pi^2|\xi|^2\le R^2} \mathcal{W}(u,u)(x,\xi)dxd\xi\\ &= \sup_{\|u\|_{L^2(\mathbb{R}^n)}=1} \iint_{2\pi(|x|^2+|\xi|^2)\le R^2} \mathcal{W}(u,u)(x,\xi)dxd\xi,\end{aligned}$$

and we have proven here that for $u\in L^2(\mathbb{R}^n)$ with norm 1

$$\begin{aligned}&\iint_{|x|^2+|\xi|^2\le \frac{a}{2\pi}=\frac{R^2}{2\pi}} \mathcal{W}(u,u)(x,\xi)dxd\xi\\ &\quad\le 1-\frac{1}{(n-1)!}\int_a^{+\infty} e^{-t}t^{n-1}dt = 1-\frac{\Gamma(n,R^2)}{\Gamma(n)},\end{aligned}$$

where the upper incomplete Gamma function $\Gamma(z, x)$ is given by

$$\Gamma(z, x) = \int_x^{+\infty} t^{z-1} e^{-t}\, dt. \tag{A.8.3}$$

This is indeed the result of [39, Theorem 1].

N.B. Let $x > 0$ be given and let $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. Then, we have

$$\Gamma(z, x) = \int_0^{+\infty} (s + x)^{z-1} e^{-s-x}\, ds = e^{-x} \int_0^{+\infty} (s + x)^{z-1} e^{-s}\, ds,$$

so that if $z = n + 1$, $n \in \mathbb{N}$, we find

$$\begin{aligned}\Gamma(n + 1, x) &= e^{-x} \int_0^{+\infty} (s + x)^n e^{-s}\, ds = e^{-x} \sum_{0 \le k \le n} \binom{n}{k} x^k \int_0^{+\infty} s^{n-k} e^{-s}\, ds \\ &= e^{-x} \sum_{0 \le k \le n} \binom{n}{k} x^k \Gamma(n + 1 - k) = n! e^{-x} \sum_{0 \le k \le n} \frac{x^k}{k!}.\end{aligned}$$

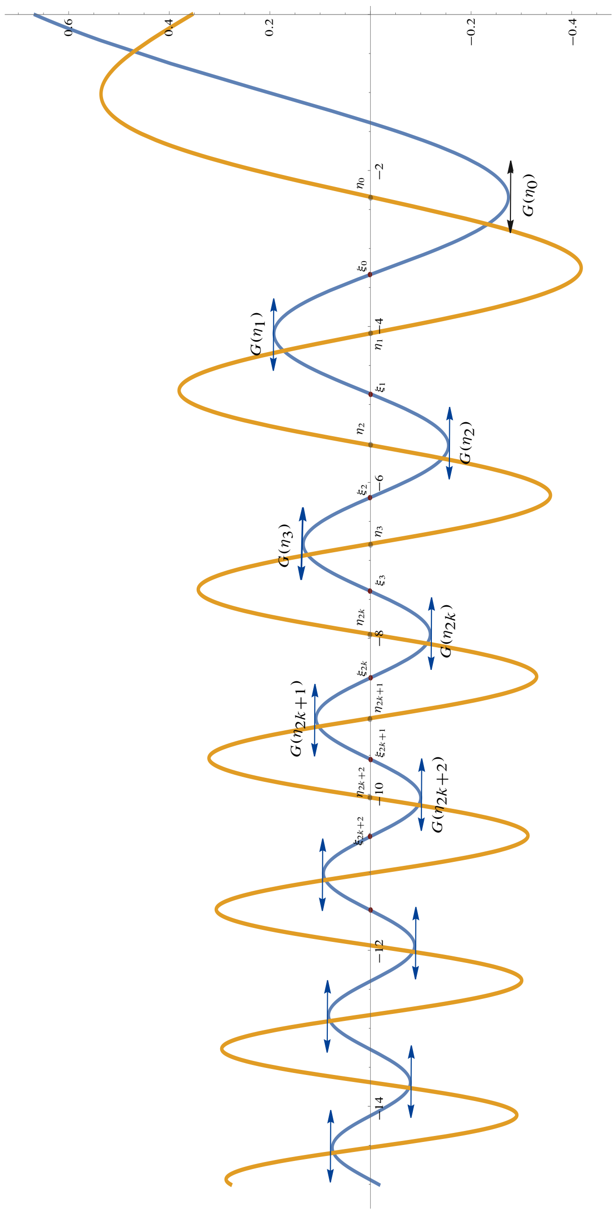

Figure A.1. The function **G** and its derivative the **Airy** function, on $\mathbb{R}_-$.