Chapter 7

Hilbert-valued compressed sensing

In this chapter, we develop Hilbert-valued compressed sensing theory. Here, rather than the classical setting of a vector in $\mathbb{C}^N$, one seeks to recover a Hilbert-valued vector in $\mathcal{V}^N$. This was considered in [52] in the for the classical sparsity model with ℓ^1-minimization. We now consider the more general weighted sparsity model and weighted ℓ^1-minimization. This model was first developed in [121] in the scalar-valued case. See also [2, 37] and [8, Chapter 6]. Note that in this chapter, we shall write $\mathcal{V}$ rather than $\mathcal{V}_h$, as is done in (4.6). Since $\mathcal{V}$ is arbitrary, all the results shown below will also apply in the case of $\mathcal{V}_h$.

7.1 Weighted sparsity and weighted best approximation

Let $\Lambda \subseteq \mathcal{F}$ and $\boldsymbol{w} = (w_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \Lambda} > \boldsymbol{0}$ be positive weights. Given a set $S \subseteq \Lambda$, we define its weighted cardinality as

$$|S|_{\boldsymbol{w}} := \sum_{i \in S} w_i^2.$$

The following two definitions extend Definitions 2.2 and 2.3 to the weighted setting.

Definition 7.1 (Weighted sparsity). Let $\Lambda \subseteq \mathcal{F}$. A $\mathcal{V}$-valued sequence $\boldsymbol{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \Lambda}$ is *weighted* $(k, \boldsymbol{w})$*-sparse* for some $k \geq 0$ and weights $\boldsymbol{w} = (w_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \Lambda} > \boldsymbol{0}$ if

$$|\mathrm{supp}(\boldsymbol{c})|_{\boldsymbol{w}} \leq k,$$

where $\mathrm{supp}(\boldsymbol{z}) = \{\boldsymbol{\nu} : \|z_{\boldsymbol{\nu}}\|_{\mathcal{V}} \neq 0\}$ is the *support* of $\boldsymbol{z}$. The set of such vectors is denoted by $\Sigma_{k,\boldsymbol{w}}$.

Definition 7.2 (Weighted best $(k, \boldsymbol{w})$-term approximation error). Let $\Lambda \subseteq \mathcal{F}$, $0 < p \leq 2$, $\boldsymbol{w} > \boldsymbol{0}$, $\boldsymbol{c} \in \ell^p_{\boldsymbol{w}}(\Lambda; \mathcal{V})$ and $k \geq 0$. The $\ell^p_{\boldsymbol{w}}$*-norm weighted best* $(k, \boldsymbol{w})$*-term approximation error* of $\boldsymbol{c}$ is

$$\sigma_k(\boldsymbol{c})_{p,\boldsymbol{w};\mathcal{V}} = \min\left\{\|\boldsymbol{c} - \boldsymbol{z}\|_{p,\boldsymbol{w};\mathcal{V}} : \boldsymbol{z} \in \Sigma_{k,\boldsymbol{w}}\right\}.$$

Notice that this is equivalent to

$$\sigma_k(\boldsymbol{c})_{p,\boldsymbol{w};\mathcal{V}} = \inf\left\{\|\boldsymbol{c} - \boldsymbol{c}_S\|_{p,\boldsymbol{w};\mathcal{V}} : S \subseteq \Lambda,\ |S|_{\boldsymbol{w}} \leq k\right\}. \tag{7.1}$$

Here and elsewhere, for a sequence $\boldsymbol{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \Lambda}$ and a set $S \subseteq \Lambda$, we define $\boldsymbol{c}_S$ as the sequence with $\boldsymbol{\nu}$th entry equal to $c_{\boldsymbol{\nu}}$ if $\boldsymbol{\nu} \in S$ and zero otherwise.

7.2 The weighted robust null space property

For the rest of this chapter, we consider the index set $\Lambda = \{1, \dots, N\}$ for some $N \in \mathbb{N}$. Our analysis of the weighted SR-LASSO problem is presented in terms of the so-called weighted robust null space property. Let $\boldsymbol{w} > \boldsymbol{0}$ and $k > 0$. A bounded linear operator $\boldsymbol{A} \in \mathcal{B}(\mathcal{V}^N, \mathcal{V}^m)$ has the *weighted robust Null Space Property (rNSP) over $\mathcal{V}$ of order $(k, \boldsymbol{w})$ with constants $0 < \rho < 1$ and $\gamma > 0$* if

$$\|\boldsymbol{x}_S\|_{2;\mathcal{V}} \leq \frac{\rho \|\boldsymbol{x}_{S^c}\|_{1,\boldsymbol{w};\mathcal{V}}}{\sqrt{k}} + \gamma \|\boldsymbol{A}\boldsymbol{x}\|_{2;\mathcal{V}}, \quad \forall \boldsymbol{x} \in \mathcal{V}^N,$$

for any $S \subseteq [N]$ with $|S|_{\boldsymbol{w}} \leq k$.

Importantly, the weighted rNSP implies distance bounds in the $\ell^1_{\boldsymbol{w}}$- and ℓ^2-norms. The following lemma is standard in the scalar case (see, e.g., [8, Lemma 6.24]). We omit the proof of its extension to the Hilbert-valued case, since it follows almost exactly the same arguments.

Lemma 7.3 (Weighted rNSP implies $\ell^1_{\boldsymbol{w}}$ and ℓ^2 distance bounds). *Suppose that $\boldsymbol{A} \in \mathcal{B}(\mathcal{V}^N, \mathcal{V}^m)$ has the weighted rNSP over $\mathcal{V}$ of order $(k, \boldsymbol{w})$ with constants $0 < \rho < 1$ and $\gamma > 0$. Let $\boldsymbol{x}, \boldsymbol{z} \in \mathcal{V}^N$. Then*

$$\begin{aligned}
\|\boldsymbol{z} - \boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}} &\leq C_1 (2\sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \|\boldsymbol{z}\|_{1,\boldsymbol{w};\mathcal{V}} - \|\boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}}) \\
&\quad + C_2 \sqrt{k} \|\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x})\|_{2;\mathcal{V}}, \\
\|\boldsymbol{z} - \boldsymbol{x}\|_{2;\mathcal{V}} &\leq \frac{C_1'}{\sqrt{k}} (2\sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \|\boldsymbol{z}\|_{1,\boldsymbol{w};\mathcal{V}} - \|\boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}}) \\
&\quad + C_2' \|\boldsymbol{A}(\boldsymbol{z} - \boldsymbol{x})\|_{2;\mathcal{V}},
\end{aligned}$$

where the constants are given by

$$C_1 = \frac{(1+\rho)}{(1-\rho)}, \quad C_2 = \frac{2\gamma}{(1-\rho)}, \quad C_1' = \left(\frac{(1+\rho)^2}{1-\rho}\right) \quad \textit{and} \quad C_2' = \left(\frac{(3+\rho)\gamma}{1-\rho}\right).$$

Lemma 7.3 can be used to show distance bounds for exact minimizers of the Hilbert-valued weighted SR-LASSO problem

$$\min_{\boldsymbol{z} \in \mathcal{V}^N} \mathcal{G}(\boldsymbol{z}), \quad \mathcal{G}(\boldsymbol{z}) := \lambda \|\boldsymbol{z}\|_{1,\boldsymbol{w};\mathcal{V}} + \|\boldsymbol{A}\boldsymbol{z} - \boldsymbol{b}\|_{2;\mathcal{V}}. \tag{7.2}$$

Fortunately, it also implies bounds for approximate minimizers, such as those obtained by a finite number of steps of the primal-dual iteration.

Lemma 7.4 (Weighted rNSP implies error bounds for inexact minimizers). *Suppose that $\boldsymbol{A} \in \mathcal{B}(\mathcal{V}^N, \mathcal{V}^m)$ has the weighted rNSP over $\mathcal{V}$ of order $(k, \boldsymbol{w})$ with constants $0 < \rho < 1$ and $\gamma > 0$. Let $\boldsymbol{x} \in \mathcal{V}^N$, $\boldsymbol{b} \in \mathcal{V}^m$ and $\boldsymbol{e} = \boldsymbol{A}\boldsymbol{x} - \boldsymbol{b} \in \mathcal{V}^m$, and consider* *the problem* (7.2) *with parameter*

$$0 < \lambda \le \frac{(1+\rho)^2}{(3+\rho)\gamma} k^{-1/2}. \tag{7.3}$$

Then, for any $\tilde{\boldsymbol{x}} \in \mathcal{V}^N$,

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}} \le C_1 \left(2\sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \frac{\mathcal{G}(\tilde{\boldsymbol{x}}) - \mathcal{G}(\boldsymbol{x})}{\lambda} \right) + \left(\frac{C_1}{\lambda} + C_2 \sqrt{k} \right) \|\boldsymbol{e}\|_{2;\mathcal{V}},$$
$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_{2;\mathcal{V}} \le \frac{C_1'}{\sqrt{k}} \left(2\sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \frac{\mathcal{G}(\tilde{\boldsymbol{x}}) - \mathcal{G}(\boldsymbol{x})}{\lambda} \right) + \left(\frac{C_1'}{\sqrt{k}\lambda} + C_2' \right) \|\boldsymbol{e}\|_{2;\mathcal{V}},$$

where C_1, C_2, C_1' *and* C_2' *are as in Lemma* 7.3.

Proof. First notice that $C_1'/C_2' \le C_1/C_2$ since $0 < \rho < 1$, where C_1, C_2, C_1' and C_2' are as in Lemma 7.3. Hence the condition on λ implies that

$$\lambda \le \min\{C_1/C_2, C_1'/C_2'\} k^{-1/2},$$

Using this lemma and this bound, we deduce that

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}} \le 2C_1 \sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \frac{C_1}{\lambda} \left(\lambda \|\tilde{\boldsymbol{x}}\|_{1,\boldsymbol{w};\mathcal{V}} + \left\| \boldsymbol{A}\tilde{\boldsymbol{x}} - \boldsymbol{b} \right\|_{2;\mathcal{V}} - \lambda \|\boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}} \right)$$
$$+ C_2 \sqrt{K} \|\boldsymbol{e}\|_{2;\mathcal{V}}.$$

The definition of $\mathcal{G}$ in (7.2) gives

$$\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_{1,\boldsymbol{w};\mathcal{V}} \le 2C_1 \sigma_k(\boldsymbol{x})_{1,\boldsymbol{w};\mathcal{V}} + \frac{C_1}{\lambda} (\mathcal{G}(\tilde{\boldsymbol{x}}) - \mathcal{G}(\boldsymbol{x}) + \|\boldsymbol{e}\|_{2;\mathcal{V}}) + C_2 \sqrt{k} \|\boldsymbol{e}\|_{2;\mathcal{V}},$$

which is the first result. The second follows in an analogous manner. ■

7.3 The weighted rNSP and weighted restricted isometry property

In the next chapter, we give explicit conditions in terms of m under which the measurement matrices (4.3) satisfy the weighted rNSP over $\mathcal{V}$. It is well known that showing the (weighted) rNSP directly can be difficult. In the classical, scalar setting, this is overcome by showing that the (weighted) rNSP is implied by the so-called (weighted) restricted isometry property. Hence, in this section, we first introduced this property and describe its relation to the (weighted) rNSP.

Let $\boldsymbol{w} > \boldsymbol{0}$ and $k > 0$. A bounded linear operator $\boldsymbol{A} \in \mathcal{B}(\mathcal{V}^N, \mathcal{V}^m)$ has the *weighted Restricted Isometry Property (RIP)* over $\mathcal{V}$ of order $(k, \boldsymbol{w})$ if there exists a constant $0 < \delta < 1$ such that

$$(1-\delta)\|\boldsymbol{z}\|_{2;\mathcal{V}}^2 \le \|\boldsymbol{A}\boldsymbol{z}\|_{2;\mathcal{V}}^2 \le (1+\delta)\|\boldsymbol{z}\|_{2;\mathcal{V}}^2, \quad \forall \boldsymbol{z} \in \Sigma_{k,\boldsymbol{w}} \subseteq \mathcal{V}^N.$$

The smallest constant such that this property holds is called the $(k, \boldsymbol{w})$th *weighted Restricted Isometry Constant (wRIC)* of $\boldsymbol{A}$, and is denoted as $\delta_{k,\boldsymbol{w}}$.

It is first convenient to show an equivalence between the scalar weighted RIP over $\mathbb{C}$ and the Hilbert-valued weighted RIP over $\mathcal{V}$.

Lemma 7.5 (Weighted RIP over $\mathbb{C}$ is equivalent to the weighted RIP over $\mathcal{V}$). *Let $\boldsymbol{w} > \boldsymbol{0}$, $k > 0$ and $\boldsymbol{A} = (a_{ij})_{i,j=1}^{m,N} \in \mathbb{C}^{m\times N}$ be a matrix. Then $\boldsymbol{A}$ satisfies the weighted RIP over $\mathbb{C}$ of order $(k, \boldsymbol{w})$ with constant $0 < \delta < 1$ if and only if the corresponding bounded linear operator $\boldsymbol{A} \in \mathcal{B}(\mathcal{V}^N, \mathcal{V}^m)$ defined by*

$$\boldsymbol{x} = (x_i)_{i=1}^N \in \mathcal{V}^N \mapsto \boldsymbol{A}\boldsymbol{x} := \left(\sum_{i=1}^N a_{ij} x_j\right)_{i=1}^m \in \mathcal{V}^m,$$

satisfies the weighted RIP over $\mathcal{V}$ of order $(k, \boldsymbol{w})$ with the same constant δ.

Proof. We follow similar arguments to [52, Remark 3.5]. First, we rewrite the equivalence as follows:

$$(1-\delta)\|\boldsymbol{x}\|_{2;\mathcal{V}}^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_{2;\mathcal{V}}^2 \leq (1+\delta)\|\boldsymbol{x}\|_{2;\mathcal{V}}^2, \quad \forall \boldsymbol{x} \in \mathcal{V}^N,\ |\mathrm{supp}(\boldsymbol{x})|_{\boldsymbol{w}} \leq k, \tag{7.4}$$

if and only if

$$(1-\delta)\|\boldsymbol{x}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}\|_2^2 \leq (1+\delta)\|\boldsymbol{x}\|_2^2, \quad \forall \boldsymbol{x} \in \mathbb{C}^N,\ |\mathrm{supp}(\boldsymbol{x})|_{\boldsymbol{w}} \leq k. \tag{7.5}$$

Suppose that (7.5) holds. Let $\boldsymbol{x} = (x_j)_{i=1}^N \in \mathcal{V}^N$ be $(k, \boldsymbol{w})$-sparse and $\{\phi_i\}_i$ be an orthonormal basis of $\mathcal{V}$. Then, for each $i \in [N]$, $x_i \in \mathcal{V}$ can be uniquely represented as

$$x_i = \sum_j \alpha_{ij}\phi_j, \quad \alpha_{ij} \in \mathbb{C}.$$

Let $\boldsymbol{x}_j = (\alpha_{ij})_{i=1}^N \in \mathbb{C}^N$. Then $\mathrm{supp}(\boldsymbol{x}_j) \subseteq \mathrm{supp}(\boldsymbol{x})$ and therefore $\boldsymbol{x}_j$ is $(k, \boldsymbol{w})$-sparse. Hence (7.5) gives

$$(1-\delta)\|\boldsymbol{x}_j\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{x}_j\|_2^2 \leq (1+\delta)\|\boldsymbol{x}_j\|_2^2. \tag{7.6}$$

Now observe that

$$\sum_j \|\boldsymbol{x}_j\|_2^2 = \sum_{i=1}^N \sum_j |\alpha_{ij}|^2 = \sum_{i=1}^N \|x_i\|_{\mathcal{V}}^2 = \|\boldsymbol{x}\|_{2;\mathcal{V}}^2$$

and

$$\sum_j \|\boldsymbol{A}\boldsymbol{x}_j\|_2^2 = \sum_j \sum_{i=1}^m \left|\sum_{k=1}^N a_{ik}\alpha_{kj}\right|^2 = \sum_{i=1}^m \left\|\sum_{k=1}^N a_{ik}x_k\right\|_{\mathcal{V}}^2 = \|\boldsymbol{A}\boldsymbol{x}\|_{2;\mathcal{V}}^2.$$

Summing (7.6) over j, we deduce that (7.4) holds.

Conversely, suppose that (7.4) holds and let $\boldsymbol{z} = (z_i)_{i=1}^N \in \mathbb{C}^N$ with $|\mathrm{supp}(\boldsymbol{z})|_{\boldsymbol{w}} \leq k$. Define $\boldsymbol{x} = (z_i \phi_i) \in \mathcal{V}^N$ and notice that $\|\boldsymbol{x}\|_{2;\mathcal{V}} = \|\boldsymbol{z}\|_2$ and $\|\boldsymbol{A}\boldsymbol{x}\|_{2;\mathcal{V}} = \|\boldsymbol{A}\boldsymbol{z}\|_2$. Since $\mathrm{supp}(\boldsymbol{x}) = \mathrm{supp}(\boldsymbol{z})$ and $|\mathrm{supp}(\boldsymbol{z})|_{\boldsymbol{w}} \leq k$, we now apply (7.4) to deduce that $(1-\delta)\|\boldsymbol{z}\|_2^2 \leq \|\boldsymbol{A}\boldsymbol{z}\|_2^2 \leq (1+\delta)\|\boldsymbol{z}\|_2^2$. We conclude that (7.5) holds. ■

The following result shows that the weighted RIP is a sufficient condition for the weighted rNSP. This result is well known in the scalar-valued case (see, e.g., [8, Theorem 6.26]). Since its extension to the Hilbert-valued case is straightforward, we omit the proof.

Lemma 7.6 (Weighted RIP implies the weighted rNSP). *Let* $\boldsymbol{w} > \boldsymbol{0}$, $k > 0$ *and suppose that* $\boldsymbol{A} \in \mathbb{C}^{m \times N}$ *has the weighted RIP over* $\mathcal{V}$ *of order* $(2k, \boldsymbol{w})$ *with constant* $\delta_{2k,\boldsymbol{w}} < (2\sqrt{2}-1)/7$. *Then* $\boldsymbol{A}$ *has the weighted rNSP of order* $(k, \boldsymbol{w})$ *over* $\mathcal{V}$ *with constants* $\rho = 2\sqrt{2}\delta_{2k,\boldsymbol{w}}/(1-\delta_{2k,\boldsymbol{w}})$ *and* $\gamma = \sqrt{1+\delta_{2k,\boldsymbol{w}}}/(1-\delta_{2k,\boldsymbol{w}})$.