Appendix A

Best polynomial approximation rates for holomorphic functions

In this appendix, we recap a series of standard best approximation error bounds for polynomial approximation of holomorphic functions. These are used in Chapter 10 to estimate the various error terms appearing in Theorems 8.2–8.4.

A.1 The finite-dimensional case

We first consider the finite-dimensional case, where $\mathcal{U} = [-1, 1]^d$ for $d < \infty$ and $f : \mathcal{U} \to \mathcal{V}$ is a Hilbert-valued function (in fact, the following results also apply in the more general setting of Banach-valued functions; however, we shall not consider this explicitly). We now summarize the various approximation error bounds in the following theorem. This result combines various well-known results in the literature. It is essentially the same as [8, Theorem 3.25]. However, we have made a number of minor edits to fit the notation and setup of this work (see Remark A.2 below).

Theorem A.1 (Best s-term decay rates; finite dimensions). *Let $d \in \mathbb{N}$, $f \in \mathcal{B}(\boldsymbol{\rho})$ for some $\boldsymbol{\rho} > \mathbf{1}$, where $\mathcal{B}(\boldsymbol{\rho})$ is as in (2.6), and $\boldsymbol{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{N}_0^d}$ be its Chebyshev or Legendre coefficients. Then the following best s-term decay rates hold:*

(i) *for any $0 < p \leq q \leq 2$ and $s \in \mathbb{N}$, there exists a lower set $S \subset \mathbb{N}_0^d$ of size $|S| \leq s$ such that*

$$\sigma_s(\boldsymbol{c})_{q;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}} \leq C \cdot s^{1/q-1/p},$$

where $\sigma_s(\boldsymbol{c})_{q;\mathcal{V}}$ is as in Definition 2.3 (with $\Lambda = \mathbb{N}_0^d$), $\boldsymbol{u}$ is as in (4.7) and $C = C(d, p, \boldsymbol{\rho}) > 0$ depends on d, p and $\boldsymbol{\rho}$ only;

(ii) *for any $0 < p \leq q \leq 2$ and $k > 0$,*

$$\sigma_k(\boldsymbol{c})_{q,\boldsymbol{u};\mathcal{V}} \leq C \cdot k^{1/q-1/p},$$

where $\sigma_k(\boldsymbol{c})_{q,\boldsymbol{u};\mathcal{V}}$ is as in Definition 7.2, $\boldsymbol{u}$ is as in (4.7) and

$$C = C(d, p, \boldsymbol{\rho}) > 0$$

depends on d, p and $\boldsymbol{\rho}$ only;

(iii) *for any $0 < p \leq 2$,*

$$0 < \gamma < (d+1)^{-1} \left(d! \prod_{j=1}^{d} \log(\rho_j) \right)^{1/d},$$

and $s \in \mathbb{N}$, there exists a lower set $S \subset \mathbb{N}_0^d$ of size $|S| \leq s$ such that

$$\sigma_s(\boldsymbol{c})_{p;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{p;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{p,\boldsymbol{u};\mathcal{V}} \leq C \cdot \exp(-\gamma s^{1/d}),$$

where $\sigma_s(\boldsymbol{c})_{p;\mathcal{V}}$ is as in Definition 2.3 (with $\Lambda = \mathbb{N}_0^d$), $\boldsymbol{u}$ is as in (4.7) and $C = C(d, \gamma, p, \boldsymbol{\rho}) > 0$ depends on d, γ, p and $\boldsymbol{\rho}$ only.

Remark A.2. There are several differences between Theorem A.1 and [8, Theorem 3.25]. A minor difference is that we do not specify the various constants C appearing in the result. Another difference is in the presentation of (iii). Here we allow arbitrary $s \geq 1$ (instead of $s \geq \bar{s}$) at the expense of a larger (and unspecified) constant C. The main difference, however, is the additional term $\|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}}$ appearing in (i). This can be shown as follows. First, one defines the sequence $\bar{\boldsymbol{c}} = (u_{\boldsymbol{\nu}}^{2/q-1} c_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathbb{N}_0^d}$ so that $\|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}} = \|\bar{\boldsymbol{c}} - \bar{\boldsymbol{c}}_S\|_{q;\mathcal{V}}$ and then uses Stechkin's inequality in lower sets (see, e.g., [8, Lemma 3.9]) to show that $\|\bar{\boldsymbol{c}} - \bar{\boldsymbol{c}}_S\|_{q;\mathcal{V}} \leq s^{1/q-1/p} \|\bar{\boldsymbol{c}}\|_{p,M;\mathcal{V}}$, where $\|\cdot\|_{p,M;\mathcal{V}}$ is the norm on the *majorant ℓ^p space* $\ell^p_M(\mathbb{N}_0^d; \mathcal{V})$ (see, e.g., [8, Definition 3.8]). Finally, it can be shown that $\|\bar{\boldsymbol{c}}\|_{p,M;\mathcal{V}} \leq C(d, p, \boldsymbol{\rho})$ using standard arguments. See, e.g., [8, Lemma 7.19] (this lemma only considers the scalar-valued case; however the extension to the Hilbert-valued case is straightforward).

Note that Theorem A.1 immediately implies Theorems 2.4 and 2.6. For the former, we note that

$$\|f - f_{S_1}\|_{L^2_\varrho(\mathcal{U};\mathcal{V})} = \|\boldsymbol{c} - \boldsymbol{c}_{S_1}\|_{2;\mathcal{V}} \quad \text{and} \quad \|f - f_{S_2}\|_{L^\infty(\mathcal{U};\mathcal{V})} \leq \|\boldsymbol{c} - \boldsymbol{c}_{S_2}\|_{1,\boldsymbol{u};\mathcal{V}}.$$

We then apply (i) with $q = 2$ or $q = 1$. For the latter, we use (iii) with $p = 1$.

A.2 The infinite-dimensional case

We now consider the infinite-dimensional case, where $d = \infty$ and $\mathcal{U} = [-1, 1]^{\mathbb{N}}$.

Theorem A.3 (Best s-term decay rates; infinite-dimensional case). *Let $d = \infty$, $0 < p < 1$, $\varepsilon > 0$, $\boldsymbol{b} \in \ell^p(\mathbb{N})$ with $\boldsymbol{b} > \boldsymbol{0}$ and $f \in \mathcal{B}(\boldsymbol{b}, \varepsilon)$, where $\mathcal{B}(\boldsymbol{b}, \varepsilon)$ is as in (2.7). Let $\boldsymbol{c} = (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu} \in \mathcal{F}}$ be the Chebyshev or Legendre coefficients of f. Then the following best s-term decay rates hold:*

(i) *For any $p \leq q < \infty$ and $s \in \mathbb{N}$, there exists a lower set $S \subset \mathcal{F}$ of size $|S| \leq s$ such that*

$$\sigma_s(\boldsymbol{c})_{q;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q;\mathcal{V}} \leq \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}} \leq C \cdot s^{1/q-1/p},$$

where $\sigma_s(\boldsymbol{c})_{q;\mathcal{V}}$ is as in Definition 2.3 (with $\Lambda = \mathcal{F}$), $\boldsymbol{u}$ is as in (4.7) and $C = C(\boldsymbol{b}, \varepsilon, p) > 0$ depends on $\boldsymbol{b}$, ε and p only.

(ii) *For any $p \le q \le 2$ and $k > 0$,*

$$\sigma_k(\boldsymbol{c})_{q,\boldsymbol{u};\mathcal{V}} \le C \cdot k^{1/q-1/p},$$

where $\sigma_k(\boldsymbol{c})_{q,\boldsymbol{u};\mathcal{V}}$ is as in Definition 7.2, $\boldsymbol{u}$ is as in (4.7) and $C = (\boldsymbol{b}, \varepsilon, p) > 0$ depends on $\boldsymbol{b}$, ε and p only.

(iii) *Suppose that $\boldsymbol{b}$ is monotonically nonincreasing. Then, for any $p \le q < \infty$ and $s \in \mathbb{N}$, there exists an anchored set $S \subset \mathcal{F}$ of size $|S| \le s$ such that*

$$\sigma_s(\boldsymbol{c})_{q;\mathcal{V}} \le \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q;\mathcal{V}} \le \|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}} \le C \cdot s^{1/q-1/p},$$

where $\sigma_s(\boldsymbol{c})_{q;\mathcal{V}}$ is as in Definition 2.3 (with $\Lambda = \mathcal{F}$), $\boldsymbol{u}$ is as in (4.7) and $C = (\boldsymbol{b}, \varepsilon, p) > 0$ depends on $\boldsymbol{b}$, ε and p only.

Remark A.4. This theorem is based on [8, Theorems 3.29 and 3.33]. Besides the term $\|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}}$, parts (i) and (iii) can be found in [8, Theorem 3.29] and [8, Theorem 3.33], respectively. As in the finite-dimensional case (see Remark A.2), the main difference is the assertion of the bound on $\|\boldsymbol{c} - \boldsymbol{c}_S\|_{q,\boldsymbol{u};\mathcal{V}}$. This can be established through similar arguments, using either the majorant ℓ^p space $\ell^p_M(\mathcal{F};\mathcal{V})$ or the anchored ℓ^p space $\ell^p_A(\mathcal{F};\mathcal{V})$ (see, e.g., [8, Definition 3.31]) and then Stechkin's inequality in lower or anchored sets (see, e.g., [8, Lemma 3.32]). See also [8, Lemma 7.23] (this lemma only considers the scalar-valued case; however the extension to the Hilbert-valued case is straightforward).

Note that neither [8, Theorem 3.29] nor [8, Theorem 3.33] asserts part (ii) of Theorem A.3. This can be shown via the weighted Stechkin's inequality (see, e.g., [8, Lemma 3.12]), which gives the bound $\sigma_k(\boldsymbol{c})_{q,\boldsymbol{u};\mathcal{V}} \le \|\boldsymbol{c}\|_{p,\boldsymbol{u};\mathcal{V}} \cdot k^{1/q-1/p}$, and then by showing that $\|\boldsymbol{c}\|_{p,\boldsymbol{u};\mathcal{V}} \le C(\boldsymbol{b}, \varepsilon, p)$. This latter fact can be obtained by the straightforward extension of [8, Lemma 7.23] to the Hilbert-valued setting.

Note that Theorem A.3 implies Theorem 2.5. This follows from (i) by setting either $q = 2$ or $q = 1$.